Exercise 1.
Take a bounded connected open set \( \Omega \subset \mathbb{R}^N \) and a function \( u \in W^{k,p}(\Omega) \) with \( k \geq 1 \) and \( 1 \leq p \leq \infty \), show that:

1. If \( \nabla u = 0 \) then \( u \) is constant.

2. If, for all multi-indices \( |\alpha| = k \), we have \( \partial^\alpha u = 0 \) then \( u \) is a polynomial of total degree\(^1\) less or equal to \( k - 1 \).

Solution 1.

**Question 1.** We will show that using a density argument. First, let’s show that if \( v \in C^\infty(\Omega) \) and \( \nabla v = 0 \) then \( v \) is constant. Let’s fix a point \( x \in \Omega \), then for all \( y \in \Omega \) there exists a smooth path \( \gamma_{x,y} : [0,1] \to \Omega \) such that \( \gamma_{x,y}(0) = x \) and \( \gamma_{x,y}(1) = y \). Then we compute

\[
u(y) = u(x) + \int_0^1 \nabla u(\gamma_{x,y}(t)) \cdot \gamma_{x,y}'(t) \, dt = u(x)
\]

which give that \( u \) is constant.

Using the same mollifier \( u_\varepsilon(x) = \varphi_\varepsilon * u(x) \) as in the lecture note. The function \( u_\varepsilon \) is defined on \( \Omega_\varepsilon = \{ x \in \Omega \mid \text{dist}(x, \partial \Omega) > \varepsilon \} \) and we have that \( \nabla u_\varepsilon(x) = \varphi_\varepsilon * (\nabla u)(x) = 0 \) on \( \Omega_\varepsilon \). Therefore, there exists a constant \( c_\varepsilon \) such that \( u_\varepsilon = c_\varepsilon \) on \( \Omega_\varepsilon \). For any \( \varepsilon_0 > 0 \), \( u_\varepsilon \equiv c_\varepsilon \to u \) in \( W^{1,p}(\Omega_{\varepsilon_0}) \). Which give that \( c_\varepsilon \) is a Cauchy sequence so there exists a constant \( c \) such that \( c_\varepsilon \to c \). So \( u \equiv c \) on \( \Omega_{\varepsilon_0} \) since it is true for all \( \varepsilon_0 > 0 \) that mean that \( u \equiv c \) on \( \Omega \).

Technically, the density argument only works for \( p < \infty \), however \( W^{k,\infty}(\Omega) \subset W^{k,1}(\Omega) \) because \( L^\infty(\Omega) \subset L^1(\Omega) \) (\( ||u||_1 \leq ||\Omega|| u||_\infty \) with \( ||\Omega|| \) the Lebesgue measure of \( \Omega \)), therefore we also get that \( u \) is constant in the case \( p = \infty \).

**Question 2.** We will show that by induction on \( k \geq 1 \). The **Question 1.** take care of the case \( k = 1 \). Take \( k \geq 2 \), let’s assume that for all multi-indices \( |\alpha| = k - 1 \), we have \( \partial^\alpha u = 0 \) then \( u \) is a polynomial of total degree less or equal to \( k - 2 \). For \( \alpha \) a multi-index such that \( |\alpha| = k - 1 \), then we have \( \nabla \partial^\alpha u = 0 \) therefore there exists \( c_\alpha \) such that \( \partial^\alpha u = c_\alpha \). We define

\[
P_{k-1} = \sum_{|\beta|=k-1} \frac{c_\beta}{\beta!} x^\beta \quad \text{with } \beta! = \prod_{n=1}^N \beta_n !.
\]

\(^1\)We recall that a multivariate polynomial of total degree \( k \) is given by \( P(x) = \sum_{|\alpha|=k} c_\alpha x^\alpha \) where \( c_\alpha \in \mathbb{R} \) and \( x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N} \) for all multi-indices \( \alpha \in \{0, \ldots, k\}^N \).
We get $\partial^\alpha (u - P_{k-1}) = 0$ because for multi-indices $\alpha \neq \beta$ such that $|\alpha| = |\beta|$ we have that $\partial^\alpha x^\beta = \partial_1^\alpha x^{\beta_1} \cdots \partial_N^\alpha x^{\beta_N} = 0$ (there exists $j$ such that $\alpha_j > \beta_j$). By the induction hypothesis there exists a polynomial $Q$ of total degree less or equal to $k - 2$ such that $u - P_{k-1} = Q$. Therefore, by induction, $u$ is a polynomial of total degree less or equal to $k - 1$.

