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Sobolev spaces: tutorials

Exercise sheet 06 with solution

Exercise 1.
Let a bounded open Lipschitz domain $\Omega \subset \mathbb{R}^N$ and a function $f \in \mathrm{L}^2(\Omega)$. We define the Poisson's equation: find $u \in \mathrm{H}^1(\Omega)$ such that

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ \partial_\nu u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where ∂_ν is the normal derivative on the boundary $\partial\Omega$.

1. Compute the weak formulation of Eq. (1).
2. Assuming that a solution of the weak formulation of Eq. (1) exists, show that it can not be unique in $\mathrm{H}^1(\Omega)$.
3. Show that $\int_\Omega f(x)\,\mathrm{d}x = 0$ is a necessary condition for Eq. (1) to admit a weak solution in $\mathrm{H}^1(\Omega)$.

We define the subspace of $\mathrm{L}^2(\Omega)$ of functions with zero mean and the corresponding subspace in $\mathrm{H}^1(\Omega)$ by

$$\mathrm{L}^2_0(\Omega) := \left\{ u \in \mathrm{L}^2(\Omega) \;\middle|\; \int_\Omega u(x)\,\mathrm{d}x = 0 \right\} \quad \text{and} \quad V = \mathrm{L}^2_0(\Omega) \cap \mathrm{H}^1(\Omega).$$

4. Show that we have the orthogonal decomposition $\mathrm{H}^1(\Omega) = \operatorname{span}\{x \mapsto 1\} \oplus_\perp V$.
5. Show that the bilinear form $\langle u, v\rangle_V := \int_\Omega \nabla u(x) \cdot \nabla v(x)\,\mathrm{d}x$ is an inner product on the space V and that the associated norm $\|u\|_V = \sqrt{\langle u, u\rangle_V}$ is equivalent to the norm $\langle \cdot, \cdot \rangle_{1,2}$.
6. For $f \in \mathrm{L}^2_0(\Omega)$, show that the weak formulation of Eq. (1) has a unique solution in V and that there exists $C > 0$ such that $\|u\|_V \leq C\,\|f\|_2$.

Solution 1.
Question 1. As usual, we multiply by $v \in \mathrm{H}^1(\Omega)$ and use the divergence theorem to get following weak formulation: find $u \in \mathrm{H}^1(\Omega)$ such that

$$\int_\Omega \nabla u(x) \cdot \nabla v(x)\,\mathrm{d}x = \int_\Omega f(x)\,v(x)\,\mathrm{d}x, \qquad \forall v \in \mathrm{H}^1(\Omega). \tag{2}$$

Question 2. Let $u \in \mathrm{H}^1(\Omega)$ be a solution of Eq. (2) and consider $w = u + 1 \in \mathrm{H}^1(\Omega)$. The function w is different from the solution u but we have $\nabla w(x) = \nabla u(x)$, for almost every $x \in \Omega$. Therefore, for all $v \in \mathrm{H}^1(\Omega)$, we compute

$$\int_\Omega \nabla w(x) \cdot \nabla v(x) \,\mathrm{d}x = \int_\Omega \nabla u(x) \cdot \nabla v(x) \,\mathrm{d}x = \int_\Omega f(x)\, v(x) \,\mathrm{d}x$$

and so, if a solution exists of Eq. (2), it is not unique.

Question 3. Take $v \equiv 1 \in \mathrm{H}^1(\Omega)$ in Eq. (2) to get

$$0 = \int_\Omega f(x) \,\mathrm{d}x.$$

Therefore, for solution to exits we must have f with zero mean.

Question 4. The sum is direct because for $u \in \mathrm{H}^1(\Omega)$, we have

$$u = \underbrace{\frac{1}{|\Omega|} \int_\Omega u(x) \,\mathrm{d}x}_{\in \operatorname{span}\{x \mapsto 1\}} + \underbrace{u - \frac{1}{|\Omega|} \int_\Omega u(x) \,\mathrm{d}x}_{\in V}$$

and, if $u \in \operatorname{span}\{x \mapsto 1\} \cap V$, the function is constant and of zero mean so $u = 0$. The subspaces $\operatorname{span}\{x \mapsto 1\}$ and V are orthogonal, take $w \in V$, we compute

$$\langle x \mapsto 1, w \rangle_{1,2} = \int_\Omega w(x) \,\mathrm{d}x = 0.$$

Question 5. The bilinear form $\langle \cdot, \cdot \rangle_V$ is symmetric and non-negative by definition. For the definiteness part, we take $u \in V$ such that $\|u\|_V = \sqrt{\langle u, u \rangle_V} = 0$, this give $\nabla u = 0$ in Ω. So u is constant and the only constant in V is 0 therefore $u = 0$. The bilinear form $\langle \cdot, \cdot \rangle_V$ is an inner product.

