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Sobolev spaces: tutorials

Exercise sheet 04 with solution

Exercise 1.
Prove the following theorem using question 1 to 3.

Theorem 1. *Assume $N \in \mathbb{N}$ and $N \geq 2$. Then there is a continuous embedding*

$$\mathrm{W}^{1,N}(\mathbb{R}^N) \hookrightarrow \mathrm{L}^q(\mathbb{R}^N)$$

precisely for $N \leq q < \infty$.

1. Find necessary conditions on the values $p, q, r \geq 1$ and $0 \leq \theta \leq 1$ for the estimate $\|u\|_q \leq C \|\nabla u\|_p^\theta \|u\|_r^{1-\theta}$ to hold.

2. For $\alpha \geq N$ and $u \in \mathrm{W}^{1,N}(\mathbb{R}^N) \cap \mathrm{L}^{\frac{(\alpha-1)N}{N-1}}(\mathbb{R}^N)$ prove that there exists $C_\alpha > 0$ such that

$$\|u\|_{\frac{\alpha N}{N-1}} \leq C_\alpha \|\nabla u\|_N^{\frac{1}{\alpha}} \|u\|_{\frac{(\alpha-1)N}{N-1}}^{1-\frac{1}{\alpha}} \tag{1}$$

using Sobolev's inequality.

(Hint: look at the proof of Theorem 6.2 in the lecture note)

3. Conclude.

(Hint: induction)

4. Counterexample: show that there exists $u \in \mathrm{W}^{1,N}(\mathbb{R}^N)$ such that $u \notin \mathrm{L}^\infty(\mathbb{R}^N)$.

(Hint: try a logarithmic singularity)

Solution 1.

Question 1. Take $u \in C_0^\infty(\mathbb{R}^N)$ such that $u \not\equiv 0$, we define $u_\lambda(x) = u(\lambda x)$ and we have the following identities

$$\|u_\lambda\|_s = \lambda^{-\frac{N}{s}} \|u\|_s \quad \text{and} \quad \|\nabla u_\lambda\|_s = \lambda^{1-\frac{N}{s}} \|\nabla u\|_s$$

for $s \geq 1$. If we assume that there exists a constant $C > 0$ such that the estimate $\|u_\lambda\|_q \leq C \|\nabla u_\lambda\|_p^\theta \|u_\lambda\|_r^{1-\theta}$ is true then it becomes

$$\|u\|_q \leq \lambda^{\frac{N}{q}+\theta\left(1-\frac{N}{p}\right)-(1-\theta)\frac{N}{r}} C \|\nabla u\|_p^\theta \|u\|_r^{1-\theta} \tag{2}$$

and if the exponent of λ is not zero, depending on its sign, taking $\lambda \to 0^+$ or $+\infty$, will yield $\|u\|_q \leq 0$ which is absurd. Therefore, the exponent of λ in Eq. (2) must be zeros which mean that the values $p, q, r \geq 1$ and $0 \leq \theta \leq 1$ must satisfy

$$\frac{1}{q} = \theta\left(\frac{1}{p} - \frac{1}{N}\right) + (1-\theta)\frac{1}{r}.$$

Question 2. We take $u \in \mathrm{W}^{1,N}(\mathbb{R}^N) \cap \mathrm{L}^{\frac{(\alpha-1)N}{N-1}}(\mathbb{R}^N)$ and we define $v = |u|^{\alpha-1} u$ with $\alpha \geq N$. First, let's show that $v \in \mathrm{W}^{1,1}(\mathbb{R}^N)$. We observe that we have $N \leq \alpha \leq \frac{(\alpha-1)N}{N-1}$ for all $\alpha \geq N$, so using the interpolation properties of the Lebesgue spaces, there exists $\theta \in [0,1]$ such that $\frac{1}{\alpha} = \theta\frac{1}{N} + (1-\theta)\frac{N-1}{(\alpha-1)N}$ and

$$\|v\|_1 = \|u\|_\alpha^\alpha \leq \|u\|_N^{\theta\alpha} \, \|u\|_{\frac{(\alpha-1)N}{N-1}}^{(1-\theta)\alpha}.$$

