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Sobolev spaces: tutorials

Exercise sheet 05 with solution

Exercise 1.

1. Take an unbounded open set $\Omega \subset \mathbb{R}^N$ such that there exists an infinite number of disjoint balls $\bigcup_{i\in\mathbb{N}} B_r(x_i) \subset \Omega$, with $(x_i)_{i\in\mathbb{N}} \in \Omega^{\mathbb{N}}$ a sequence and $r > 0$ a constant such that $B_r(x_i) \cap B_r(x_j) = \varnothing$ for all $i \neq j$. Show that there can not exist a compact embedding $\mathrm{W}^{k,p}(\Omega) \to \mathrm{L}^q(\Omega)$ for any integer k and $1 \leq p, q \leq \infty$.

   (Hint: translation)

2. Take a bounded open set $\Omega \subset \mathbb{R}^N$ and a real number $1 \leq p < N$. Show that the continuous embedding $\mathrm{W}_0^{1,p}(\Omega) \hookrightarrow \mathrm{L}^{p^*}(\Omega)$ is not compact, where $p^* = \frac{Np}{N-p}$.

   (Hint: scaling)

Solution 1.

Question 1. Fix an integer k and reals $1 \leq p, q \leq \infty$. Take the sequence $u_i : x \mapsto \varphi(x - x_i)$ with $\varphi \in C_0^\infty(B_r(0))$ nontrivial. For all $i \in \mathbb{N}$, we have $u_i \in \mathrm{W}^{k,p}(\Omega)$ and

$$\|u_i\|_{k,p} = \|\varphi\|_{k,p}$$

so the sequence $(u_i)_{i\in\mathbb{N}}$ is bounded in $\mathrm{W}^{k,p}(\Omega)$. For $i \neq j$, we compute

$$\|u_i - u_j\|_q^q = \int_\Omega |u_i - u_j|^q \,\mathrm{d}x = \int_{B_r(x_i)} |u_i|^q \,\mathrm{d}x + \int_{B_r(x_j)} |u_j|^q \,\mathrm{d}x = 2\,\|\varphi\|_q^q \tag{1}$$

because $\mathrm{supp}(u_i) \cap \mathrm{supp}(u_i) = B_r(x_i) \cap B_r(x_j) = \varnothing$. Equation (1) implies that it can not exists a convergent subsequence $u_{\sigma(i)}$ in $\mathrm{L}^q(\Omega)$. Therefore, an embedding $\mathrm{W}^{k,p}(\Omega) \to \mathrm{L}^q(\Omega)$ can not be compact.

Question 2. There exists an infinite sequence of disjoint balls $(B_{r_i}(x_i))_{i\in\mathbb{N}}$ such that

$$\overline{\bigcup_{i\in\mathbb{N}} B_{r_i}(x_i)} \subset \Omega \quad \text{and} \quad B_{r_i}(x_i) \cap B_{r_j}(x_j) = \varnothing, \text{ for } i \neq j,$$

with $(x_i)_{i\in\mathbb{N}} \in \Omega^{\mathbb{N}}$ a sequence of points and $(r_i)_{i\in\mathbb{N}} \in (0, +\infty)^{\mathbb{N}}$ a sequence of radii. From the boundedness of Ω, the sequence $(r_i)_{i\in\mathbb{N}}$ tends to 0 as $i \to +\infty$ and therefore there exists $R > 0$ such that $r_i \leq R$, for all $i \in \mathbb{N}$. Take a nontrivial $\varphi \in C_0^\infty(B_1(0))$ and define the sequence

$$u_i : x \mapsto \begin{cases} r_i^{1-\frac{N}{p}} \varphi\left(\dfrac{x - x_i}{r_i}\right) & \text{if } x \in B_{r_i}(x_i) \\ 0 & \text{otherwise} \end{cases}.$$

