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Sobolev spaces: tutorials

Exercise sheet 03 with solution

Exercise 1.
Take a bounded connected open set $\Omega \subset \mathbb{R}^N$ and a function $u \in \mathrm{W}^{k,p}(\Omega)$ with $k \geq 1$ and $1 \leq p \leq \infty$, show that:

1. If $\nabla u = 0$ then u is constant.
2. If, for all multi-indices $|\alpha| = k$, we have $\partial^\alpha u = 0$ then u is a polynomial of total degree[1] less or equal to $k-1$.

Solution 1.
Question 1. We will show that using a density argument. First, let's show that if $v \in C^\infty(\Omega)$ and $\nabla v = 0$ then v is constant. Let's fix a point $x \in \Omega$, then for all $y \in \Omega$ there exists a smooth path $\gamma_{x,y} : [0,1] \to \Omega$ such that $\gamma_{x,y}(0) = x$ and $\gamma_{x,y}(1) = y$. Then we compute

$$u(y) = u(x) + \int_0^1 \nabla u(\gamma_{x,y}(t)) \cdot \gamma_{x,y}'(t) \,\mathrm{d}t = u(x)$$

which give that u is constant.

Using the same mollifier $u_\varepsilon(x) = \varphi_\varepsilon * u(x)$ as in the lecture note. The function u_ε is defined on $\Omega_\varepsilon = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$ and we have that $\nabla u_\varepsilon(x) = \varphi_\varepsilon * (\nabla u)(x) = 0$ on Ω_ε. Therefore, there exists a constant c_ε such that $u_\varepsilon = c_\varepsilon$ on Ω_ε. For any $\varepsilon_0 > 0$, $u_\varepsilon \equiv c_\varepsilon \to u$ in $\mathrm{W}^{1,p}(\Omega_{\varepsilon_0})$. Which give that c_ε is a Cauchy sequence so there exists a constant c such that $c_\varepsilon \to c$. So $u \equiv c$ on Ω_{ε_0} since it is true for all $\varepsilon_0 > 0$ that mean that $u \equiv c$ on Ω.

Technically, the density argument only works for $p < \infty$, however $\mathrm{W}^{k,\infty}(\Omega) \subset \mathrm{W}^{k,1}(\Omega)$ because $\mathrm{L}^\infty(\Omega) \subset \mathrm{L}^1(\Omega)$ ($\|u\|_1 \leq |\Omega| \, \|u\|_\infty$ with $|\Omega|$ the Lebesgue measure of Ω), therefor we also get that u is constant in the case $p = \infty$.

Question 2. We will show that by induction on $k \geq 1$. The *Question 1.* take care of the case $k = 1$. Take $k \geq 2$, let's assume that for all multi-indices $|\alpha| = k-1$, we have $\partial^\alpha u = 0$ then u is a polynomial of total degree less or equal to $k-2$. For α a multi-index such that $|\alpha| = k-1$, then we have $\nabla \partial^\alpha u = 0$ therefore there exists c_α such that $\partial^\alpha u = c_\alpha$. We define

$$P_{k-1} = \sum_{|\beta|=k-1} \frac{c_\beta}{\beta!} x^\beta \qquad \text{with } \beta! = \prod_{n=1}^{N} \beta_n!.$$

[1] We recall that a multivariate polynomial of total degree k is given by $P(x) = \sum_{|\alpha|=k} c_\alpha x^\alpha$ where $c_\alpha \in \mathbb{R}$ and $x^\alpha = x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}$ for all multi-indices $\alpha \in \{0, \ldots, k\}^N$.

We get $\partial^\alpha (u - P_{k-1}) = 0$ because for multi-indices $\alpha \neq \beta$ such that $|\alpha| = |\beta|$ we have that $\partial^\alpha x^\beta = \partial_1^{\alpha_1} x^{\beta_1} \cdots \partial_1^{\alpha_N} x^{\beta_N} = 0$ (there exists j such that $\alpha_j > \beta_j$). By the induction hypothesis there exists a polynomial Q of total degree less or equal to $k-2$ such that $u - P_{k-1} = Q$. Therefore, by induction, u is a polynomial of total degree less or equal to $k-1$.

Remark 1. By the local nature of the derivative Exercise 1 stay true for $u \in \mathrm{W}^{k,p}_{\mathrm{loc}}(\mathbb{R}^N)$.

