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July 23, 2021

Sobolev spaces: tutorials

Exercise sheet 07 with solution

Exercise 1.
Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Show that there is no trace operator such that $\mathrm{W}^{1,N}(\Omega) \to \mathrm{L}^\infty(\partial\Omega)$.

(Clue: *In the past, the answers shine*)

Solution 1.
We take the same function u as in *Exercise sheet 04, Exercise 1, Question 4*, to recall this function is $u : \mathbb{R}^N \to \mathbb{R}$ defined by

$$\phi : x \mapsto \left|\ln\left(|x|\right)\right|^\delta \chi(x),$$

for $0 < \delta < 1 - \frac{1}{N}$ and $\chi \in C_0^\infty\left(B_{\frac{1}{2}}(0)\right)$ such that $\chi \equiv 1$ on $B_{\frac{1}{4}}(0)$. This function has the property that $\phi \in \mathrm{W}^{1,N}(\mathbb{R}^N)$ and $\phi \notin \mathrm{L}^\infty(\mathbb{R}^N)$ because of the singularity at $x = 0$. We define the sequence of smooth function $\psi_n : \overline{\Omega} \to \mathbb{R}$ by

$$\psi_n(x) = \phi(x - x_n), \quad \forall x \in \Omega,$$

for a sequence $(x_n)_{n\in\mathbb{N}} \in \left(\mathbb{R}^N \setminus \overline{\Omega}\right)^{\mathbb{N}}$ such that $x_n \to x^* \in \partial\Omega$.

By contradiction, if we have a continuous trace operator $\gamma : \mathrm{W}^{1,N}(\Omega) \to \mathrm{L}^\infty(\partial\Omega)$ that mean there exists a contant $C > 0$ such that

$$\|\gamma u\|_{\mathrm{L}^\infty(\partial\Omega)} \leq C \|u\|_{\mathrm{W}^{1,N}(\Omega)}, \quad \forall u \in \mathrm{W}^{1,N}(\Omega),$$

and the trace correspond to the boundary restriction for function in $\mathrm{W}^{1,N}(\Omega) \cap C(\overline{\Omega})$. We compute

$$\max_{\partial\omega} |\psi_n| = \|\gamma\psi_n\|_{\mathrm{L}^\infty(\partial\Omega)} \leq C \|\psi_n\|_{\mathrm{W}^{1,N}(\Omega)} \leq C \|\phi\|_{\mathrm{W}^{1,N}(\mathbb{R}^N)}.$$

The right hand side is bounded but the left hand part is unbounded as $n \to +\infty$ which is a contraction, therefore, there is no trace operator such that $\mathrm{W}^{1,N}(\Omega) \to \mathrm{L}^\infty(\partial\Omega)$.

Exercise 2.
Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain, $a \in C(\overline{\Omega})$ a nontrivial function with $a \geq 0$, $b \in C(\partial\Omega)$ a nontrivial function with $b \geq 0$, and $1 \leq p < \infty$. Show that the following quantities are norms and equivalent to the $\|\cdot\|_{1,p}$ norm on $\mathrm{W}^{1,p}(\Omega)$:

1. $\|u\|_a = \left(\int_\Omega |\nabla u(x)|^p \,\mathrm{d}x + \int_\Omega a(x)\,|u(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}}$;
2. $\|u\|_b = \left(\int_\Omega |\nabla u(x)|^p \,\mathrm{d}x + \int_{\partial\Omega} b(x)\,|\gamma u(x)|^p \,\mathrm{d}\sigma(x)\right)^{\frac{1}{p}}$.

(Hint: Theorem 10.5)

Solution 2.

The two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ are non-negative, homogeneous of degree 1, and satisfy the triangular inequality via the Minkowski's inequality.

Question 1. For the definiteness: if $\|u\|_a = 0$ then $\|\nabla u\|_p = 0$ which implies that u is constant and $\int_\Omega a\,|u|^p \,\mathrm{d}x = 0$ implies that the constant is zero.