**Remark 1.** By the local nature of the derivative Exercise 1 stay true for $u \in W^{k,p}_{\text{loc}}(\mathbb{R}^N)$.

**Exercise 2.**

We define the open half-space $\mathbb{R}^N_+ := \{(x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_N > 0\}$ and an extension operator $E : L^p(\mathbb{R}^N_+) \to L^p(\mathbb{R}^N)$ define by

$$Eu(x) = \begin{cases} u(x) & \text{if } x \in \mathbb{R}^N_+ \\ u(x', -x_N) & \text{if } x' \in \mathbb{R}^{N-1} \text{ and } x_N < 0 \end{cases}.$$

1. Show that $E : W^{1,p}(\mathbb{R}^N_+) \to W^{1,p}(\mathbb{R}^N)$ is a linear continuous operator.

2. Show that there exists $u \in W^{2,p}(\mathbb{R}^N_+)$ such that $Eu \notin W^{2,p}(\mathbb{R}^N)$. Search $u$ of the form $u(x) = \varphi(x') f(x_N)$ with $\varphi \in C^\infty_0(\mathbb{R}^{N-1})$ and $f \in C^2(\mathbb{R}_+)$ with $f'(0) \neq 0$.

Now, we define an extension operator $F : L^p(\mathbb{R}^N_+) \to L^p(\mathbb{R}^N)$ define by

$$Fu(x) = \begin{cases} u(x) & \text{if } x_N > 0 \\ a u(x', -\lambda x_N) + b u(x', -\mu x_N) & \text{if } x_N < 0 \end{cases}$$

with $a, b \in \mathbb{R}$ and $\lambda, \mu \in \mathbb{R}_+$.

3. For a multi-index $\alpha \in \mathbb{N}^N$ such that $|\alpha| \leq 2$, we define $v_\alpha$ by

$$v_\alpha(x) = \begin{cases} \partial^\alpha u(x) & \text{if } x_N > 0 \\ a(-\lambda)^{\alpha_N} \partial^\alpha u(x', -\lambda x_N) + b(-\mu)^{\alpha_N} \partial^\alpha u(x', -\mu x_N) & \text{if } x_N < 0 \end{cases}.$$

Show that $\partial^\alpha (Fu) = v_\alpha$ if, and only if,

$$\begin{cases} a + b = 1 \\ -\lambda a - \mu b = 1 \end{cases}.$$

4. Show that there exist $a, b \in \mathbb{R}$ and $\lambda, \mu \in \mathbb{R}_+$ such that $F : W^{2,p}(\mathbb{R}^N_+) \to W^{2,p}(\mathbb{R}^N)$ is a linear continuous operator.

**Solution 2.**

We define useful notation for the corrections, the lower half-space $\mathbb{R}^N_- := \{ x \in \mathbb{R}^N \mid x_N < 0 \}$ and the normal $\nu : x' \in \mathbb{R}^{N-1} \mapsto (0, \ldots, 0, 1)^T$.

**Question 1.** Since the operator $E$ extend the function across the interface $\mathbb{R}^{N-1} \times \{0\}$, we need to carefully compute the weak derivative. For $1 \leq i \leq N - 1$ and $\varphi \in C^\infty_0(\mathbb{R}^N)$, using the Green’s first identity, we compute

$$\int_{\mathbb{R}^N} Eu(x) \partial_i \varphi(x) \, dx = \int_{\mathbb{R}^N_+} u(x) \partial_i \varphi(x) \, dx + \int_{\mathbb{R}^N_-} u(x', -x_N) \partial_i \varphi(x) \, dx$$
\[- \int_{\mathbb{R}^N} \partial_i u(x) \varphi(x) \, dx - \int_{\mathbb{R}^N} \partial_i u(x', -x_N) \varphi(x) \, dx \]

there is no boundary term because the \( \nu_i = 0 \) for \( 1 \leq i \leq N - 1 \). We obtain

\[
\partial_i E u(x) = \begin{cases} 
\partial_i u(x) & \text{if } x_N > 0 \\
\partial_i u(x', -x_N) & \text{if } x_N < 0.
\end{cases}
\]