Exercise 2.
We define the open half-space $\mathbb{R}^N_+ := \left\{(x_1, \dots, x_N) \in \mathbb{R}^N \;\middle|\; x_N > 0\right\}$ and an extension operator $E : \mathrm{L}^p(\mathbb{R}^N_+) \to \mathrm{L}^p(\mathbb{R}^N)$ define by

$$Eu(x) = \begin{cases} u(x) & \text{if } x \in \mathbb{R}^N_+ \\ u(x', -x_N) & \text{if } x' \in \mathbb{R}^{N-1} \text{ and } x_N < 0 \end{cases}.$$

1. Show that $E : \mathrm{W}^{1,p}(\mathbb{R}^N_+) \to \mathrm{W}^{1,p}(\mathbb{R}^N)$ is a linear continuous operator.
2. Show that there exists $u \in \mathrm{W}^{2,p}(\mathbb{R}^N_+)$ such that $Eu \notin \mathrm{W}^{2,p}(\mathbb{R}^N)$. Search u of the form $u(x) = \varphi(x')\, f(x_N)$ with $\varphi \in C_0^\infty(\mathbb{R}^{N-1})$ and $f \in C^2(\mathbb{R}_+)$ with $f'(0) \neq 0$.

Now, we define a extension operator $F : \mathrm{L}^p(\mathbb{R}^N_+) \to \mathrm{L}^p(\mathbb{R}^N)$ define by

$$Fu(x) = \begin{cases} u(x) & \text{if } x_N > 0 \\ a\, u(x', -\lambda\, x_N) + b\, u(x', -\mu\, x_N) & \text{if } x_N < 0 \end{cases}$$

with $a, b \in \mathbb{R}$ and $\lambda, \mu \in \mathbb{R}_+$.

3. For a multi-index $\alpha \in \mathbb{N}^N$ such that $|\alpha| \leq 2$, we define v_α by

$$v_\alpha(x) = \begin{cases} \partial^\alpha u(x) & \text{if } x_N > 0 \\ a(-\lambda)^{\alpha_N}\, \partial^\alpha u(x', -\lambda\, x_N) + b(-\mu)^{\alpha_N}\, \partial^\alpha u(x', -\mu\, x_N) & \text{if } x_N < 0 \end{cases}.$$

   Show that $\partial^\alpha(Fu) = v_\alpha$ if, and only if,

$$\begin{cases} a + b = 1 \\ -\lambda a - \mu b = 1 \end{cases}.$$

4. Show that there exist $a, b \in \mathbb{R}$ and $\lambda, \mu \in \mathbb{R}_+$ such that $F : \mathrm{W}^{2,p}(\mathbb{R}^N_+) \to \mathrm{W}^{2,p}(\mathbb{R}^N)$ is a linear continuous operator.

Solution 2.
We define useful notation for the corrections, the lower half-space $\mathbb{R}^N_- := \left\{x \in \mathbb{R}^N \;\middle|\; x_N < 0\right\}$ and the normal $\nu : x' \in \mathbb{R}^{N-1} \mapsto (0, \dots, 0, 1)^\intercal$.

Question 1. Since the operator E extend the function across the interface $\mathbb{R}^{N-1} \times \{0\}$, we need to carefully compute the weak derivative. For $1 \leq i \leq N-1$ and $\varphi \in C_0^\infty(\mathbb{R}^N)$, using the Green's first identity, we compute

$$\int_{\mathbb{R}^N} Eu(x)\, \partial_i \varphi(x)\, \mathrm{d}x = \int_{\mathbb{R}^N_+} u(x)\, \partial_i \varphi(x)\, \mathrm{d}x + \int_{\mathbb{R}^N_-} u(x', -x_N)\, \partial_i \varphi(x)\, \mathrm{d}x$$

$$= -\int_{\mathbb{R}^N_+} \partial_i u(x)\,\varphi(x)\,\mathrm{d}x - \int_{\mathbb{R}^N_-} \partial_i u(x', -x_N)\,\varphi(x)\,\mathrm{d}x$$

there is no boundary term because the $\nu_i = 0$ for $1 \le i \le N-1$. We obtain

$$\partial_i Eu(x) = \begin{cases} \partial_i u(x) & \text{if } x_N > 0 \\ \partial_i u(x', -x_N) & \text{if } x_N < 0 \end{cases}.$$