We compute $\nabla v = \alpha \, |u|^{\alpha-1} \nabla u$, from this, using the Hölder inequality with $p = N$, we get

$$\|\nabla v\|_1 = \alpha \, \left\| |u|^{\alpha-1} \nabla u \right\|_1 \leq \alpha \, \|\nabla u\|_N \, \|u\|_{\frac{(\alpha-1)N}{N-1}}^{\alpha-1}. \tag{3}$$

We apply Theorem 6.2 in the lecture note with $p = 1$ to v, we get

$$\|v\|_{\frac{N}{N-1}} \leq \frac{1}{\sqrt{N}} \|\nabla v\|_1 \quad \text{which gives} \quad \| |u|^\alpha \|_{\frac{N}{N-1}} \leq \frac{\alpha}{\sqrt{N}} \left\| |u|^{\alpha-1} \nabla u \right\|_1$$

after rearranging the norms and Eq. (3), we obtain

$$\|u\|_{\frac{\alpha N}{N-1}}^\alpha \leq \frac{\alpha}{\sqrt{N}} \, \|\nabla u\|_N \, \|u\|_{\frac{(\alpha-1)N}{N-1}}^{\alpha-1}$$

then taking the power $\frac{1}{\alpha}$, we finally get

$$\|u\|_{\frac{\alpha N}{N-1}} \leq \left(\frac{\alpha}{\sqrt{N}}\right)^{\frac{1}{\alpha}} \, \|\nabla u\|_N^{\frac{1}{\alpha}} \, \|u\|_{\frac{(\alpha-1)N}{N-1}}^{1-\frac{1}{\alpha}}.$$

Question 3. First, from Eq. (1), we deduce

$$\|u\|_{\frac{\alpha N}{N-1}} \leq C_\alpha \left(\|\nabla u\|_N + \|u\|_{\frac{(\alpha-1)N}{N-1}} \right). \tag{4}$$

using the Young's inequality[1]. Then we will show, by induction, that for all $k \in \mathbb{N}$, there exists a constant $D_k > 0$ such that

$$\|u\|_{\frac{(N+k)N}{N-1}} \leq D_k \, \|u\|_{1,N}, \qquad \forall u \in \mathrm{W}^{1,N}(\mathbb{R}^N).$$

Base case: for $k = 0$, we apply Eq. (4) with $\alpha = N \geq 2 - \frac{1}{N}$, we obtain

$$\|u\|_{\frac{N^2}{N-1}} \leq C_N \left(\|\nabla u\|_N + \|u\|_N \right) \leq D_0 \, \|u\|_{1,N}$$

with $D_0 = 2C_N$ a constant[2].

[1] For $a, b \geq 0$ and $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, we have $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$.

[2] Recall that $\|u\|_N + \|\nabla u\|_N \leq 2\,\|u\|_{1,N}$.

Inductive step: Take $k \geq 1$ and we assume that for all $\ell < k$ there exists a constant $D_\ell > 0$ such that

$$\|u\|_{\frac{(N+\ell)N}{N-1}} \leq D_\ell \|u\|_{1,N}, \qquad \forall u \in \mathrm{W}^{1,N}(\mathbb{R}^N).$$

we apply Eq. (4) with $\alpha = N + k \geq 2 - \frac{1}{N}$, we obtain

$$\begin{aligned}
\|u\|_{\frac{(N+k)N}{N-1}} &\leq C_{N+k}\left(\|\nabla u\|_N + \|u\|_{\frac{(N+k-1)N}{N-1}}\right) \\
&\leq C_{N+k}\left(\|\nabla u\|_N + D_{k-1}\|u\|_{1,N}\right) \qquad \text{(inductive hypothesis for } \ell = k-1) \\
&\leq C_{N+k}\left(1 + D_{k-1}\right)\|u\|_{1,N}
\end{aligned}$$

and set $D_k = C_{N+k}\left(1 + D_{k-1}\right)$.
Conclusion: for all $k \in \mathbb{N}$, there exists a constant $D_k > 0$ such that

$$\|u\|_{\frac{(N+k)N}{N-1}} \leq D_k \|u\|_{1,N}, \qquad \forall u \in \mathrm{W}^{1,N}(\mathbb{R}^N).$$

