Sobolev spaces: tutorials
Exercise sheet 04 with solution

Exercise 1.
Prove the following theorem using question 1 to 3.

**Theorem 1.** Assume $N \in \mathbb{N}$ and $N \geq 2$. Then there is a continuous embedding
\[
W^{1,N}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)
\]
precisely for $N \leq q < \infty$.

1. Find necessary conditions on the values $p, q, r \geq 1$ and $0 \leq \theta \leq 1$ for the estimate
\[
\|u\|_q \leq C \|\nabla u\|_p^\theta \|u\|_r^{1-\theta}
\]
to hold.

2. For $\alpha \geq N$ and $u \in W^{1,N}(\mathbb{R}^N) \cap L^{(\alpha-1)N/(\alpha-N)}(\mathbb{R}^N)$ prove that there exists $C_\alpha > 0$ such that
\[
\|u\|_{\frac{\alpha N}{\alpha - 1}} \leq C_\alpha \|\nabla u\|_{\frac{N}{\alpha-N}} \|u\|_{\frac{1}{1-(\alpha-1)N}}
\]
using Sobolev's inequality.

(Hint: look at the proof of Theorem 6.2 in the lecture note)

3. Conclude.

(Hint: induction)

4. Counterexample: show that there exists $u \in W^{1,N}(\mathbb{R}^N)$ such that $u \not\in L^\infty(\mathbb{R}^N)$.

(Hint: try a logarithmic singularity)

**Solution 1.**

**Question 1.** Take $u \in C^\infty_0(\mathbb{R}^N)$ such that $u \neq 0$, we define $u_\lambda(x) = u(\lambda x)$ and we have the following identities
\[
\|u_\lambda\|_s = \lambda^{-\frac{N}{s}} \|u\|_s \quad \text{and} \quad \|\nabla u_\lambda\|_s = \lambda^{1-\frac{N}{s}} \|\nabla u\|_s
\]
for $s \geq 1$. If we assume that there exists a constant $C > 0$ such that the estimate
\[
\|u_\lambda\|_q \leq C \|\nabla u_\lambda\|_p^\theta \|u_\lambda\|_r^{1-\theta}
\]
is true then it becomes
\[
\|u\|_q \leq \lambda^{\frac{N}{q}+\theta(1-\frac{N}{q})-(1-\theta)\frac{N}{r}} \quad C \|\nabla u\|_p^\theta \|u\|_r^{1-\theta}
\]
(2)
and if the exponent of \( \lambda \) is not zero, depending on its sign, taking \( \lambda \to 0^+ \) or \( +\infty \), will yield \( \|u\|_q \leq 0 \) which is absurd. Therefore, the exponent of \( \lambda \) in Eq. (2) must be zeros which mean that the values \( p, q, r \geq 1 \) and \( 0 \leq \theta \leq 1 \) must satisfy

\[
\frac{1}{q} = \theta \left( \frac{1}{p} - \frac{1}{N} \right) + (1 - \theta) \frac{1}{r}.
\]

**Question 2.** We take \( u \in W^{1,N}([\mathbb{R}^N]) \cap L^{\frac{(\alpha-1)N}{\alpha-1}}(\mathbb{R}^N) \) and we define \( v = |u|^{\alpha-1} u \) with \( \alpha \geq N \). First, let’s show that \( v \in W^{1,1}(\mathbb{R}^N) \). We observe that we have \( N \leq \alpha \leq \frac{(\alpha-1)N}{N-1} \) for all \( \alpha \geq N \), so using the interpolation properties of the Lebesgue spaces, there exists \( \theta \in [0,1] \) such that

\[
\frac{1}{\alpha} = \theta \frac{1}{N} + (1 - \theta) \frac{N-1}{(\alpha-1)N}.
\]

We compute \( \nabla v = \alpha |u|^{\alpha-1} \nabla u \), from this, using the Hölder inequality with \( p = N \), we get

\[
\|\nabla v\|_1 = \alpha \| |u|^{\alpha-1} \nabla u\|_1 \leq \alpha \|\nabla u\|_N \|u|^{1-\frac{1}{N}}\|_N.
\]

We apply Theorem 6.2 in the lecture note with \( p = 1 \) to \( v \), we get

\[
\|v\|_{\frac{N}{N-1}} \leq \frac{1}{\sqrt{N}} \|\nabla v\|_1 \quad \text{which gives} \quad \|u\|^\alpha_{\frac{N}{N-1}} \leq \frac{\alpha}{\sqrt{N}} \|u|^{\alpha-1} \nabla u\|_1
\]

after rearranging the norms and Eq. (3), we obtain

\[
\|u\|^\alpha_{\frac{N}{N-1}} \leq \frac{\alpha}{\sqrt{N}} \|\nabla u\|_N \|u|^{\alpha-1}_{\frac{N-1}{(\alpha-1)N}}
\]

then taking the power \( \frac{1}{\alpha} \), we finally get

\[
\|u\|_{\frac{N\alpha}{N-1}} \leq \left( \frac{\alpha}{\sqrt{N}} \right)^{\frac{1}{\alpha}} \|\nabla u\|^{\frac{1}{\alpha}}_{\frac{N}{N-1}} \|u|^{\frac{1}{\alpha}}_{\frac{N-1}{(\alpha-1)N}}.
\]