For the last derivative and $\varphi \in C_0^\infty(\mathbb{R}^N)$, using the Green's first identity, we compute

$$\begin{aligned}\int_{\mathbb{R}^N} Eu(x)\,\partial_N\varphi(x)\,\mathrm{d}x &= \int_{\mathbb{R}^N_+} u(x)\,\partial_N\varphi(x)\,\mathrm{d}x + \int_{\mathbb{R}^N_-} u(x', -x_N)\,\partial_N\varphi(x)\,\mathrm{d}x \\ &= -\int_{\mathbb{R}^N_+} \partial_N u(x)\,\varphi(x)\,\mathrm{d}x + \int_{\mathbb{R}^N_-} \partial_N u(x', -x_N)\,\varphi(x)\,\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^{N-1}} u(x', 0)\,\varphi(x', 0)\,\mathrm{d}x' + \int_{\mathbb{R}^{N-1}} u(x', 0)\,\varphi(x', 0)\,\mathrm{d}x' \\ &= -\int_{\mathbb{R}^N_+} \partial_N u(x)\,\varphi(x)\,\mathrm{d}x + \int_{\mathbb{R}^N_-} \partial_N u(x', -x_N)\,\varphi(x)\,\mathrm{d}x\end{aligned}$$

which give

$$\partial_N Eu(x) = \begin{cases} \partial_N u(x) & \text{if } x_N > 0 \\ -\partial_N u(x', -x_N) & \text{if } x_N < 0 \end{cases}.$$

From those expressions, we have $\|Eu\|_{\mathrm{W}^{1,p}(\mathbb{R}^N)} = 2^{\frac{1}{p}}\,\|u\|_{\mathrm{W}^{1,p}(\mathbb{R}^N_+)}$ which tell us that the operator $E : \mathrm{W}^{1,p}(\mathbb{R}^N_+) \to \mathrm{W}^{1,p}(\mathbb{R}^N)$ is a linear continuous.

Question 2. We take $\varphi \in C_0^\infty(\mathbb{R}^{N-1})$ and define the function $u(x', x_N) = \varphi(x')\ \mathsf{e}^{-x_N}$. The function $u \in C^\infty(\mathbb{R}^N_+)$ and all the derivative up to order 2 are p-integrable, so we have $u \in \mathrm{W}^{2,p}(\mathbb{R}^N_+)$. The extension Eu is $x \mapsto \varphi(x')\ \mathsf{e}^{-|x_N|}$. We will show that the second derivative $\partial_N^2 Eu$ is not in $\mathrm{L}^p(\mathbb{R})$ and therefore, we have $Eu \notin \mathrm{W}^{2,p}(\mathbb{R}^N)$.

Now, we show that the second derivative $\partial_N^2 Eu$ is not in $\mathrm{L}^p(\mathbb{R})$, to do that we compute $\partial_N^2\left(\mathsf{e}^{-|x_N|}\right)$. From the lecture, we have $\partial_N\left(\mathsf{e}^{-|x_N|}\right) = -\operatorname{sgn}(x_N)\ \mathsf{e}^{-|x_N|}$ then, for $\psi \in C_0^\infty(\mathbb{R})$, we compute the second weak derivative

$$\begin{aligned}\int_{\mathbb{R}} \left(-\operatorname{sgn}(x_N)\ \mathsf{e}^{-|x_N|}\right)\ \psi'(x_N)\,\mathrm{d}x_N &= -\int_{\mathbb{R}_+} \mathsf{e}^{-x_N}\psi'(x_N)\,\mathrm{d}x_N + \int_{\mathbb{R}_-} \mathsf{e}^{x_N}\psi'(x_N)\,\mathrm{d}x_N \\ &= -\int_{\mathbb{R}_+} \mathsf{e}^{-x_N}\psi(x_N)\,\mathrm{d}x_N - \int_{\mathbb{R}_-} \mathsf{e}^{x_N}\psi(x_N)\,\mathrm{d}x_N - 2\psi(0) \\ &= -\,2\psi(0) - \int_{\mathbb{R}} \mathsf{e}^{-|x_N|}\,\psi(x_N)\,\mathrm{d}x_N.\end{aligned}$$

There do not exists a function $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$ such that $\int_{\mathbb{R}} f(x_N)\,\psi(x_N)\,\mathrm{d}x_N = 2\psi(0)$, for all $\psi \in C_0^\infty(\mathbb{R})$. To see that, we consider the sequence of function $\psi_n \in C_0^\infty(\mathbb{R})$, for $n \ge 1$, such that $\psi_n \equiv 1$ for $x \in \left[-\frac{1}{2n}, \frac{1}{2n}\right]$, $\operatorname{supp}(\psi_n) \subset \left[-\frac{1}{n}, \frac{1}{n}\right]$, and $0 \le \psi_n \le 1$. We have

$$2\psi_n(0) = \int_{\mathbb{R}} f(x_N)\,\psi_n(x_N)\,\mathrm{d}x_N = \int_{-1}^{1} f(x_N)\,\psi_n(x_N)\,\mathrm{d}x_N$$

and, using the dominated convergence theorem on the sequence $(f\psi_n)_{n\in\mathbb{N}}$ dominated by $f \in \mathrm{L}^1((-1,1))$ and it pointwise converge to 0 almost everywhere, we obtain

$$0 = \lim_{n\to+\infty} \int_{\mathbb{R}} f(x_N)\,\psi_n(x_N)\,\mathrm{d}x_N = \lim_{n\to+\infty} 2\psi_n(0) = 2$$

which is absurd, so $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$ does not exists.

