Exercise 1.
Prove the following theorem using question 1 to 3.

Theorem 1. Assume $N \in \mathbb{N}$ and $N \geq 2$. Then there is a continuous embedding

$$W^{1,N} (\mathbb{R}^N) \hookrightarrow L^q (\mathbb{R}^N)$$

precisely for $N \leq q < \infty$.

1. Find necessary conditions on the values $p, q, r \geq 1$ and $0 \leq \theta \leq 1$ for the estimate

$$\|u\|_q \leq C \|\nabla u\|_p^\theta \|u\|_r^{1-\theta}$$

1.  to hold.

2. For $\alpha \geq N$ and $u \in W^{1,N} (\mathbb{R}^N) \cap L^{\frac{(\alpha-1)N}{\alpha-1}} (\mathbb{R}^N)$ prove that there exists $C_\alpha > 0$ such that

$$\|u\|_{\frac{\alpha N}{N-1}} \leq C_\alpha \|\nabla u\|_{\frac{N}{N}} \|u\|_{\frac{(\alpha-1)N}{\alpha-1}}$$

using Sobolev’s inequality.

(Hint: look at the proof of Theorem 6.2 in the lecture note)

3. Conclude.

   (Hint: induction)

4. Counterexample: show that there exists $u \in W^{1,N} (\mathbb{R}^N)$ such that $u \not\in L^\infty (\mathbb{R}^N)$.

   (Hint: try a logarithmic singularity)

Exercise 2.
Fix $1 \leq p \leq \infty$ and let $u \in W^{1,p} (\mathbb{R})$.

1. Show that there exists $\tilde{u} \in C^0 (\mathbb{R})$ such that $u = \tilde{u}$ almost everywhere on $\mathbb{R}$.

   (Hint: fundamental theorem of calculus)

2. Show that $u \in L^\infty (\mathbb{R})$ by proving that there exists a constant $C > 0$ such that

$$|u(x)| \leq C \|u\|_p^{\frac{1}{p}} \|u'\|_p^{\frac{1}{p}}.$$