**Question 3.** First, from Eq. (1), we deduce

\[
\|u\|_{\frac{N\alpha}{N-1}} \leq C_{\alpha} \left( \|\nabla u\|_N + \|u|^{\alpha-1}_{\frac{N-1}{(\alpha-1)N}} \right).
\]

using the Young’s inequality\(^1\). Then we will show, by induction, that for all \( k \in \mathbb{N} \), there exists a constant \( D_k > 0 \) such that

\[
\|u\|_{\frac{(N+k)\alpha}{N-1}} \leq D_k \|u\|_{1,N}, \quad \forall u \in W^{1,N}(\mathbb{R}^N).
\]

**Base case:** for \( k = 0 \), we apply Eq. (4) with \( \alpha = N \geq 2 - \frac{1}{N} \), we obtain

\[
\|u\|_{\frac{N^2}{N-1}} \leq C_N \left( \|\nabla u\|_N + \|u\|_N \right) \leq D_0 \|u\|_{1,N}
\]

with \( D_0 = 2C_N \) a constant\(^2\).

\(^1\)For \( a, b \geq 0 \) and \( p, q \geq 1 \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \), we have \( ab \leq \frac{a^p}{p} + \frac{b^q}{q} \).

\(^2\)Recall that \( \|u\|_N + \|\nabla u\|_N \leq 2\|u\|_{1,N} \).
Therefore, we have a continuous embedding for all \( \forall u \in W^{1,N}(\mathbb{R}^N) \).

Inductive step: Take \( k \geq 1 \) and we assume that for all \( \ell < k \) there exists a constant \( D_\ell > 0 \) such that
\[
\|u\|_{\frac{(N+k)N}{N-1}} \leq D_\ell \|u\|_{1,N}, \quad \forall u \in W^{1,N}(\mathbb{R}^N).
\]
we apply Eq. (4) with \( \alpha = N + k \geq 2 - \frac{1}{N} \), we obtain
\[
\|u\|_{\frac{(N+k)N}{N-1}} \leq C_{N+k} \left( \|\nabla u\|_N + \|u\|_{\frac{(N+k-1)N}{N-1}} \right)
\]
\[
\leq C_{N+k} \left( \|\nabla u\|_N + D_{k-1} \|u\|_{1,N} \right)
\]
\[
\leq C_{N+k} (1 + D_{k-1}) \|u\|_{1,N}
\]
and set \( D_k = C_{N+k} (1 + D_{k-1}) \).

Conclusion: for all \( k \in \mathbb{N} \), there exists a constant \( D_k > 0 \) such that
\[
\|u\|_{\frac{(N+k)N}{N-1}} \leq D_k \|u\|_{1,N}, \quad \forall u \in W^{1,N}(\mathbb{R}^N).
\]

Then, we use the interpolation properties of the Lebesgue spaces. For \( N \leq q < \infty \), we take the smallest integer \( k \geq \frac{qN}{N-1} - N > 0 \) so we have \( \frac{N-1}{(N+k)N} \leq \frac{1}{q} \leq \frac{1}{N} \). There exists \( \theta \in [0,1] \) such that \( \frac{1}{q} = \theta \frac{1}{N} + (1-\theta) \frac{N-1}{(N+k)N} \) and, using the generalized Hölder’s inequality, we obtain
\[
\|u\|_q \leq \|u\|_{1,N}^{\theta} \|u\|_{\frac{(N+k)N}{N-1}}^{1-\theta} \leq \|u\|_{1,N}^{\theta} D_k^{1-\theta} \|u\|_{1,N}^{1-\theta} \leq D_k^{1-\theta} \|u\|_{1,N}
\]
therefore, we have a continuous embedding \( W^{1,N}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N) \) precisely for \( N \leq q < \infty \).