We compute

$$\|u_i\|_p = r_i \|\varphi\|_p \qquad \text{and} \qquad \|\partial_n u_i\|_p = \|\partial_n \varphi\|_p , \quad \text{for } 1 \le n \le N,$$

which give the boundedness of the sequence $(u_i)_{i\in\mathbb{N}}$ in the space $\mathrm{W}^{1,p}(\Omega)$,

$$\|u_i\|_{k,p}^p = \|u_i\|_p^p + \sum_{n=1}^{N} \|\partial_n u_i\|_p^p \le \max(1,R)^p \|\varphi\|_{k,p}^p .$$

Then, for $i \neq j$, we compute

$$\|u_i - u_j\|_{p^*}^{p^*} = \int_{B_{r_i}(x_i)} |u_i|^{p^*} \,\mathrm{d}x + \int_{B_{r_j}(x_j)} |u_j|^{p^*} \,\mathrm{d}x = 2\|\varphi\|_{p^*}^{p^*}$$

because $\operatorname{supp}(u_i) \cap \operatorname{supp}(u_i) = B_r(x_i) \cap B_r(x_j) = \varnothing$. So it can not exists a convergent subsequence $u_{\sigma(i)}$ in $\mathrm{L}^{p^*}(\Omega)$. Therefore, the continuous embedding $\mathrm{W}^{1,p}(\Omega) \to \mathrm{L}^{p^*}(\Omega)$ can not be compact.

Exercise 2.
We denote by μ the Minkowski content[1] of $K \subset \mathbb{R}^N$ a compact set, defined by

$$\mu(K) = \liminf_{\varepsilon\to 0} \frac{\left|\left\{x \in \mathbb{R}^N \;\middle|\; 0 < \operatorname{dist}(x,K) < \varepsilon\right\}\right|}{\varepsilon}.$$

The isoperimetric inequality: there exists a constant $C_N > 0$ such that for all smooth compact sets $K \subset \mathbb{R}^N$, we have

$$|K|^{\frac{N-1}{N}} \le C_N\, \mu(K). \tag{2}$$

To prove the isoperimetric inequality, we define the function

$$\phi(x) = \begin{cases} 1 - \dfrac{\operatorname{dist}(x,K)}{\varepsilon} & \text{if } \operatorname{dist}(x,K) \le \varepsilon \\ 0 & \text{otherwise} \end{cases}.$$

We assume that we have proved that $\phi \in \mathrm{W}^{1,1}(\mathbb{R}^N)$ and $|\nabla\phi(x)| \le \varepsilon^{-1}$ for all $x \in \mathbb{R}^N$ such that $0 < \operatorname{dist}(x,K) < \varepsilon$.

1. Show Eq. (2) using Sobolev's inequality on ϕ.
2. Show that the constant C_N in Eq. (2) must satisfy $C_N \ge N^{-1} |B_1(0)|^{-\frac{1}{N}}$.

(Hind: take K a ball)

Solution 2.
Question 1. We use the Theorem 6.2 in the lecture note with $p = 1$ on ϕ, we get $\|\phi\|_{\frac{N}{N-1}} \le C_N \|\nabla\phi\|_1$. Using the fact $0 \le \mathbb{1}_K \le \phi$, we compute

$$|K|^{\frac{N-1}{N}} = \|\mathbb{1}_K\|_{\frac{N}{N-1}} \le \|\phi\|_{\frac{N}{N-1}} \le C_N \|\nabla\phi\|_1$$

and

$$\|\nabla\phi\|_1 = \int_{\{x\in\mathbb{R}^N \;|\; 0<\operatorname{dist}(x,K)<\varepsilon\}} |\nabla\phi| \,\mathrm{d}x \le \frac{\left|\left\{x \in \mathbb{R}^N \;\middle|\; 0 < \operatorname{dist}(x,K) < \varepsilon\right\}\right|}{\varepsilon}.$$

[1]The Minkowski content generalize the notion of surface area to higher dimensions.

So, we get

$$|K|^{\frac{N-1}{N}} \leq C_N \frac{\left|\left\{x \in \mathbb{R}^N \;\middle|\; 0 < \operatorname{dist}(x, K) < \varepsilon\right\}\right|}{\varepsilon}$$

and taking the $\liminf_{\varepsilon \to 0}$ gives the result.