For the last derivative and \( \varphi \in C_0^\infty(\mathbb{R}^N) \), using the Green’s first identity, we compute

\[
\int_{\mathbb{R}^N} E u(x) \partial_N \varphi(x) \, dx = \int_{\mathbb{R}^N} u(x) \partial_N \varphi(x) \, dx + \int_{\mathbb{R}^N} u(x', -x_N) \partial_N \varphi(x) \, dx
\]

\[
= - \int_{\mathbb{R}^N} \partial_N u(x) \varphi(x) \, dx + \int_{\mathbb{R}^N} \partial_N u(x', -x_N) \varphi(x) \, dx
\]

\[
- \int_{\mathbb{R}^{N-1}} u(x', 0) \varphi(x', 0) \, dx' + \int_{\mathbb{R}^{N-1}} u(x', 0) \varphi(x', 0) \, dx'
\]

\[
= - \int_{\mathbb{R}^N} \partial_N u(x) \varphi(x) \, dx + \int_{\mathbb{R}^N} \partial_N u(x', -x_N) \varphi(x) \, dx
\]

which give

\[
\partial_N E u(x) = \begin{cases} 
\partial_N u(x) & \text{if } x_N > 0 \\
-\partial_N u(x', -x_N) & \text{if } x_N < 0.
\end{cases}
\]

From those expressions, we have \( \|E u\|_{W^{1,p}(\mathbb{R}^N)} = 2^{1/p} \|u\|_{W^{1,p}(\mathbb{R}^N)} \) which tell us that the operator \( E : W^{1,p}(\mathbb{R}^N) \to W^{1,p}(\mathbb{R}^N) \) is a linear continuous.

**Question 2.** We take \( \varphi \in C_0^\infty(\mathbb{R}^{N-1}) \) and define the function \( u(x', x_N) = \varphi(x') \exp^{-x_N} \).

The function \( u \in C_0^\infty(\mathbb{R}_+^N) \) and all the derivative up to order 2 are \( p \)-integrable, so we have \( u \in \text{W}^{2,p}(\mathbb{R}_+^N) \). The extension \( E u \) is \( x \mapsto \varphi(x') \exp^{-|x_N|} \). We will show that the second derivative \( \partial_N^2 E u \) is not in \( L^p(\mathbb{R}) \) and therefore, we have \( E u \notin \text{W}^{2,p}(\mathbb{R}_+^N) \).

Now, we show that the second derivative \( \partial_N^2 E u \) is not in \( L^p(\mathbb{R}) \), to do that we compute \( \partial_N^2 (\exp^{-|x_N|}) \). From the lecture, we have \( \partial_N (\exp^{-|x_N|}) = -\text{sgn}(x_N) \exp^{-|x_N|} \) then, for \( \psi \in C_0^\infty(\mathbb{R}) \), we compute the second weak derivative

\[
\int_{\mathbb{R}} (-\text{sgn}(x_N) \exp^{-|x_N|}) \psi'(x_N) \, dx_N = - \int_{\mathbb{R}_+} \exp^{-x_N} \psi'(x_N) \, dx_N + \int_{\mathbb{R}_-} \exp^{x_N} \psi'(x_N) \, dx_N
\]

\[
= - \int_{\mathbb{R}_+} \exp^{-x_N} \psi(x_N) \, dx_N - \int_{\mathbb{R}_-} \exp^{x_N} \psi(x_N) \, dx_N - 2\psi(0)
\]

\[
= - 2\psi(0) - \int_{\mathbb{R}} \exp^{-|x_N|} \psi(x_N) \, dx_N.
\]

