Karlsruhe Institute of Technology

Institute for Analysis
PD Dr. Rainer Mandel
Dr. Zoïs Moitier

June 11, 2021

Sobolev spaces: tutorials

Exercise sheet 04

Exercise 1.
Prove the following theorem using question 1 to 3.

Theorem 1. *Assume $N \in \mathbb{N}$ and $N \geq 2$. Then there is a continuous embedding*

$$\mathrm{W}^{1,N}(\mathbb{R}^N) \hookrightarrow \mathrm{L}^q(\mathbb{R}^N)$$

precisely for $N \leq q < \infty$.

1. Find necessary conditions on the values $p, q, r \geq 1$ and $0 \leq \theta \leq 1$ for the estimate $\|u\|_q \leq C \|\nabla u\|_p^\theta \|u\|_r^{1-\theta}$ to hold.

2. For $\alpha \geq N$ and $u \in \mathrm{W}^{1,N}(\mathbb{R}^N) \cap \mathrm{L}^{\frac{(\alpha-1)N}{N-1}}(\mathbb{R}^N)$ prove that there exists $C_\alpha > 0$ such that

$$\|u\|_{\frac{\alpha N}{N-1}} \leq C_\alpha \|\nabla u\|_N^{\frac{1}{\alpha}} \|u\|_{\frac{(\alpha-1)N}{N-1}}^{1-\frac{1}{\alpha}} \tag{1}$$

using Sobolev's inequality.

(Hint: look at the proof of Theorem 6.2 in the lecture note)

3. Conclude.

(Hint: induction)

4. Counterexample: show that there exists $u \in \mathrm{W}^{1,N}(\mathbb{R}^N)$ such that $u \notin \mathrm{L}^\infty(\mathbb{R}^N)$.

(Hint: try a logarithmic singularity)

Exercise 2.
Fix $1 \leq p \leq \infty$ and let $u \in \mathrm{W}^{1,p}(\mathbb{R})$.

1. Show that there exists $\widetilde{u} \in C^0(\mathbb{R})$ such that $u = \widetilde{u}$ almost everywhere on $\mathbb{R}$.

(Hint: fundamental theorem of calculus)

2. Show that $u \in \mathrm{L}^\infty(\mathbb{R})$ by proving that there exists a constant $C > 0$ such that

$$|u(x)| \leq C \|u\|_p^{1-\frac{1}{p}} \|u'\|_p^{\frac{1}{p}}.$$