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July 9, 2021

Sobolev spaces: tutorials

Exercise sheet 06

Exercise 1.
Let a bounded open Lipschitz domain $\Omega \subset \mathbb{R}^N$ and a function $f \in \mathrm{L}^2(\Omega)$. We define the Poisson's equation: find $u \in \mathrm{H}^1(\Omega)$ such that

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ \partial_\nu u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where ∂_ν is the normal derivative on the boundary $\partial\Omega$.

1. Compute the weak formulation of Eq. (1).
2. Assuming that a solution of the weak formulation of Eq. (1) exists, show that it can not be unique in $\mathrm{H}^1(\Omega)$.
3. Show that $\int_\Omega f(x)\,\mathrm{d}x = 0$ is a necessary condition for Eq. (1) to admit a weak solution in $\mathrm{H}^1(\Omega)$.

We define the subspace of $\mathrm{L}^2(\Omega)$ of functions with zero mean and the corresponding subspace in $\mathrm{H}^1(\Omega)$ by

$$\mathrm{L}^2_0(\Omega) := \left\{ u \in \mathrm{L}^2(\Omega) \;\middle|\; \int_\Omega u(x)\,\mathrm{d}x = 0 \right\} \quad \text{and} \quad V = \mathrm{L}^2_0(\Omega) \cap \mathrm{H}^1(\Omega).$$

4. Show that we have the orthogonal decomposition $\mathrm{H}^1(\Omega) = \operatorname{span}\{x \mapsto 1\} \oplus_\perp V$.
5. Show that the bilinear form $\langle u, v\rangle_V := \int_\Omega \nabla u(x) \cdot \nabla v(x)\,\mathrm{d}x$ is an inner product on the space V and that the associated norm $\|u\|_V = \sqrt{\langle u, u\rangle_V}$ is equivalent to the norm $\langle \cdot, \cdot\rangle_{1,2}$.
6. For $f \in \mathrm{L}^2_0(\Omega)$, show that the weak formulation of Eq. (1) has a unique solution in V and that there exists $C > 0$ such that $\|u\|_V \leq C\,\|f\|_2$.

Exercise 2.
Let $I = (0, 2\pi)$. The optimal Wirtinger inequality in one dimension is

$$\left\| f - \frac{1}{2\pi}\int_0^{2\pi} f(x)\,\mathrm{d}x \right\|_2 \leq \|f'\|_2\,, \qquad \text{for } f \in \mathrm{H}^1(I). \tag{2}$$

1. Prove Eq. (2) for $f \in C^\infty([0, 2\pi])$ using Fourier series.
2. Prove Eq. (2) for $f \in \mathrm{H}^1(I)$.
3. Characterize the functions that satisfy the equality of Eq. (2).