Now let's show that $\|\cdot\|_V$ and $\|\cdot\|_{1,2}$ are equivalent. We directly have $\|u\|_V \leq \|u\|_{1,2}$ for all $u \in V$. For the converse inequality, using the Wirtinger inequality for $u \in V$, we get

$$\|u\|_2 = \left\| u - \frac{1}{|\Omega|} \int_\Omega u(x) \,\mathrm{d}x \right\|_2 \leq C \|\nabla u\|_2 = C \|u\|_V$$

and by definition of the norms $\langle \cdot, \cdot \rangle_{1,2}$ and we compute

$$\|u\|_{1,2}^2 = \|u\|_2^2 + \|\nabla u\|_2^2 \leq \left(C^2 + 1\right) \|\nabla u\|_2^2 = \left(C^2 + 1\right) \|u\|_V^2$$

so there exists $D > 1$ such that $\|u\|_V \leq \|u\|_{1,2} \leq D \|u\|_V$ for all $u \in V$.

Question 6. We define the associated problem to Eq. (2) on the space V by: find $u \in V$ such that

$$\int_\Omega \nabla u(x) \cdot \nabla w(x) \,\mathrm{d}x = \int_\Omega f(x)\, w(x) \,\mathrm{d}x, \qquad \forall w \in V. \tag{3}$$

We verify the hypothesis of the Lax-Milgram theorem on the Hilbert space V:

(i) we have $\left| \int_\Omega \nabla u(x) \cdot \nabla w(x) \,\mathrm{d}x \right| \leq \|u\|_V \|w\|_V$ for $u, w \in V$;

(ii) we have $\int_\Omega |\nabla u(x)|^2 \,\mathrm{d}x \geq \|u\|_V^2$ for $u \in V$;

(iii) we have $\left|\int_\Omega f(x)\, w(x)\, \mathrm{d}x\right| \leq \|f\|_2 \|w\|_2 \leq \|f\|_2 C \|w\|_V$ for $w \in V$.

Using the Lax-Milgram theorem on Eq. (3), there exists a unique $u \in V$ that satisfy Eq. (3) and $\|u\|_V \leq C \|f\|_2$. Now, we show that the solution u of Eq. (3) is also a solution of Eq. (2), for $v \in \mathrm{H}^1(\Omega)$ from the decomposition $\mathrm{H}^1(\Omega) = \mathrm{span}\{x \mapsto 1\} \oplus V$ there exists $(c, w) \in \mathrm{span}\{x \mapsto 1\} \times V$ such that $v = c + w$ and we compute

$$\begin{aligned}
\int_\Omega \nabla u(x) \cdot \nabla v(x)\, \mathrm{d}x &= \int_\Omega \nabla u(x) \cdot \nabla w(x)\, \mathrm{d}x && (\nabla c = 0)\\
&= \int_\Omega f(x)\, w(x)\, \mathrm{d}x && (u \text{ solution of Eq. (3)})\\
&= \int_\Omega f(x)\, v(x)\, \mathrm{d}x. && \left(\int_\Omega f(x)\, \mathrm{d}x = 0\right)
\end{aligned}$$

So, for $f \in \mathrm{L}^2_0(\Omega)$, u is the unique solution in V that satisfy Eq. (2).

Remark 1. Using Lax-Milgram theorem, Eq. (3) has an unique solution for $f \in \mathrm{L}^2(\Omega)$ contrary to Eq. (2) how has a solution if, and only if, $f \in \mathrm{L}^2(\Omega)$ with zero mean and the solution is not unique in $\mathrm{H}^1(\Omega)$. We say that Eq. (3) is well-posed and Eq. (2) is ill-posed in the Hadamard sense, see `en.wikipedia.org/wiki/Well-posed_problem`.

Exercise 2.
Let $I = (0, 2\pi)$. The optimal Wirtinger inequality in one dimension is

$$\left\| f - \frac{1}{2\pi} \int_0^{2\pi} f(x)\, \mathrm{d}x \right\|_2 \leq \|f'\|_2\,, \qquad \text{for } f \in \mathrm{H}^1(I). \tag{4}$$

1. Prove Eq. (4) for $f \in C^\infty([0, 2\pi])$ using Fourier series.
2. Prove Eq. (4) for $f \in \mathrm{H}^1(I)$.
3. Characterize the functions that satisfy the equality of Eq. (4).

Solution 2.
Question 1. Since $f \in C^\infty([0, 2\pi])$, the Fourier series of f and f' absolutely converge, and we have

$$f(x) = a_0 + \sum_{n=1}^{+\infty} a_n \cos(nx) + b_n \sin(nx) \quad \text{and} \quad f'(x) = \sum_{n=1}^{+\infty} -n\, a_n \sin(nx) + n\, b_n \cos(nx).$$

