Sobolev spaces: tutorials
Exercise sheet 06 with solution

Exercise 1.
Let a bounded open Lipschitz domain $\Omega \subset \mathbb{R}^N$ and a function $f \in L^2(\Omega)$. We define the Poisson’s equation: find $u \in H^1(\Omega)$ such that

$$\begin{cases}
\Delta u = f & \text{in } \Omega \\
\partial_\nu u = 0 & \text{on } \partial \Omega
\end{cases}$$

(1)

where $\partial_\nu$ is the normal derivative on the boundary $\partial \Omega$.

1. Compute the weak formulation of Eq. (1).

2. Assuming that a solution of the weak formulation of Eq. (1) exists, show that it cannot be unique in $H^1(\Omega)$.

3. Show that $\int_{\Omega} f(x) \, dx = 0$ is a necessary condition for Eq. (1) to admit a weak solution in $H^1(\Omega)$.

We define the subspace of $L^2(\Omega)$ of functions with zero mean and the corresponding subspace in $H^1(\Omega)$ by

$$L^2_0(\Omega) := \left\{ u \in L^2(\Omega) \mid \int_{\Omega} u(x) \, dx = 0 \right\} \quad \text{and} \quad V = L^2_0(\Omega) \cap H^1(\Omega).$$

4. Show that we have the orthogonal decomposition $H^1(\Omega) = \text{span}\{x \mapsto 1\} \oplus V$.

5. Show that the bilinear form $\langle u, v \rangle_V := \int_{\Omega} \nabla u(x) \cdot \nabla v(x) \, dx$ is an inner product on the space $V$ and that the associated norm $\|u\|_V = \sqrt{\langle u, u \rangle_V}$ is equivalent to the norm $\langle \cdot, \cdot \rangle_{1,2}$.

6. For $f \in L^2_0(\Omega)$, show that the weak formulation of Eq. (1) has a unique solution in $V$ and that there exists $C > 0$ such that $\|u\|_V \leq C \|f\|_2$.

Solution 1.

Question 1. As usual, we multiply by $v \in H^1(\Omega)$ and use the divergence theorem to get the following weak formulation: find $u \in H^1(\Omega)$ such that

$$\int_{\Omega} \nabla u(x) \cdot \nabla v(x) \, dx = \int_{\Omega} f(x) v(x) \, dx, \quad \forall v \in H^1(\Omega).$$

(2)
We verify the hypothesis of the Lax-Milgram theorem on the Hilbert space $\mathcal{H}$ so there exists and by definition of the norms definiteness part, we take $\text{span}$ and, if $u$ is constant and the only constant in $\Omega$, therefore, for solution to exits we must have $f$ with zero mean.

Question 3. Take $v \equiv 1 \in \mathcal{H}$ in Eq. (2) to get
$$0 = \int_\Omega f(x) \, dx.$$ Therefore, for solution to exits we must have $f$ with zero mean.

Question 4. The sum is direct because for $u \in \mathcal{H}$, we have
$$u = \frac{1}{|\Omega|} \int_\Omega u(x) \, dx + u - \frac{1}{|\Omega|} \int_\Omega u(x) \, dx$$ and, if $u \in \text{span}\{x \mapsto 1\} \cap \mathcal{V}$, the function is constant and of zero mean so $u = 0$. The subspaces $\text{span}\{x \mapsto 1\}$ and $\mathcal{V}$ are orthogonal, take $w \in \mathcal{V}$, we compute
$$\langle x \mapsto 1, w \rangle_{1,2} = \int_\Omega w(x) \, dx = 0.$$

Question 5. The bilinear form $\langle \cdot, \cdot \rangle_{\mathcal{V}}$ is symmetric and non-negative by definition. For the definiteness part, we take $u \in \mathcal{V}$ such that $\|u\|_{\mathcal{V}} = \sqrt{\langle u, u \rangle_{\mathcal{V}}} = 0$, this give $\nabla u = 0$ in $\Omega$. So $u$ is constant and the only constant in $\mathcal{V}$ is $0$ therefore $u = 0$. The bilinear form $\langle \cdot, \cdot \rangle_{\mathcal{V}}$ is an inner product.

Now let’s show that $\|\cdot\|_{\mathcal{V}}$ and $\|\cdot\|_{1,2}$ are equivalent. We directly have $\|u\|_{\mathcal{V}} \leq \|u\|_{1,2}$ for all $u \in \mathcal{V}$. For the converse inequality, using the Wirtinger inequality for $\mathcal{V}$, we get
$$\|u\|_2 = \left\| u - \frac{1}{|\Omega|} \int_\Omega u(x) \, dx \right\|_2 \leq C \|\nabla u\|_2 = C \|u\|_{\mathcal{V}},$$ and by definition of the norms $\langle \cdot, \cdot \rangle_{1,2}$ and we compute
$$\|u\|_{1,2}^2 = \|u\|_2^2 + \|\nabla u\|_2^2 \leq (C^2 + 1) \|\nabla u\|_2^2 = (C^2 + 1) \|u\|_{\mathcal{V}}^2.$$ so there exists $D > 1$ such that $\|u\|_{\mathcal{V}} \leq \|u\|_{1,2} \leq D \|u\|_{\mathcal{V}}$ for all $u \in \mathcal{V}$.