Question 2. Fix $K = B_r(x)$ with $x \in \mathbb{R}^N$ and $r > 0$. We compute the Lebesgue measure of K, we have

$$|K| = \int_{B_r(x)} 1 \, \mathrm{d}y = r^N \int_{B_1(0)} 1 \, \mathrm{d}z = r^N \omega_N$$

with $\omega_N = |B_1(0)|$. We compute the Minkowski content of K, we have

$$\begin{aligned}
\mu(K) &= \liminf_{\varepsilon \to 0} \frac{\left|\left\{y \in \mathbb{R}^N \;\middle|\; r < |y - x| < r + \varepsilon\right\}\right|}{\varepsilon} \\
&= \liminf_{\varepsilon \to 0} \frac{|B_{r+\varepsilon}(x)| - |B_r(x)|}{\varepsilon} \\
&= \liminf_{\varepsilon \to 0} \frac{(r+\varepsilon)^N - r^N}{\varepsilon} \omega_N \\
&= N r^{N-1} \omega_N.
\end{aligned}$$

Now we combine the two expression to get

$$r^{N-1} \omega_N^{1 - \frac{1}{N}} \leq C_N N r^{N-1} \omega_N \qquad \Longrightarrow \qquad N^{-1} \omega_N^{-\frac{1}{N}} \leq C_N$$

which gives the result.

Remark 1. Actually, $N^{-1} \omega_N^{-\frac{1}{N}}$ is the optimal constant of the isoperimetric inequality.

Exercise 3.
Let E, F, and G be three Banach spaces.

1. For $K : E \to F$ a compact operator and $L : F \to G$ a bounded operator, show that $L \circ K$ is a compact operator.
2. For $L : E \to F$ a bounded operator and $K : F \to G$ a compact operator, show that $K \circ L$ is a compact operator.
3. Let $K_n : E \to F$ be a sequence of compact operator that converge to K with respect to the operator norm $|||A|||_{E \to F} = \sup_{x \in E \setminus \{0\}} \frac{\|Ax\|_F}{\|x\|_E}$, show that K is compact.

Solution 3.
Question 1. Take a bounded sequence $(e_n)_{n \in \mathbb{N}} \in E^{\mathbb{N}}$. Since K is compact, there exists a strictly increasing map $n \mapsto \sigma(n)$ such that the subsequence $\left(K e_{\sigma(n)}\right)_{n \in \mathbb{N}}$ converge to $f \in F$. Since L is a bounded operator, $\left(L \circ K e_{\sigma(n)}\right)_{n \in \mathbb{N}}$ converge to Lf. Therefore, $L \circ K$ is compact.

Question 2. Take a bounded sequence $(e_n)_{n \in \mathbb{N}} \in E^{\mathbb{N}}$. Since L is bounded, the sequence $\left(L e_{\sigma(n)}\right)_{n \in \mathbb{N}}$ is bounded in F. Since K is compact, there exists a strictly increasing map $n \mapsto \sigma(n)$ such that the subsequence $\left(L \circ K e_{\sigma(n)}\right)_{n \in \mathbb{N}}$ converge to $g \in G$. Therefore, $K \circ L$ is compact.

Question 3. We will use a so called diagonal extraction argument. Take a bounded sequence $(e_i)_{i \in \mathbb{N}} \in E^{\mathbb{N}}$ such that $\|e_i\|_E \leq R$. Since K_0 is compact, there exists a strictly increasing map