Question 3. For $\alpha \in \mathbb{N}^N$ multi-index such that $|\alpha| \leq 2$, we consider three cases depending on the value of α_N. If $\alpha_N = 0$, the computation of the ∂^α weak derivative is straight forward using the integration by part and the fact that $\nu_i = 0$ for $i \in \{1, \dots, N-1\}$. If $\alpha_N = 1$, and for $\varphi \in C_0^\infty(\mathbb{R}^N)$, we compute

$$\begin{aligned}
\int_{\mathbb{R}^N} Fu\, \partial^\alpha \varphi \,\mathrm{d}x &= \int_{\mathbb{R}^N_+} u\, \partial^{\alpha'} \partial_N \varphi \,\mathrm{d}x + \int_{\mathbb{R}^N_-} [a\, u(x', -\lambda x_N) + b\, u(x', -\mu x_N)]\; \partial^{\alpha'} \partial_N \varphi \,\mathrm{d}x \\
&= \int_{\mathbb{R}^N_+} \partial^{\alpha'} u\, \partial_N \varphi \,\mathrm{d}x + \int_{\mathbb{R}^N_-} \left[a \partial^{\alpha'} u(x', -\lambda x_N) + b \partial^{\alpha'} u(x', -\mu x_N)\right] \partial_N \varphi \,\mathrm{d}x \\
&= - \int_{\mathbb{R}^N_+} \partial^{\alpha'} \partial_N u\, \varphi \,\mathrm{d}x + \int_{\mathbb{R}^N_-} \left[\lambda a \partial^{\alpha'} \partial_N u(x', -\lambda x_N) + \mu b \partial^{\alpha'} \partial_N u(x', -\mu x_N)\right] \varphi \,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^{N-1}} \partial^{\alpha'} u\, \varphi \,\mathrm{d}x' + \int_{\mathbb{R}^{N-1}} [a+b]\, \partial^{\alpha'} u(x', 0)\, \varphi \,\mathrm{d}x' \\
&= - \int_{\mathbb{R}^N} v_\alpha\, \varphi \,\mathrm{d}x + \int_{\mathbb{R}^{N-1}} [a+b-1]\, u(x', 0)\, \varphi \,\mathrm{d}x'
\end{aligned}$$

therefore, we have $\partial^\alpha Fu = v_\alpha$ if, and only if, $a + b = 1$. If $\alpha_N = 2$, from the previous computation, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} Fu\, \partial^\alpha \varphi \,\mathrm{d}x &= \int_{\mathbb{R}^N_+} u\, \partial^{\alpha'} \partial_N^2 \varphi \,\mathrm{d}x + \int_{\mathbb{R}^N_-} [a\, u(x', -\lambda x_N) + b\, u(x', -\mu x_N)]\; \partial^{\alpha'} \partial_N^2 \varphi \,\mathrm{d}x \\
&= - \int_{\mathbb{R}^N_+} \partial^{\alpha'} \partial_N u\, \partial_N \varphi \,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N_-} \left[\lambda a \partial^{\alpha'} \partial_N u(x', -\lambda x_N) + \mu b \partial^{\alpha'} \partial_N u(x', -\mu x_N)\right] \partial_N \varphi \,\mathrm{d}x \\
&= \int_{\mathbb{R}^N} v_\alpha\, \varphi \,\mathrm{d}x + \int_{\mathbb{R}^N_+} [1 + \lambda a + \mu b]\, \partial^{\alpha'} \partial_N u(x', 0) \varphi(x', 0) \,\mathrm{d}x'
\end{aligned}$$

therefore, we have $\partial^\alpha Fu = v_\alpha$ if, and only if, $-\lambda a - \mu b = 1$.