Question 4. For \( 0 < \delta < 1 - \frac{1}{N} \) and \( \chi \in C_0^\infty \left(B \left(0, \frac{1}{2}\right)\right) \) such that \( \chi \equiv 1 \) on \( B \left(0, \frac{1}{4}\right) \), we define the function
\[
u: x \mapsto |\ln(|x|)|^\delta \chi(x).
\]
First, let’s show that \( u \not\in L^\infty(\mathbb{R}^N) \). Take \( M > 0 \), we have
\[
B(0,\rho_M) \subset u^{-1} ((M, +\infty)) \quad \text{with} \quad \rho_M = \min \left( \exp \left(-M \frac{1}{\delta}\right), \frac{1}{4} \right)
\]
and \( B(0,\rho_M) \) has non zero measure, therefore, \( u \not\in L^\infty(\mathbb{R}^N) \).

Now let’s check that \( u \in W^{1,N}(\mathbb{R}^N) \).
\[
\|u\|_{W^{1,N}} = \int_{\mathbb{R}^N} |\ln(|x|)|^{\delta N} |\chi(x)|^N \, dx
\]
\[
= \int_{B(0,\frac{1}{4})} |\ln(|x|)|^{\delta N} \, dx + \int_{B(0,\frac{1}{4}) \setminus B(0,\frac{1}{2})} |\ln(|x|)|^{\delta N} |\chi(x)|^N \, dx
\]
\[
= A_N \int_0^{\frac{1}{4}} |\ln(r)|^{\delta N} r^{N-1} \, dr + C_\chi
\]
\[
= \left( A_N \int_{S^{N-1}} d\sigma(\omega) \right)
\]
this is finite because the function \( r \mapsto |\ln(|x|)|^{\delta N} r^{N-1} \) is continuous on \( [0,\frac{1}{2}] \). Then we compute the weak derivative of \( u \) to be
\[
\nabla u(x) = -\delta \frac{x}{|x|^2} |\ln(|x|)|^{\delta-1} \chi(x) + |\ln(|x|)|^\delta \nabla \chi(x).
\]
The function \( x \mapsto |\ln(|x|)|^{\delta} \nabla \chi(x) \) is in \( C_0^\infty (\mathbb{R}^N) \) so its \( L^N \) norm is finite. For the other part of the gradient, as before, we compute
\[
\int_{\mathbb{R}^N} \frac{|\chi(x)|^N}{|x|^N |\ln(|x|)|^{(1-\delta)N}} \, dx = \int_{B(0,\frac{1}{4})} \frac{1}{|x|^N |\ln(|x|)|^{(1-\delta)N}} \, dx + C'_\chi
\]
\[ = A_N \int_0^1 \frac{1}{r \log(r)^{(1-\delta)N}} \, dr + C' \]

since we have \((1 - \delta)N > 1\) because \(0 < \delta < 1 - \frac{1}{r}\), the first integral is finite (Bertrand’s integral\(^3\)), so \(\nabla u \in L^N(\mathbb{R}^N)\). Therefore, we have \(u \in W^{1,N}(\mathbb{R}^N)\).

**Exercise 2.**

Fix \(1 \leq p \leq \infty\) and let \(u \in W^{1,p}(\mathbb{R})\).

1. Show that there exists \(\tilde{u} \in C^0(\mathbb{R})\) such that \(u = \tilde{u}\) almost everywhere on \(\mathbb{R}\).

   (Hint: fundamental theorem of calculus)

2. Show that \(u \in L^\infty(\mathbb{R})\) by proving that there exists a constant \(C > 0\) such that

\[ |u(x)| \leq C \|u\|_p^{1-\frac{1}{p}} \|u'\|_p^{\frac{1}{p}}. \]

**Solution 2.**

**Question 1.** We define the function \(v\) by

\[ v : x \mapsto \int_0^x u'(t) \, dt \]

it is well defined since we have the estimate \(|v(x)| \leq |x|^{1-\frac{1}{p}} \|u'\|_p\), using the Hölder inequality. To show that the function \(v\) is continuous on \(\mathbb{R}\), we fix a point \(x \in \mathbb{R}\) and will show that it is continuous at \(x\). We define the interval \(I = (x - 1, x + 1)\) and a sequence \((y_n)_{n \in \mathbb{N}} \in I^n\) such that \(y_n \to x\) as \(n \to +\infty\). We have

\[ v(y_n) - v(x) = \int_x^{y_n} u'(t) \, dt \to 0 \]

using the dominated convergence theorem on the sequence of function \(t \mapsto \mathbf{1}_{[x,y_n]}(t) u'(t)\) which is dominated by \(|u'| \in L^1(I) \ (\int_I |u'(t)| \, dt \leq 2^{1-\frac{1}{p}} \|u'\|_p)\) and \(\mathbf{1}_{[x,y_n]}(t) u'(t) \to 0\) as \(n \to +\infty\) for almost every \(t \in I\).