Now we show that the norm $\|\cdot\|_a$ is equivalent to $\|\cdot\|_{1,p}$. We directly have

$$\|u\|_a \leq \max\left(1, \max_{\overline{\Omega}} a^{\frac{1}{p}}\right) \|u\|_{1,p}\,.$$

We show the converse inequality $\|u\|_{1,p} \leq C_a \|u\|_a$ by contradiction. We assume that there exists a sequence $(u_n)_{n\in\mathbb{N}} \in \mathrm{W}^{1,p}(\Omega)^{\mathbb{N}}$ such that

$$\|u_n\|_{1,p} = 1 \quad \text{and} \quad \lim_{n\to+\infty} \|u_n\|_a = 0.$$

The sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{W}^{1,p}(\Omega)$ so by the Rellich-Kondrachov theorem, there exists a converging subsequence $u_{\sigma(n)} \to u$ in $\mathrm{L}^p(\Omega)$. Furthermore, $\lim_{n\to+\infty} \left\|u_{\sigma(n)}\right\|_a = 0$ implies that $\lim_{n\to+\infty} \left\|\nabla u_{\sigma(n)}\right\|_p = 0$ and, for $\phi \in C_0^\infty(\Omega)$ and $i \in \{1, \ldots, N\}$, we compute

$$0 = \lim_{n\to+\infty} \int_\Omega \partial_i u_{\sigma(n)}(x)\phi(x)\,\mathrm{d}x = -\lim_{n\to+\infty} \int_\Omega u_{\sigma(n)}(x)\partial_i\phi(x)\,\mathrm{d}x = -\int_\Omega u(x)\partial_i\phi(x)\,\mathrm{d}x$$

therefore $u \equiv c$ a constant. From $\lim_{n\to+\infty} \|u_n\|_a = 0$, we deduce

$$0 = \lim_{n\to+\infty} \int_\Omega a(x)\left|u_{\sigma(n)}\right|^p \mathrm{d}x = \int_\Omega a(x)\,|c|^p \,\mathrm{d}x$$

which give $u \equiv c = 0$ and, from $\|u_n\|_{1,p} = 1$, we have the contradiction

$$1 = \underbrace{\left\|\nabla u_{\sigma(n)}\right\|_p^p}_{\to 0} + \underbrace{\left\|u_{\sigma(n)}\right\|_p^p}_{\to 0}.$$

So by contradiction, there exists a constant $C_a > 0$ such that $\|u\|_{1,p} \leq C_a \|u\|_a$ for all $u \in \mathrm{W}^{1,p}(\Omega)$ and we have

$$C_a^{-1}\|u\|_{1,p} \leq \|u\|_a \leq \max\left(1, \max_{\overline{\Omega}} a\right)^{\frac{1}{p}} \|u\|_{1,p}\,.$$

Question 2. For the definiteness: if $\|u\|_b = 0$ then $\|\nabla u\|_p = 0$ which implies that u is constant and $\int_{\partial\Omega} b\,|\gamma u|^p \,\mathrm{d}\sigma(x) = 0$ implies that the constant is zero. Now we show that the norm $\|\cdot\|_b$ is equivalent to $\|\cdot\|_{1,p}$. We directly have

$$\|u\|_b \leq \left(1 + |||\gamma|||^p_{\mathrm{W}^{1,p}(\Omega)\to\mathrm{L}^p(\partial\Omega)} \max_{\partial\Omega} b\right)^{\frac{1}{p}} \|u\|_{1,p}\,.$$

We show the converse inequality $\|u\|_{1,p} \leq C_b \|u\|_b$ by contradiction. We assume that there exists a sequence $(u_n)_{n\in\mathbb{N}} \in \mathrm{W}^{1,p}(\Omega)^{\mathbb{N}}$ such that

$$\|u_n\|_{1,p} = 1 \quad \text{and} \quad \lim_{n\to+\infty} \|u_n\|_b = 0.$$

The sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{W}^{1,p}(\Omega)$ so by the Rellich-Kondrachov theorem, there exists a converging subsequence $u_{\sigma(n)} \to u$ in $\mathrm{L}^p(\Omega)$. Furthermore, $\lim_{n\to+\infty} \left\|u_{\sigma(n)}\right\|_b = 0$ implies that $\lim_{n\to+\infty} \left\|\nabla u_{\sigma(n)}\right\|_p = 0$ and, for $\phi \in C_0^\infty(\Omega)$ and $i \in \{1, \dots, N\}$, we compute

$$0 = \lim_{n\to+\infty} \int_\Omega \partial_i u_{\sigma(n)}(x)\phi(x)\,\mathrm{d}x = -\lim_{n\to+\infty} \int_\Omega u_{\sigma(n)}(x)\partial_i\phi(x)\,\mathrm{d}x = -\int_\Omega u(x)\partial_i\phi(x)\,\mathrm{d}x$$

therefore $u \equiv c$ a constant. From $\lim_{n\to+\infty} \|u_n\|_b = 0$, we deduce

$$0 = \lim_{n\to+\infty} \int_\Omega b(x) \left|\gamma u_{\sigma(n)}\right|^p \mathrm{d}\sigma(x) = \int_\Omega b(x)\,|c|^p\,\mathrm{d}x$$

which give $u \equiv c = 0$ and, from $\|u_n\|_{1,p} = 1$, we have a contradiction

$$1 = \underbrace{\left\|\nabla u_{\sigma(n)}\right\|_p^p}_{\to 0} + \underbrace{\left\|u_{\sigma(n)}\right\|_p^p}_{\to 0}.$$