Then, we use the interpolation properties of the Lebesgue spaces. For $N \leq q < \infty$, we take the smallest integer $k \geq \frac{qN}{N-1} - N > 0$ so we have $\frac{N-1}{(N+k)N} \leq \frac{1}{q} \leq \frac{1}{N}$. There exists $\theta \in [0,1]$ such that $\frac{1}{q} = \theta\frac{1}{N} + (1-\theta)\frac{N-1}{(N+k)N}$ and, using the generalized Hölder's inequality, we obtain

$$\|u\|_q \leq \|u\|_N^\theta \|u\|_{\frac{(N+k)N}{N-1}}^{1-\theta} \leq \|u\|_{1,N}^\theta D_k^{1-\theta}\|u\|_{1,N}^{1-\theta} \leq D_k^{1-\theta}\|u\|_{1,N}$$

therefore, we have a continuous embedding $\mathrm{W}^{1,N}(\mathbb{R}^N) \hookrightarrow \mathrm{L}^q(\mathbb{R}^N)$ precisely for $N \leq q < \infty$.

Question 4. For $0 < \delta < 1 - \frac{1}{N}$ and $\chi \in C_0^\infty\left(B\left(0, \frac{1}{2}\right)\right)$ such that $\chi \equiv 1$ on $B\left(0, \frac{1}{4}\right)$, we define the function

$$u : x \mapsto |\ln(|x|)|^\delta \chi(x).$$

First, let's show that $u \notin \mathrm{L}^\infty(\mathbb{R}^N)$. Take $M > 0$, we have

$$B(0, \rho_M) \subset u^{-1}\left((M, +\infty)\right) \qquad \text{with } \rho_M = \min\left(\exp\left(-M^{\frac{1}{\delta}}\right), \frac{1}{4}\right)$$

and $B(0, \rho_M)$ has non zero measure, therefore, $u \notin \mathrm{L}^\infty(\mathbb{R}^N)$.

Now let's check that $u \in \mathrm{W}^{1,N}(\mathbb{R}^N)$.

$$\begin{aligned}
\|u\|_N^N &= \int_{\mathbb{R}^N} |\ln(|x|)|^{\delta N} |\chi(x)|^N \,\mathrm{d}x \\
&= \int_{B\left(0,\frac{1}{4}\right)} |\ln(|x|)|^{\delta N} \,\mathrm{d}x + \int_{B\left(0,\frac{1}{2}\right)\setminus B\left(0,\frac{1}{4}\right)} |\ln(|x|)|^{\delta N} |\chi(x)|^N \,\mathrm{d}x \\
&= A_N \int_0^{\frac{1}{4}} |\ln(r)|^{\delta N} r^{N-1} \,\mathrm{d}r + C_\chi \qquad \left(A_N = \int_{\mathbb{S}^{N-1}} \mathrm{d}\sigma(\omega)\right)
\end{aligned}$$

this is finite because the function $r \mapsto |\ln(r)|^{\delta N} r^{N-1}$ is continuous on $\left[0, \frac{1}{2}\right]$. Then we compute the weak derivative of u to be

$$\nabla u(x) = -\delta\frac{x}{|x|^2}|\ln(|x|)|^{\delta-1}\chi(x) + |\ln(|x|)|^\delta \nabla\chi(x).$$

