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Sobolev spaces: tutorials

Exercise sheet 05

Exercise 1.

1. Take an unbounded open set $\Omega \subset \mathbb{R}^N$ such that there exists an infinite number of disjoint balls $\bigcup_{i\in\mathbb{N}} B_r(x_i) \subset \Omega$, with $(x_i)_{i\in\mathbb{N}} \in \Omega^{\mathbb{N}}$ a sequence and $r > 0$ a constant such that $B_r(x_i) \cap B_r(x_j) = \varnothing$ for all $i \neq j$. Show that there can not exist a compact embedding $\mathrm{W}^{k,p}(\Omega) \to \mathrm{L}^q(\Omega)$ for any integer k and $1 \leq p, q \leq \infty$.

   (Hint: translation)

2. Take a bounded open set $\Omega \subset \mathbb{R}^N$ and a real number $1 \leq p < N$. Show that the continuous embedding $\mathrm{W}^{1,p}_0(\Omega) \hookrightarrow \mathrm{L}^{p^*}(\Omega)$ is not compact, where $p^* = \frac{Np}{N-p}$.

   (Hint: scaling)

Exercise 2.

We denote by μ the Minkowski content[1] of $K \subset \mathbb{R}^N$ a compact set, defined by

$$\mu(K) = \liminf_{\varepsilon\to 0} \frac{\left|\left\{x \in \mathbb{R}^N \;\middle|\; 0 < \operatorname{dist}(x, K) < \varepsilon\right\}\right|}{\varepsilon}.$$

The isoperimetric inequality: there exists a constant $C_N > 0$ such that for all smooth compact sets $K \subset \mathbb{R}^N$, we have

$$|K|^{\frac{N-1}{N}} \leq C_N\, \mu(K). \tag{1}$$

To prove the isoperimetric inequality, we define the function

$$\phi(x) = \begin{cases} 1 - \dfrac{\operatorname{dist}(x, K)}{\varepsilon} & \text{if } \operatorname{dist}(x, K) \leq \varepsilon \\ 0 & \text{otherwise} \end{cases}.$$

We assume that we have proved that $\phi \in \mathrm{W}^{1,1}(\mathbb{R}^N)$ and $|\nabla\phi(x)| \leq \varepsilon^{-1}$ for all $x \in \mathbb{R}^N$ such that $0 < \operatorname{dist}(x, K) < \varepsilon$.

1. Show Eq. (1) using Sobolev's inequality on ϕ.

2. Show that the constant C_N in Eq. (1) must satisfy $C_N \geq N^{-1} |B_1(0)|^{-\frac{1}{N}}$.

   (Hind: take K a ball)

[1]The Minkowski content generalize the notion of surface area to higher dimensions.

Exercise 3.
Let E, F, and G be three Banach spaces.

1. For $K : E \to F$ a compact operator and $L : F \to G$ a bounded operator, show that $L \circ K$ is a compact operator.

2. For $L : E \to F$ a bounded operator and $K : F \to G$ a compact operator, show that $K \circ L$ is a compact operator.

3. Let $K_n : E \to F$ be a sequence of compact operator that converge to K with respect to the operator norm $|||A|||_{E\to F} = \sup_{x\in E\setminus\{0\}} \frac{\|Ax\|_F}{\|x\|_E}$, show that K is compact.

Exercise 4.
Let $\Omega \subset \mathbb{R}^N$ be a bounded open set.

1. Let $k \in C^{0,\alpha}(\overline{\Omega} \times \overline{\Omega})$ be an α-Hölder function ($\alpha > 0$), and show that the operator

$$\begin{array}{rccc} K : & C(\overline{\Omega}) & \to & C(\overline{\Omega}) \\ & u & \mapsto & \displaystyle\int_\Omega k(x,y)\, u(y)\, \mathrm{d}y \end{array}$$

is compact.

2. Let $g \in \mathrm{L}^1(\mathbb{R}^N)$, show that the operator

$$\begin{array}{rccc} G : & \mathrm{L}^p(\Omega) & \to & \mathrm{L}^p(\Omega) \\ & u & \mapsto & g * u \end{array}$$

is compact for $1 \le p < \infty$.