So by contradiction, there exists a constant $C_b > 0$ such that $\|u\|_{1,p} \le C_b \|u\|_b$ for all $u \in \mathrm{W}^{1,p}(\Omega)$ and we have

$$C_b^{-1}\|u\|_{1,p} \le \|u\|_b \le \left(1 + \|\!|\gamma|\!\|^p_{\mathrm{W}^{1,p}(\Omega)\to\mathrm{L}^p(\partial\Omega)} \max_{\partial\Omega} b\right)^{\frac{1}{p}} \|u\|_{1,p}\,.$$

Exercise 3.
Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain, $f \in \mathrm{L}^2(\Omega)$ a function, and $\alpha \in C(\partial\Omega)$ a nontrivial function with $\alpha(x) \ge 0$ for almost every $x \in \partial\Omega$. We denote $a : \mathrm{H}^1(\Omega) \times \mathrm{H}^1(\Omega) \to \mathbb{R}$ a bilinear form, and $\ell : \mathrm{L}^2(\Omega) \to \mathbb{R}$ a linear form define by

$$\begin{aligned} a(u,v) &:= \int_\Omega \nabla u(x)\cdot\nabla v(x)\,\mathrm{d}x + \int_{\partial\Omega} \alpha(x)\,\gamma u(x)\,\gamma v(x)\,\mathrm{d}\sigma(x), \\ \ell(v) &:= \int_\Omega f(x)\,v(x)\,\mathrm{d}x. \end{aligned}$$

1. Show that there exists $u \in \mathrm{H}^1(\Omega)$ such that $a(u,v) = \ell(v)$, for all $v \in \mathrm{H}^1(\Omega)$.
2. Assume that the solution u of *Question 1* is in $\mathrm{H}^2(\Omega)$. Find the PDE that u satisfies.

Solution 3.
Question 1. From previous exercise a is coercive. From the trace theorem a and ℓ are continues. By the Lax-Milgram theorem, there exists a unique $u \in \mathrm{H}^1(\Omega)$ such that $a(u,v) = \ell(v)$, for all $v \in \mathrm{H}^1(\Omega)$.

Question 2. For $v \in \mathrm{H}^1(\Omega)$, we have $a(u,v) = \ell(v)$ which give

$$\int_\Omega \nabla u(x)\cdot\nabla v(x)\,\mathrm{d}x + \int_{\partial\Omega} \alpha(x)\,\gamma u(x)\,\gamma v(x)\,\mathrm{d}\sigma(x) = \int_\Omega f(x)\,v(x)\,\mathrm{d}x$$

we deduce the expression

$$\int_\Omega \left[-\Delta u(x) - f(x)\right]\,v(x)\,\mathrm{d}x + \int_{\partial\Omega} \left[\partial_n u(x) + \alpha(x)\gamma u(x)\right]\,\gamma v(x)\,\mathrm{d}\sigma(x) = 0 \qquad (1)$$

with ∂_n the exterior normal derivative. First take $v \in C_0^\infty(\Omega)$, Eq. (1) become

$$\int_\Omega [-\Delta u(x) - f(x)]\ v(x)\,\mathrm{d}x = 0$$

and $-\Delta u(x) - f(x) \in \mathrm{L}^2(\Omega)$, therefore, we obtain $-\Delta u(x) = f(x)$, for almost everywhere $x \in \Omega$. Now, taking $w \in C^\infty(\partial\Omega)$ and $v \in \mathrm{H}^1(\Omega)$ such that $w = \gamma v$, Eq. (1) reduce to

$$\int_{\partial\Omega} [\partial_n u(x) + \gamma u(x)]\ w(x)\,\mathrm{d}\sigma(x) = 0$$

and $\partial_n u(x) + \gamma u(x) \in \mathrm{L}^2(\partial\Omega)$ therefore $\partial_n u(x) + \gamma u(x) = 0$, for almost everywhere $x \in \partial\Omega$. So the PDE is

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \partial_n u + \alpha u = 0 & \text{in } \partial\Omega \end{cases}.$$

Remark 1. A boundary condition of the form $\beta\partial_n u + \alpha u = g$ is usually referred as a *Robin boundary condition* or *impedance boundary conditions*.