$i \mapsto \sigma_0(i)$ such that the subsequence $\left(K_0 e_{\sigma_0(i)}\right)_{i\in\mathbb{N}}$ converge to $f_0 \in F$. Since K_1 is compact, there exists a strictly increasing map $i \mapsto \sigma_1(i)$ such that the subsequence $\left(K_1 e_{\sigma_0\circ\sigma_1(i)}\right)_{i\in\mathbb{N}}$ converge to $f_1 \in F$. So, by induction, there exists a sequence of strictly increasing map $i \mapsto \sigma_n(i)$ such that the subsequence $\left(K_n e_{\sigma_0\circ\sigma_1\circ\cdots\circ\sigma_n(i)}\right)_{i\in\mathbb{N}}$ converge to $f_n \in F$, for all $n \in \mathbb{N}$. We define the map $\rho : n \mapsto \sigma_0 \circ \cdots \circ \sigma_n(n)$ (this is the diagonal extraction). For $i < j$, we have $j \leq \sigma_{i+1} \circ \cdots \circ \sigma_j(j)$ because $\sigma_{i+1} \circ \cdots \circ \sigma_j$ is a strictly increasing map $\mathbb{N} \to \mathbb{N}$, which gives

$$\rho(i) = \sigma_0 \circ \cdots \circ \sigma_i(i) < \sigma_0 \circ \cdots \circ \sigma_i \circ \sigma_{i+1} \circ \cdots \circ \sigma_j(j) = \rho(j)$$

so ρ is a strictly increasing map. Now, we want to show that the sequence $\left(K e_{\rho(n)}\right)_{n\in\mathbb{N}}$ is a Cauchy sequence. Let fix $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that for all $i \geq N$, we have $\|\!|K_i - K|\!\|_{E\to F} < \varepsilon$. Since $\left(K_N e_{\rho(i)}\right)_{i\in\mathbb{N}}$ is convergence, it is a Cauchy sequence, there exits $M \in \mathbb{N}$ such that, for all $i, j \geq M$, we have $\left\|K_N e_{\rho(i)} - K_N e_{\rho(j)}\right\|_F < \varepsilon$. For $i, j \geq \max(N, M)$, we compute

$$\begin{aligned}
\left\|K e_{\rho(i)} - K e_{\rho(j)}\right\|_F &\leq \left\|K e_{\rho(i)} - K_N e_{\rho(i)}\right\|_F + \left\|K_N e_{\rho(i)} - K_N e_{\rho(j)}\right\|_F + \left\|K e_{\rho(j)} - K_N e_{\rho(j)}\right\|_F \\
&\leq \|\!|K - K_N|\!\|_{E\to F} R + \varepsilon + \|\!|K - K_N|\!\|_{E\to F} R \\
&\leq (2R + 1)\varepsilon
\end{aligned}$$

therefore the sequence $\left(K e_{\rho(i)}\right)_{i\in\mathbb{N}}$ is a Cauchy sequence, so it converge, and K is compact.

Exercise 4.
Let $\Omega \subset \mathbb{R}^N$ be a bounded open set.

1. Let $k \in C^{0,\alpha}(\overline{\Omega} \times \overline{\Omega})$ be an α-Hölder function ($\alpha > 0$), and show that the operator

$$\begin{array}{rccc} K : & C(\overline{\Omega}) & \to & C(\overline{\Omega}) \\ & u & \mapsto & \displaystyle\int_\Omega k(x, y)\, u(y)\, \mathrm{d}y \end{array}$$

   is compact.

2. Let $g \in \mathrm{L}^1(\mathbb{R}^N)$, show that the operator

$$\begin{array}{rccc} G : & \mathrm{L}^p(\Omega) & \to & \mathrm{L}^p(\Omega) \\ & u & \mapsto & g * u \end{array}$$

   is compact for $1 \leq p < \infty$.

Solution 4.

Question 1. Let $(u_i)_{i\in\mathbb{N}} \in C(\overline{\Omega})^{\mathbb{N}}$ be a bounded sequence of continuous functions, with the bound $\|u_i\|_\infty \leq R$ for all $i \in \mathbb{N}$. We want to show that the sequence $(Ku_i)_{i\in\mathbb{N}}$ has a converging subsequence using the Arzelá-Ascoli theorem (Theorem 9.3 in the lecture note). The set $\overline{\Omega}$ is bounded and closed, the sequence $(Ku_i)_{i\in\mathbb{N}}$ is point wise bounded

$$|Ku_i(x)| \leq \left|\int_\Omega k(x, y)\, u_i(y)\, \mathrm{d}y\right| \leq |\Omega| \max_{\overline{\Omega}\times\overline{\Omega}} |k|\; \|u_i\|_\infty \leq R\; |\Omega| \max_{\overline{\Omega}\times\overline{\Omega}} |k|\,.$$