There do not exist a function \( f \in L^1_{\text{loc}}(\mathbb{R}) \) such that \( \int_{\mathbb{R}} f(x_N) \psi(x_N) \, dx_N = 2\psi(0) \), for all \( \psi \in C_0^\infty(\mathbb{R}) \). To see that, we consider the sequence of function \( \psi_n \in C_0^\infty(\mathbb{R}) \), for \( n \geq 1 \), such that \( \psi_n \equiv 1 \) for \( x \in [-\frac{1}{2n}, \frac{1}{2n}] \), \( \text{supp}(\psi_n) \subset [-\frac{1}{n}, \frac{1}{n}] \), and \( 0 \leq \psi_n \leq 1 \). We have

\[
2\psi_n(0) = \int_{\mathbb{R}} f(x_N) \psi_n(x_N) \, dx_N = \int_{-1}^1 f(x_N) \psi_n(x_N) \, dx_N
\]

and, using the dominated convergence theorem on the sequence \( (f \psi_n)_{n \in \mathbb{N}} \) dominated by \( f \in L^1((-1, 1)) \) and it pointwise converge to 0 almost everywhere, we obtain

\[
0 = \lim_{n \to +\infty} \int_{\mathbb{R}} f(x_N) \psi_n(x_N) \, dx_N = \lim_{n \to +\infty} 2\psi_n(0) = 2
\]
which is absurd, so \( f \in L^1_{\text{loc}}(\mathbb{R}) \) does not exist.

**Question 3.** For \( \alpha \in \mathbb{N}^N \) multi-index such that \( |\alpha| \leq 2 \), we consider three cases depending on the value of \( \alpha_N \). If \( \alpha_N = 0 \), the computation of the \( \partial^\alpha \) weak derivative is straightforward using the integration by part and the fact that \( \nu_i = 0 \) for \( i \in \{1, \ldots, N-1\} \). If \( \alpha_N = 1 \), and for \( \varphi \in C_0^\infty(\mathbb{R}^N) \), we compute

\[
\int_{\mathbb{R}^N} Fu \partial^\alpha \varphi \, dx = \int_{\mathbb{R}^N} u \partial^{\alpha'} \partial_N \varphi \, dx + \int_{\mathbb{R}^N} \left[a u(x', -\lambda x_N) + b u(x', -\mu x_N)\right] \partial^{\alpha'} \partial_N \varphi \, dx
\]

\[
= \int_{\mathbb{R}^N} \partial^{\alpha'} u \partial_N \varphi \, dx + \int_{\mathbb{R}^N} \left[a \partial^{\alpha'} u(x', -\lambda x_N) + b \partial^{\alpha'} u(x', -\mu x_N)\right] \partial_N \varphi \, dx
\]

\[
= -\int_{\mathbb{R}^N} \partial^{\alpha'} u \partial_N \varphi \, dx + \int_{\mathbb{R}^N} \left[\lambda a \partial^{\alpha'} \partial_N u(x', -\lambda x_N) + \mu b \partial^{\alpha'} \partial_N u(x', -\mu x_N)\right] \varphi \, dx
\]

\[
\leq -\int_{\mathbb{R}^N} \partial^{\alpha'} u \varphi \, dx + \int_{\mathbb{R}^N} \left[a + b\right] \partial^{\alpha'} u(x', 0) \varphi \, dx'
\]

therefore, we have \( \partial^\alpha Fu = v_\alpha \) if, and only if, \( a + b = 1 \). If \( \alpha_N = 2 \), from the previous computation, we have

\[
\int_{\mathbb{R}^N} Fu \partial^\alpha \varphi \, dx = \int_{\mathbb{R}^N} u \partial^{\alpha'} \partial_N^2 \varphi \, dx + \int_{\mathbb{R}^N} \left[a u(x', -\lambda x_N) + b u(x', -\mu x_N)\right] \partial^{\alpha'} \partial_N^2 \varphi \, dx
\]

\[
= -\int_{\mathbb{R}^N} \partial^{\alpha'} \partial_N u \partial_N \varphi \, dx
\]

\[
+ \int_{\mathbb{R}^N} \left[\lambda a \partial^{\alpha'} \partial_N u(x', -\lambda x_N) + \mu b \partial^{\alpha'} \partial_N u(x', -\mu x_N)\right] \partial_N \varphi \, dx
\]

\[
= \int_{\mathbb{R}^N} v_\alpha \varphi \, dx + \int_{\mathbb{R}^N} \left[1 + \lambda a + \mu b\right] \partial^{\alpha'} \partial_N u(x', 0) \varphi(x', 0) \, dx'
\]

therefore, we have \( \partial^\alpha Fu = v_\alpha \) if, and only if, \( -\lambda a - \mu b = 1 \).