Using the orthogonal properties of the functions $x \mapsto \cos(nx)$ and $x \mapsto \sin(nx)$ on $[0, 2\pi]$, we compute

$$\begin{aligned}
\left\| f - \frac{1}{2\pi} \int_0^{2\pi} f(x)\, \mathrm{d}x \right\|_2^2 &= \left\| \sum_{n=1}^{+\infty} a_n \cos(n\,\cdot) + b_n \sin(n\,\cdot) \right\|_2^2\\
&= \sum_{n=1}^{+\infty} \int_0^{2\pi} |a_n|^2 \cos^2(nx) + |b_n|^2 \sin^2(nx)\, \mathrm{d}x\\
&= \pi \sum_{n=1}^{+\infty} |a_n|^2 + |b_n|^2
\end{aligned}$$

and

$$\begin{aligned}
\|f'\|_2^2 &= \left\| \sum_{n=1}^{+\infty} -n\, a_n \cos(n\,\cdot) + n\, b_n \sin(n\,\cdot) \right\|_2^2 \\
&= \sum_{n=1}^{+\infty} \int_0^{2\pi} n^2 \left|a_n\right|^2 \cos^2(nx) + n^2 \left|b_n\right|^2 \sin^2(nx)\,\mathrm{d}x \\
&= \pi \sum_{n=1}^{+\infty} n^2 \left|a_n\right|^2 + n^2 \left|b_n\right|^2
\end{aligned}$$

for $n \geq 1$, we have $\pi \sum_{n=1}^{+\infty} |a_n|^2 + |b_n|^2 \leq \pi \sum_{n=1}^{+\infty} n^2 |a_n|^2 + n^2 |b_n|^2$ which give

$$\left\| f - \frac{1}{2\pi} \int_0^{2\pi} f(x)\,\mathrm{d}x \right\|_2 \leq \|f'\|_2\,, \qquad \text{for } f \in C^\infty([0, 2\pi]).$$

Question 2. By density, for $f \in \mathrm{H}^1(I)$, there exists a sequence $(f_k)_{k\in\mathbb{N}} \in C^\infty([0, 2\pi])^{\mathbb{N}}$ such that $f_k \to f$ as $k \to +\infty$ in $\mathrm{H}^1(I)$. We directly have that $f_k \to f$ and $f_k' \to f'$ in $\mathrm{L}^2(I)$ as $k \to +\infty$. For the convergence of the mean, we compute

$$\left| \int_0^{2\pi} f_k(x)\,\mathrm{d}x - \int_0^{2\pi} f(x)\,\mathrm{d}x \right| \leq \int_0^{2\pi} |f_k - f|\,\mathrm{d}x \leq \sqrt{2\pi}\, \|f_k - f\|_2$$

so $f_k - \frac{1}{2\pi}\int_0^{2\pi} f_k(x)\,\mathrm{d}x \to f - \frac{1}{2\pi}\int_0^{2\pi} f(x)\,\mathrm{d}x$ as $k \to +\infty$ in $\mathrm{L}^2(I)$. Therefore, by passing to the limit in $\left\| f_k - \frac{1}{2\pi}\int_0^{2\pi} f_k(x)\,\mathrm{d}x \right\|_2 \leq \|f_k'\|_2$, we get

$$\left\| f - \frac{1}{2\pi} \int_0^{2\pi} f(x)\,\mathrm{d}x \right\|_2 \leq \|f'\|_2\,, \qquad \text{for } f \in \mathrm{H}^1(I).$$

Question 3. From *Question 1*, we see that the functions $x \mapsto 1$, $x \mapsto \cos(x)$, and $x \mapsto \sin(x)$ realize the equality in Eq. (4) and that it remains true for any linear combination of the three functions. We define the subspace $E \subset \mathrm{H}^1(I)$ by

$$E = \operatorname{span}\left(x \mapsto 1, x \mapsto \cos(x), x \mapsto \sin(x)\right)$$

and we want to show that $f \in \mathrm{H}^1(I)$ realize the equality of Eq. (4) if, and only if, $f \in E$. If $f \in E$ then $f \in \mathrm{H}^1(I)$ realize the equality of Eq. (4) is a direct computation. The reverse implication require a bit more works. First, we define the orthogonal complement $W \subset \mathrm{H}^1(I)$ such that $\mathrm{H}^1(I) = E \oplus_\perp W$. The set $W \cap C^\infty(I)$ is compose of function such that their Fourier series coefficient $a_0 = a_1 = b_1 = 0$. For such function, we can redo *Question 1* and found an improved Wirtinger inequality of the form

$$\|w\|_2 \leq \frac{1}{2} \|w'\|_2\,, \qquad \text{for } w \in W \cap C^\infty(I)$$

then, using *Question 2* on W, we get

$$\|w\|_2 \leq \frac{1}{2} \|w'\|_2\,, \qquad \text{for } w \in W.$$

Assume $f \in \mathrm{H}^1(I)$ realize the equality in Eq. (4). We can write $f = e + w$ with $(e, w) \in E \times W$ then we compute

$$\left\| e - \frac{1}{2\pi} \int_0^{2\pi} e(x)\,\mathrm{d}x \right\|_2^2 + \|w\|_2^2 = \left\| f - \frac{1}{2\pi} \int_0^{2\pi} f(x)\,\mathrm{d}x \right\|_2^2$$

$$
\begin{aligned}
&= \|f'\|_2^2 \\
&= \|e'\|_2^2 + \|w'\|_2^2 \\
&\geq \left\| e - \frac{1}{2\pi} \int_0^{2\pi} e(x)\,\mathrm{d}x \right\|_2^2 + 4\,\|w\|_2^2
\end{aligned}
$$

which give $\|w\|_2 \geq 2\,\|w\|_2$ so $w = 0$ and $f \in E$.