Question 6. We define the associated problem to Eq. (2) on the space $\mathcal{V}$ by: find $u \in \mathcal{V}$ such that
$$\int_\Omega \nabla u(x) \cdot \nabla w(x) \, dx = \int_\Omega f(x) \, w(x) \, dx, \quad \forall w \in \mathcal{V}. \quad (3)$$ We verify the hypothesis of the Lax-Milgram theorem on the Hilbert space $\mathcal{V}$:
(i) we have $\left| \int_\Omega \nabla u(x) \cdot \nabla w(x) \, dx \right| \leq \|u\|_{\mathcal{V}} \|w\|_{\mathcal{V}}$ for $u, w \in \mathcal{V}$;
(ii) we have $\int_\Omega |\nabla u(x)|^2 \, dx \geq \|u\|_{\mathcal{V}}^2$ for $u \in \mathcal{V};$
(iii) we have \(|\int_\Omega f(x) w(x) \, dx| \leq \|f\|_2 \|w\|_2 \leq \|f\|_2 C \|w\|_V\) for \(w \in V\).

Using the Lax-Milgram theorem on Eq. (3), there exists a unique \(u \in V\) that satisfy Eq. (3) and \(\|u\|_V \leq C \|f\|_2\). Now, we show that the solution \(u\) of Eq. (3) is also a solution of Eq. (2), for \(v \in H^1(\Omega)\) from the decomposition \(H^1(\Omega) = \text{span}\{x \mapsto 1\} \oplus V\) there exists \((c, w) \in \text{span}\{x \mapsto 1\} \times V\) such that \(v = c + w\) and we compute

\[
\int_\Omega \nabla u(x) \cdot \nabla v(x) \, dx = \int_\Omega \nabla u(x) \cdot \nabla w(x) \, dx \quad (\nabla c = 0)
\]

\[
= \int_\Omega f(x) w(x) \, dx \quad (u \text{ solution of Eq. (3)})
\]

\[
= \int_\Omega f(x) v(x) \, dx. \quad \left(\int_\Omega f(x) \, dx = 0\right)
\]

So, for \(f \in L^2_0(\Omega)\), \(u\) is the unique solution in \(V\) that satisfy Eq. (2).

**Remark 1.** Using Lax-Milgram theorem, Eq. (3) has an unique solution for \(f \in L^2(\Omega)\) contrary to Eq. (2) how has a solution if, and only if, \(f \in L^2(\Omega)\) with zero mean and the solution is not unique in \(H^1(\Omega)\). We say that Eq. (3) is well-posed and Eq. (2) is ill-posed in the Hadamard sense, see [en.wikipedia.org/wiki/Well-posed_problem](http://en.wikipedia.org/wiki/Well-posed_problem).

**Exercise 2.**

Let \(I = (0, 2\pi)\). The optimal Wirtinger inequality in one dimension is

\[
\left\| f - \frac{1}{2\pi} \int_0^{2\pi} f(x) \, dx \right\|_2 \leq \|f\|_2, \quad \text{for } f \in H^1(I).
\]

1. Prove Eq. (4) for \(f \in C^\infty([0, 2\pi])\) using Fourier series.

2. Prove Eq. (4) for \(f \in H^1(I)\).

3. Characterize the functions that satisfy the equality of Eq. (4).

**Solution 2.**

**Question 1.** Since \(f \in C^\infty([0, 2\pi])\), the Fourier series of \(f\) and \(f'\) absolutely converge, and we have

\[
f(x) = a_0 + \sum_{n=1}^{+\infty} a_n \cos(nx) + b_n \sin(nx) \quad \text{and} \quad f'(x) = \sum_{n=1}^{+\infty} -n a_n \sin(nx) + n b_n \cos(nx).
\]

