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Sobolev spaces: tutorials

Exercise sheet 07

Exercise 1.
Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Show that there is no trace operator such that $\mathrm{W}^{1,N}(\Omega) \to \mathrm{L}^\infty(\partial\Omega)$.

(Clue: *In the past, the answers shine*)

Exercise 2.
Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain, $a \in C(\overline{\Omega})$ a nontrivial function with $a \geq 0$, $b \in C(\partial\Omega)$ a nontrivial function with $b \geq 0$, and $1 \leq p < \infty$. Show that the following quantities are norms and equivalent to the $\|\cdot\|_{1,p}$ norm on $\mathrm{W}^{1,p}(\Omega)$:

1. $\|u\|_a = \left(\int_\Omega |\nabla u(x)|^p \,\mathrm{d}x + \int_\Omega a(x)\,|u(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}}$;
2. $\|u\|_b = \left(\int_\Omega |\nabla u(x)|^p \,\mathrm{d}x + \int_{\partial\Omega} b(x)\,|\gamma u(x)|^p \,\mathrm{d}\sigma(x)\right)^{\frac{1}{p}}$.

(Hint: Theorem 10.5)

Exercise 3.
Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain, $f \in \mathrm{L}^2(\Omega)$ a function, and $\alpha \in C(\partial\Omega)$ a nontrivial function with $\alpha(x) \geq 0$ for almost every $x \in \partial\Omega$. We denote $a : \mathrm{H}^1(\Omega) \times \mathrm{H}^1(\Omega) \to \mathbb{R}$ a bilinear form, and $\ell : \mathrm{L}^2(\Omega) \to \mathbb{R}$ a linear form define by

$$\begin{aligned} a(u, v) &:= \int_\Omega \nabla u(x) \cdot \nabla v(x) \,\mathrm{d}x + \int_{\partial\Omega} \alpha(x)\,\gamma u(x)\,\gamma v(x)\,\mathrm{d}\sigma(x), \\ \ell(v) &:= \int_\Omega f(x)\,v(x)\,\mathrm{d}x. \end{aligned}$$

1. Show that there exists $u \in \mathrm{H}^1(\Omega)$ such that $a(u, v) = \ell(v)$, for all $v \in \mathrm{H}^1(\Omega)$.
2. Assume that the solution u of *Question 1* is in $\mathrm{H}^2(\Omega)$. Find the PDE that u satisfies.