The function $x \mapsto |\ln(|x|)|^\delta \nabla\chi(x)$ is in $C_0^\infty(\mathbb{R}^N)$ so its $\mathrm{L}^N(\mathbb{R}^N)$ norm is finite. For the other part of the gradient, as before, we compute

$$\int_{\mathbb{R}^N} \frac{|\chi(x)|^N}{|x|^N |\ln(|x|)|^{(1-\delta)N}} \,\mathrm{d}x = \int_{B\left(0,\frac{1}{4}\right)} \frac{1}{|x|^N |\ln(|x|)|^{(1-\delta)N}} \,\mathrm{d}x + C'_\chi$$

$$= A_N \int_0^{\frac{1}{4}} \frac{1}{r\,|\ln(r)|^{(1-\delta)N}}\,\mathrm{d}r + C'_\chi$$

since we have $(1-\delta)N > 1$ because $0 < \delta < 1 - \frac{1}{N}$, the first integral is finite (Bertrand's integral[3]), so $\nabla u \in \mathrm{L}^N(\mathbb{R}^N)$. Therefore, we have $u \in \mathrm{W}^{1,N}(\mathbb{R}^N)$.

Exercise 2.
Fix $1 \le p \le \infty$ and let $u \in \mathrm{W}^{1,p}(\mathbb{R})$.

1. Show that there exists $\widetilde{u} \in C^0(\mathbb{R})$ such that $u = \widetilde{u}$ almost everywhere on $\mathbb{R}$.

   (Hint: fundamental theorem of calculus)

2. Show that $u \in \mathrm{L}^\infty(\mathbb{R})$ by proving that there exists a constant $C > 0$ such that

$$|u(x)| \le C\,\|u\|_p^{1-\frac{1}{p}}\,\|u'\|_p^{\frac{1}{p}}\,.$$

Solution 2.
Question 1. We define the function v by

$$v : x \mapsto \int_0^x u'(t)\,\mathrm{d}t$$

it is well define since we have the estimate $|v(x)| \le |x|^{1-\frac{1}{p}}\,\|u'\|_p$, using the Hölder inequality. To show that the function v is continuous on $\mathbb{R}$, we fix a point $x \in \mathbb{R}$ and will show that it is continuous at x. We define the interval $I = (x-1, x+1)$ and a sequence $(y_n)_{n\in\mathbb{N}} \in I^{\mathbb{N}}$ such that $y_n \to x$ as $n \to +\infty$. We have

$$v(y_n) - v(x) = \int_x^{y_n} u'(t)\,\mathrm{d}t = \int_I \mathbb{1}_{[x,y_n]}(t)\,u'(t)\,\mathrm{d}t \underset{n\to+\infty}{\longrightarrow} 0$$

using the dominated convergence theorem on the sequence of function $t \mapsto \mathbb{1}_{[x,y_n]}(t)\,u'(t)$ which is dominated by $|u'| \in \mathrm{L}^1(I)$ ($\int_I |u'(t)|\,\mathrm{d}t \le 2^{1-\frac{1}{p}}\,\|u'\|_p$) and $\mathbb{1}_{[x,y_n]}(t)\,u'(t) \to 0$ as $n \to +\infty$ for almost every $t \in I$.

We want to show that the weak derivative of v is u'. Take $\varphi \in C_0^\infty(\mathbb{R})$, there exists $a > 0$ such that $\mathrm{supp}(\varphi) \subset [-a, a]$, we compute

$$\begin{aligned}
\int_{\mathbb{R}} v(x)\,\varphi'(x)\,\mathrm{d}x &= \int_{x=-a}^{x=a} \left[\int_{t=0}^{t=x} u'(t)\,\mathrm{d}t\right] \varphi'(x)\,\mathrm{d}x \\
&= -\int_{x=0}^{x=-a} \left[\int_{t=0}^{t=x} u'(t)\,\varphi'(x)\,\mathrm{d}t\right] \mathrm{d}x + \int_{x=0}^{x=a} \left[\int_{t=0}^{t=x} u'(t)\,\varphi'(x)\,\mathrm{d}t\right] \mathrm{d}x,
\end{aligned}$$

using the Fubini's theorem on both integrals, give

$$\int_{\mathbb{R}} v(x)\,\varphi'(x)\,\mathrm{d}x = -\int_{t=0}^{t=-a} \left[\int_{x=t}^{x=-a} u'(t)\,\varphi'(x)\,\mathrm{d}t\right] \mathrm{d}x + \int_{t=0}^{t=a} \left[\int_{x=t}^{x=a} u'(t)\,\varphi'(x)\,\mathrm{d}t\right] \mathrm{d}x$$