The function k is α-Hölder so there exists a constant $C > 0$ such that for all $x, y, x', y' \in \overline{\Omega}$, we have

$$|k(x, y) - k(x', y')| \leq C\, |(x, y) - (x', y')|^\alpha\,.$$

Let $\varepsilon > 0$ and $x, y \in \overline{\Omega}$ such that $|x-y| < \varepsilon^{\frac{1}{\alpha}}$, we compute

$$\begin{aligned} |Ku_i(x) - Ku_i(y)| &\leq \left| \int_\Omega \left(k(x,z) - k(y,z)\right) u_i(z) \, \mathrm{d}z \right| \\ &\leq \int_\Omega |k(x,z) - k(y,z)| \, |u_i(z)| \, \mathrm{d}z \\ &\leq |\Omega| \, C \, |x-y|^\alpha \, \|u_i\|_\infty \\ &\leq |\Omega| \, C R \varepsilon \end{aligned}$$

for all $i \in \mathbb{N}$. Therefore, the sequence $(Ku_i)_{i\in\mathbb{N}}$ is equicontinuous which give that there exists a converging subsequence and so the operator K is compact.

Question 2. Let $(u_i)_{i\in\mathbb{N}} \in \mathrm{L}^p(\Omega)^{\mathbb{N}}$ be a bouded sequence of continuous functions, with the bound $\|u_i\|_p \leq R$ for all $i \in \mathbb{N}$. We want to applye the Fréchet-Kolmogorov theorem (Theorem 9.9 of the lecture note) to the set $(Gu_i)_{i\in\mathbb{N}}$.

(i): The set $(Gu_i)_{i\in\mathbb{N}}$ is bounded in $\mathrm{L}^p(\Omega)$, we compute

$$\|Gu_i\|_p \leq \|g\|_1 \, \|u_i\|_p \leq \|g\|_1 \, R.$$

(ii): Let $\varepsilon > 0$, using Theorem 9.7[2] in the lecture note there exits $\delta > 0$ such that for all $|h| < \delta$, we have $\|g(\cdot + h) - g\|_1 < \varepsilon$. For $|h| < \delta$, we compute

$$\begin{aligned} \|Gu_i(\cdot + h) - Gu_i\|_p &= \|g * u_i(\cdot + h) - g * u_i\|_p \\ &= \|[g(\cdot + h) - g] * u_i\|_p \\ &\leq \|g(\cdot + h) - g\|_1 \, \|u_i\|_p \\ &\leq R\varepsilon. \end{aligned}$$

(iii): Let $\varepsilon > 0$, there exits $\rho_\varepsilon > 0$ such that $\int_{|x|>\rho_\varepsilon} |g| < \varepsilon$. We note $\omega_\varepsilon = \overline{B_{\rho_\varepsilon}(0)}$ and we compute

$$\|Gu_i\|_{\mathrm{L}^p(\Omega\setminus\omega_\varepsilon)} \leq \|g\|_{\mathrm{L}^1(\mathbb{R}^N\setminus\omega_\varepsilon)} \, \|u_i\|_{\mathrm{L}^p(\Omega\setminus\omega_\varepsilon)} \leq \|g\|_{\mathrm{L}^1(\mathbb{R}^N\setminus\omega_\varepsilon)} \, R \leq R\varepsilon.$$

Therefore the set $(Gu_i)_{i\in\mathbb{N}}$ is precompact in $\mathrm{L}^p(\mathbb{R}^N)$ so the operator G is compact.

[2]Let $f \in \mathrm{L}^p(\mathbb{R}^N)$. Then $f(\cdot + h) \to f$ in $\mathrm{L}^p(\mathbb{R}^N)$ as $|h| \to 0$.