**Question 4.** We write the system of equation \( a + b = 1 \) and \( -\lambda a - \mu b = 1 \) with matrix, we obtain

\[
M(\lambda, \mu) \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{with} \quad M(\lambda, \mu) = \begin{pmatrix} 1 & -1 \\ -\lambda & -\mu \end{pmatrix}.
\]

The determinant of \( M(\lambda, \mu) \) is \( \lambda - \mu \), if we take any \( 0 < \lambda < \mu \) the matrix \( M(\lambda, \mu) \) is invertible and there exists \( a, b \) such that \( a + b = 1 \) and \( -\lambda a - \mu b = 1 \). For a multi-index \( \alpha \) such that \( |\alpha| \leq 2 \), we compute

\[
\| \partial^\alpha Fu \|_{L^p(\mathbb{R}^N)} = \left\| \begin{array}{c} u 1_{\mathbb{R}^N_+} + (a(-\lambda)^{\alpha_N} \partial^\alpha u(x', -\lambda x_N) + b(-\mu)^{\alpha_N} \partial^\alpha u(x', -\mu x_N)) 1_{\mathbb{R}^N_+} \\
\end{array} \right\|_{L^p(\mathbb{R}^N)}
\]

\[
\leq \| \partial^\alpha u \|_{L^p(\mathbb{R}^N_+)} + \| a(-\lambda)^{\alpha_N} \partial^\alpha u(x', -\lambda x_N) + b(-\mu)^{\alpha_N} \partial^\alpha u(x', -\mu x_N) \|_{L^p(\mathbb{R}^N_+)}
\]

\[
\leq \| \partial^\alpha u \|_{L^p(\mathbb{R}^N_+)} + |a| \lambda^{\alpha_N} \| \partial^\alpha u(x', -\lambda x_N) \|_{L^p(\mathbb{R}^N_+)} + |b| \mu^{\alpha_N} \| \partial^\alpha u(x', -\mu x_N) \|_{L^p(\mathbb{R}^N_+)}
\]

\[
\leq \| \partial^\alpha u \|_{L^p(\mathbb{R}^N_+)} + |a| \lambda^{\alpha_N-1} \| \partial^\alpha u \|_{L^p(\mathbb{R}^N_+)} + |b| \mu^{\alpha_N} \| \partial^\alpha u \|_{L^p(\mathbb{R}^N_+)}
\]

\[
\leq C_\alpha \| \partial^\alpha u \|_{L^p(\mathbb{R}^N_+)}
\]

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with $C_\alpha = 1 + |a| \lambda^{\alpha N-1} + |b| \mu^{\alpha N-1}$. From those estimations, we obtain

$$\| Fu \|_{W^{2,p}(\mathbb{R}^N)} = \left( \sum_{|\alpha| \leq 2} \| \partial^\alpha F u \|_{L^p(\mathbb{R}^N)}^p \right)^{\frac{1}{p}} \leq C \| u \|_{W^{2,p}(\mathbb{R}^N)}$$

with $C = \max \{ C_\alpha \mid \alpha \in \mathbb{N}^N, |\alpha| \leq 2 \}$ and it gives that $F : W^{2,p}(\mathbb{R}^N_+) \to W^{2,p}(\mathbb{R}^N)$ is a linear continuous operator.

**Remark 2.** The construction of the extension operator $F$ can be generalised to have an extension operator from $W^{k,p}(\mathbb{R}^N_+) \to W^{k,p}(\mathbb{R}^N)$ for an integer $k \geq 1$. Define a extension operator $G : L^p(\mathbb{R}^N_+) \to L^p(\mathbb{R}^N)$ define by

$$Gu(x) = \begin{cases} u(x) & \text{if } x_N > 0 \\ \sum_{j=0}^{k-1} a_j u(x', -\lambda_j x_N) & \text{if } x_N < 0 \end{cases}$$

and show that there exists $(\lambda_j)_{j \in \{0, \ldots, k-1\}} \in (\mathbb{R}_+)^k$ and $(a_j)_{j \in \{0, \ldots, k-1\}} \in \mathbb{R}^k$ such that $G : W^{k,p}(\mathbb{R}^N_+) \to W^{k,p}(\mathbb{R}^N)$ is a linear continuous operator.