[3] We can transform the Bertrand's integral into a Riemann's integral by using the change variable $t = -\ln(r)$, which yield

$$\int_0^{\frac{1}{4}} \frac{1}{r\,|\ln(r)|^{(1-\delta)N}}\,\mathrm{d}r = \int_{\ln(4)}^{+\infty} \frac{1}{t^{(1-\delta)N}}\,\mathrm{d}t < \infty \qquad \text{if, and only if,} \qquad (1-\delta)N > 1.$$

$$
\begin{aligned}
&= -\int_{t=-a}^{t=0} u'(t)\left[\varphi'(t) - \underbrace{\varphi'(-a)}_{=0}\right] \mathrm{d}x + \int_{t=0}^{t=a} u'(t)\left[\underbrace{\varphi'(a)}_{=0} - \varphi'(t)\right] \mathrm{d}x \\
&= -\int_{t=-a}^{t=a} u'(t)\,\varphi'(t)\,\mathrm{d}t \\
&= -\int_{\mathbb{R}} u'(t)\,\varphi'(t)\,\mathrm{d}t.
\end{aligned}
$$

Therefore, we have $(u-v)' = 0$, using Exercise sheet 03 Exercise 1., there exists a constant c such that $u(x) = c + v(x)$ for almost every $x \in \mathbb{R}$. By setting $\widetilde{u} \coloneqq c + v \in C^0(\mathbb{R})$, we have answer the question.

Question 2. This question is trivial for $p = \infty$ so we only consider $1 \leq p < \infty$. For $u \in \mathrm{W}^{1,p}(\mathbb{R})$, we define $v = |u|^{p-1} u$. By definition, we have $v \in \mathrm{L}^1(\mathbb{R})$ and we have $v' \in \mathrm{L}^1(\mathbb{R})$ because $v' = p\,|u|^{p-1} u'$ and

$$
\|v'\|_1 = p\left\||u|^{p-1} u'\right\|_1 \leq p\left\||u|^{p-1}\right\|_{\frac{p}{p-1}} \|u'\|_p = p\,\|u\|_p^{p-1}\,\|u'\|_p \tag{5}
$$

therefore $v \in \mathrm{W}^{1,1}(\mathbb{R})$. We define $\widetilde{v} \in C^0(\mathbb{R})$ by

$$
\widetilde{v} : x \mapsto \int_{-\infty}^{x} v'(t)\,\mathrm{d}t
$$

using the same reasoning than *Question 1.*, there exists a constant c such that $v = c + \widetilde{v}$ almost everywhere. However, since we have $\widetilde{v}(x) \to 0$ as $x \to -\infty$ so $v(x) \to c$ as $x \to -\infty$ and v can be in $\mathrm{L}^1(\mathbb{R})$ only if $c = 0$. So, we get that $v = \widetilde{v}$ almost everywhere. From Eq. (5), we get

$$
|u(x)|^p = |v(x)| \leq \|v'\|_1 \leq p\,\|u\|_p^{p-1}\,\|u'\|_p
$$

and taking the power $\frac{1}{p}$, we obtain

$$
|u(x)| \leq p^{\frac{1}{p}}\,\|u\|_p^{1-\frac{1}{p}}\,\|u'\|_p^{\frac{1}{p}}
$$

which give

$$
\|u\|_\infty \leq p^{\frac{1}{p}}\,\|u\|_p^{1-\frac{1}{p}}\,\|u'\|_p^{\frac{1}{p}} \leq p^{\frac{1}{p}}\left(\|u\|_p + \|u'\|_p\right) \leq 2p^{\frac{1}{p}}\,\|u\|_{1,p}\,. \tag{6}
$$

Finally, we have a continuous embedding $\mathrm{W}^{1,p}(\mathbb{R}) \hookrightarrow \mathrm{L}^\infty(\mathbb{R})$.

Remark 1. The constant in Eq. (6) is independent of p because $p^{\frac{1}{p}} \leq \mathsf{e}^{\frac{1}{\mathsf{e}}}$ for all $p \geq 1$.