Question 4. We write the system of equation $a + b = 1$ and $-\lambda a - \mu b = 1$ with matrix, we obtain

$$M(\lambda, \mu) \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{with } M(\lambda, \mu) = \begin{pmatrix} 1 & 1 \\ -\lambda & -\mu \end{pmatrix}.$$

The determinant of $M(\lambda, \mu)$ is $\lambda - \mu$, if we take any $0 < \lambda < \mu$ the matrix $M(\lambda, \mu)$ is inversible and there exists a, b such that $a + b = 1$ and $-\lambda a - \mu b = 1$. For a multi-index α such that $|\alpha| \leq 2$, we compute

$$\begin{aligned}
&\|\partial^\alpha Fu\|_{\mathrm{L}^p(\mathbb{R}^N)} \\
&\qquad = \left\| u\, \mathbb{1}_{\mathbb{R}^N_+} + (a(-\lambda)^{\alpha_N}\, \partial^\alpha u(x', -\lambda\, x_N) + b(-\mu)^{\alpha_N}\, \partial^\alpha u(x', -\mu\, x_N))\, \mathbb{1}_{\mathbb{R}^N_-} \right\|_{\mathrm{L}^p(\mathbb{R}^N)} \\
&\qquad \leq \|\partial^\alpha u\|_{\mathrm{L}^p(\mathbb{R}^N_+)} + \|a(-\lambda)^{\alpha_N}\, \partial^\alpha u(x', -\lambda\, x_N) + b(-\mu)^{\alpha_N}\, \partial^\alpha u(x', -\mu\, x_N)\|_{\mathrm{L}^p(\mathbb{R}^N_-)} \\
&\qquad \leq \|\partial^\alpha u\|_{\mathrm{L}^p(\mathbb{R}^N_+)} + |a|\, \lambda^{\alpha_N}\, \|\partial^\alpha u(x', -\lambda\, x_N)\|_{\mathrm{L}^p(\mathbb{R}^N_-)} + |b|\, \mu^{\alpha_N}\, \|\partial^\alpha u(x', -\mu\, x_N)\|_{\mathrm{L}^p(\mathbb{R}^N_-)} \\
&\qquad \leq \|\partial^\alpha u\|_{\mathrm{L}^p(\mathbb{R}^N_+)} + |a|\, \lambda^{\alpha_N - 1}\, \|\partial^\alpha u\|_{\mathrm{L}^p(\mathbb{R}^N_+)} + |b|\, \mu^{\alpha_N}\, \|\partial^\alpha u\|_{\mathrm{L}^p(\mathbb{R}^N_+)} \\
&\qquad \leq C_\alpha\, \|\partial^\alpha u\|_{\mathrm{L}^p(\mathbb{R}^N_+)}
\end{aligned}$$

with $C_\alpha = 1 + |a|\,\lambda^{\alpha_N - 1} + |b|\,\mu^{\alpha_N - 1}$. From those estimations, we obtaint

$$\|Fu\|_{\mathrm{W}^{2,p}(\mathbb{R}^N)} = \left(\sum_{|\alpha|\leq 2} \|\partial^\alpha Fu\|^p_{\mathrm{L}^p(\mathbb{R}^N)}\right)^{\frac{1}{p}} \leq \left(\sum_{|\alpha|\leq 2} C^p_\alpha \|\partial^\alpha u\|^p_{\mathrm{L}^p(\mathbb{R}^N_+)}\right)^{\frac{1}{p}} \leq C\,\|u\|_{\mathrm{W}^{2,p}(\mathbb{R}^N_+)}$$

with $C = \max\left\{C_\alpha \;\middle|\; \alpha \in \mathbb{N}^N,\ |\alpha| \leq 2\right\}$ and it give that $F : \mathrm{W}^{2,p}(\mathbb{R}^N_+) \to \mathrm{W}^{2,p}(\mathbb{R}^N)$ is a linear continuous operator.

Remark 2. The construction of the extension operator F can be generalise to have an extension operator from $\mathrm{W}^{k,p}(\mathbb{R}^N_+)$ to $\mathrm{W}^{k,p}(\mathbb{R}^N)$ for an integer $k \geq 1$. Define a extension operator $G : \mathrm{L}^p(\mathbb{R}^N_+) \to \mathrm{L}^p(\mathbb{R}^N)$ define by

$$Gu(x) = \begin{cases} u(x) & \text{if } x_N > 0 \\ \displaystyle\sum_{j=0}^{k-1} a_j u(x', -\lambda_j x_N) & \text{if } x_N < 0 \end{cases}$$

and show that there exists $(\lambda_j)_{j\in\{0,\ldots,k-1\}} \in (\mathbb{R}_+)^k$ and $(a_j)_{j\in\{0,\ldots,k-1\}} \in \mathbb{R}^k$ such that $G : \mathrm{W}^{k,p}(\mathbb{R}^N_+) \to \mathrm{W}^{k,p}(\mathbb{R}^N)$ is a linear continuous operator.