We want to show that the weak derivative of \(v\) is \(u'\). Take \(\varphi \in C^\infty_c(\mathbb{R})\), there exists \(\alpha > 0\) such that \(\text{supp}(\varphi) \subseteq [-\alpha, \alpha]\), we compute

\[
\int_{\mathbb{R}} v(x) \varphi(x) \, dx = \int_{x=-\alpha}^{x=\alpha} \left[ \int_0^{t=x} u'(t) \, dt \right] \varphi'(x) \, dx = -\int_{x=0}^{x=\alpha} \left[ \int_0^{t=x} u'(t) \varphi'(x) \, dx \right] \, dx + \int_{x=0}^{x=\alpha} \left[ \int_0^{t=x} u'(t) \varphi'(x) \, dx \right] \, dx,
\]

using the Fubini’s theorem on both integrals, give

\[
\int_{\mathbb{R}} v(x) \varphi(x) \, dx = -\int_{t=-\alpha}^{t=\alpha} \left[ \int_{x=t}^{x=\alpha} u'(t) \varphi'(x) \, dx \right] \, dx + \int_{t=0}^{t=\alpha} \left[ \int_{x=t}^{x=\alpha} u'(t) \varphi'(x) \, dx \right] \, dx
\]

\(^3\)We can transform the Bertrand’s integral into a Riemann’s integral by using the change variable \(t = -\ln(r)\), which yield

\[
\int_0^1 \frac{1}{r \log(r)^{(1-\delta)N}} \, dr = \int_{\log(4)}^{+\infty} \frac{1}{t^{(1-\delta)N}} \, dt < \infty \quad \text{if, and only if,} \quad (1 - \delta)N > 1.
\]
\[
\begin{align*}
&= - \int_{t=-a}^{t=0} u'(t) \left[ \varphi'(t) - \varphi'(-a) \right] dx + \int_{t=0}^{t=a} u'(t) \left[ \varphi'(a) - \varphi'(t) \right] dx \\
&= - \int_{t=-a}^{t=a} u'(t) \varphi'(t) \, dt \\
&= - \int_{\mathbb{R}} u'(t) \varphi'(t) \, dt.
\end{align*}
\]

Therefore, we have \((u - v)' = 0\), using Exercise sheet 03 Exercise 1., there exists a constant \(c\) such that \(u(x) = c + v(x)\) for almost every \(x \in \mathbb{R}\). By setting \(\bar{u} := c + v \in C^0(\mathbb{R})\), we have answer the question.

**Question 2.** This question is trivial for \(p = \infty\) so we only consider \(1 \leq p < \infty\). For \(u \in W^{1,p}(\mathbb{R})\), we define \(v = |u|^{p-1} u\). By definition, we have \(v \in L^1(\mathbb{R})\) and we have \(v' \in L^1(\mathbb{R})\) because \(v' = p|u|^{p-1} u'\) and

\[
||v'||_1 = p |||u|^{p-1} u'||_1 \leq p ||u|^{p-1}||_{L^p} ||u'||_p = p ||u||^p ||u'||_p \tag{5}
\]

therefore \(v \in W^{1,1}(\mathbb{R})\). We define \(\bar{v} \in C^0(\mathbb{R})\) by

\[
\bar{v} : x \mapsto \int_{-\infty}^{x} v'(t) \, dt
\]

using the same reasoning than **Question 1.**, there exists a constant \(c\) such that \(v = c + \bar{v}\) almost everywhere. However, since we have \(\bar{v}(x) \to 0\) as \(x \to -\infty\) so \(v(x) \to c\) as \(x \to -\infty\) and \(v\) can be in \(L^1(\mathbb{R})\) only if \(c = 0\). So, we get that \(v = \bar{v}\) almost everywhere. From Eq. (5), we get

\[
|u(x)|^p = |v(x)| \leq ||v'||_1 \leq p ||u||^{p-1} ||u'||_p
\]

and taking the power \(\frac{1}{p}\), we obtain

\[
|u(x)| \leq \left( \frac{p}{\bar{p}} \right)^{\frac{1}{p}} ||u||^{1 - \frac{1}{p}} ||u'||^{\frac{1}{p}}
\]

which give

\[
||u||_\infty \leq \left( \frac{p}{\bar{p}} \right)^{\frac{1}{p}} ||u||^{1 - \frac{1}{p}} ||u'||^{\frac{1}{p}} \leq \left( \frac{p}{\bar{p}} \right)^{\frac{1}{p}} \left( ||u||_p + ||u'||_p \right) \leq 2 \left( \frac{p}{\bar{p}} \right)^{\frac{1}{p}} ||u||_{1,p}. \tag{6}
\]

Finally, we have a continuous embedding \(W^{1,p}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})\).

**Remark 1.** The constant in Eq. (6) is independent of \(p\) because \(\frac{1}{p} \leq \frac{1}{\bar{p}}\) for all \(p \geq 1\).