Using the orthogonal properties of the functions \(x \mapsto \cos(nx)\) and \(x \mapsto \sin(nx)\) on \([0, 2\pi]\), we compute

\[
\left\| f - \frac{1}{2\pi} \int_0^{2\pi} f(x) \, dx \right\|_2^2 = \left\| \sum_{n=1}^{+\infty} a_n \cos(n\cdot) + b_n \sin(n\cdot) \right\|_2^2
\]

\[
= \sum_{n=1}^{+\infty} \int_0^{2\pi} |a_n|^2 \cos^2(nx) + |b_n|^2 \sin^2(nx) \, dx
\]

\[
= \pi \sum_{n=1}^{+\infty} |a_n|^2 + |b_n|^2
\]
and 
\[ \| f' \|_2^2 = \left\| \sum_{n=1}^{+\infty} -n a_n \cos(n \cdot) + n b_n \sin(n \cdot) \right\|_2^2 = \sum_{n=1}^{+\infty} \int_0^{2\pi} n^2 |a_n|^2 \cos^2(n x) + n^2 |b_n|^2 \sin^2(n x) \, dx = \pi \sum_{n=1}^{+\infty} n^2 |a_n|^2 + n^2 |b_n|^2 \]

for \( n \geq 1 \), we have \( \pi \sum_{n=1}^{+\infty} |a_n|^2 + |b_n|^2 \leq \pi \sum_{n=1}^{+\infty} n^2 |a_n|^2 + n^2 |b_n|^2 \) which give 
\[ \left\| f - \frac{1}{2\pi} \int_0^{2\pi} f(x) \, dx \right\|_2 \leq \| f' \|_2, \quad \text{for } f \in C^\infty([0,2\pi]). \]

**Question 2.** By density, for \( f \in H^1(I) \), there exists a sequence \( (f_k)_{k \in \mathbb{N}} \subset C^\infty([0,2\pi])^N \) such that \( f_k \to f \) as \( k \to +\infty \) in \( H^1(I) \). We directly have that \( f_k \to f \) and \( f_k' \to f' \) in \( L^2(I) \) as \( k \to +\infty \). For the convergence of the mean, we compute 
\[ \left| \int_0^{2\pi} f_k(x) \, dx - \int_0^{2\pi} f(x) \, dx \right| \leq \int_0^{2\pi} |f_k - f| \, dx \leq \sqrt{2\pi} \| f_k - f \|_2 \]
so \( f_k - \frac{1}{2\pi} \int_0^{2\pi} f_k(x) \, dx \to f - \frac{1}{2\pi} \int_0^{2\pi} f(x) \, dx \) as \( k \to +\infty \) in \( L^2(I) \). Therefore, by passing to the limit in \( \left\| f_k - \frac{1}{2\pi} \int_0^{2\pi} f_k(x) \, dx \right\|_2 \leq \| f_k' \|_2 \), we get 
\[ \left\| f - \frac{1}{2\pi} \int_0^{2\pi} f(x) \, dx \right\|_2 \leq \| f' \|_2, \quad \text{for } f \in H^1(I). \]

**Question 3.** From Question 1, we see that the functions \( x \mapsto 1, x \mapsto \cos(x) \), and \( x \mapsto \sin(x) \) realize the equality in Eq. (4) and that it remains true for any linear combination of the three functions. We define the subspace \( E \subset H^1(I) \) by 
\[ E = \text{span} \left( x \mapsto 1, x \mapsto \cos(x), x \mapsto \sin(x) \right) \]
and we want to show that \( f \in H^1(I) \) realize the equality of Eq. (4) if, and only if, \( f \in E \). If \( f \in E \) then \( f \in H^1(I) \) realize the equality of Eq. (4) is a direct computation. The reverse implication require a bit more works. First, we define the orthogonal complement \( W \subset H^1(I) \) such that \( H^1(I) = E \perp W \). The set \( W \cap C^\infty(I) \) is compose of function such that their Fourier series coefficient \( a_0 = a_1 = b_1 = 0 \). For such function, we can redo Question 1 and found an improved Wirtinger inequality of the form 
\[ \| w \|_2 \leq \frac{1}{2} \| w' \|_2, \quad \text{for } w \in W \cap C^\infty(I) \]
then, using Question 2 on \( W \), we get 
\[ \| w \|_2 \leq \frac{1}{2} \| w' \|_2, \quad \text{for } w \in W. \]
Assume \( f \in H^1(I) \) realize the equality in Eq. (4). We can write \( f = e + w \) with \( (e, w) \in E \times W \) then we compute 
\[ \left\| e - \frac{1}{2\pi} \int_0^{2\pi} e(x) \, dx \right\|_2^2 + \| w \|_2^2 = \left\| f - \frac{1}{2\pi} \int_0^{2\pi} f(x) \, dx \right\|_2^2 \]


\[
\begin{align*}
\|f'\|^2_2 &= \|e'\|^2_2 + \|w'\|^2_2 \\
&\geq \left\| e - \frac{1}{2\pi} \int_0^{2\pi} e(x) \, dx \right\|^2_2 + 4 \|w\|^2_2
\end{align*}
\]

which give \(\|w\|_2 \geq 2 \|w\|_2\) so \(w = 0\) and \(f \in E\).