Sobolev spaces: tutorials

Exercise sheet 07 with solution

Exercise 1.
Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Show that there is no trace operator such that $W^{1,N}(\Omega) \rightarrow L^\infty(\partial \Omega)$.

(Clue: In the past, the answers shine)

Solution 1.
We take the same function $u$ as in Exercise sheet 04, Exercise 1, Question 4, to recall this function is $u : \mathbb{R}^N \rightarrow \mathbb{R}$ defined by

$$\phi : x \mapsto | \ln(|x|) |^\delta \chi(x),$$

for $0 < \delta < 1 - \frac{1}{N}$ and $\chi \in C_0^\infty \left( B_{\frac{1}{4}}(0) \right)$ such that $\chi \equiv 1$ on $B_{\frac{1}{4}}(0)$. This function has the property that $\phi \in W^{1,N}(\mathbb{R}^N)$ and $\phi \notin L^\infty(\mathbb{R}^N)$ because of the singularity at $x = 0$. We define the sequence of smooth function $\psi_n : \overline{\Omega} \rightarrow \mathbb{R}$ by

$$\psi_n(x) = \phi(x - x_n), \quad \forall x \in \Omega,$$

for a sequence $(x_n)_{n \in \mathbb{N}} \in (\mathbb{R}^N \setminus \overline{\Omega})^N$ such that $x_n \rightarrow x^* \in \partial \Omega$.

By contradiction, if we have a continuous trace operator $\gamma : W^{1,N}(\Omega) \rightarrow L^\infty(\partial \Omega)$ that mean there exists a contant $C > 0$ such that

$$\| \gamma u \|_{L^\infty(\partial \Omega)} \leq C \| u \|_{W^{1,N}(\Omega)}, \quad \forall u \in W^{1,N}(\Omega),$$

and the trace correspond to the boundary restriction for function in $W^{1,N}(\Omega) \cap C(\overline{\Omega})$. We compute

$$\max_{\partial \omega} | \psi_n | = \| \gamma \psi_n \|_{L^\infty(\partial \Omega)} \leq C \| \psi_n \|_{W^{1,N}(\Omega)} \leq C \| \phi \|_{W^{1,N}(\mathbb{R}^N)}.$$

The right hand side is bounded but the left hand part is unbounded as $n \rightarrow +\infty$ which is a contraction, therefore, there is no trace operator such that $W^{1,N}(\Omega) \rightarrow L^\infty(\partial \Omega)$.

Exercise 2.
Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain, $a \in C(\overline{\Omega})$ a nontrivial function with $a \geq 0$, $b \in C(\partial \Omega)$ a nontrivial function with $b \geq 0$, and $1 \leq p < \infty$. Show that the following quantities are norms and equivalent to the $\| \cdot \|_{1,p}$ norm on $W^{1,p}(\Omega)$:

1. $\| u \|_a = \left( \int_{\Omega} | \nabla u(x) |^p \, dx + \int_{\Omega} a(x) | u(x) |^p \, dx \right)^{\frac{1}{p}}$;

2. $\| u \|_b = \left( \int_{\Omega} | \nabla u(x) |^p \, dx + \int_{\partial \Omega} b(x) | \gamma u(x) |^p \, d\sigma(x) \right)^{\frac{1}{p}}$. 
Solution 2.
The two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ are non-negative, homogeneous of degree 1, and satisfy the triangular inequality via the Minkowski’s inequality.

Question 1. For the definiteness: if $\|u\|_a = 0$ then $\|\nabla u\|_p = 0$ which implies that $u$ is constant and $\int_{\Omega} a |u|^p \, dx = 0$ implies that the constant is zero.

Now we show that the norm $\|\cdot\|_a$ is equivalent to $\|\cdot\|_{1,p}$. We directly have

$$\|u\|_a \leq \max \left( 1, \max_{\Omega} \frac{1}{a} \right) \|u\|_{1,p}.$$ 

We show the converse inequality $\|u\|_{1,p} \leq C_a \|u\|_a$ by contradiction. We assume that there exists a sequence $(u_n)_{n \in \mathbb{N}} \in W^{1,p}(\Omega)^N$ such that

$$\|u_n\|_{1,p} = 1 \quad \text{and} \quad \lim_{n \to +\infty} \|u_n\|_a = 0.$$ 

The sequence $(u_n)_{n \in \mathbb{N}}$ is bounded in $W^{1,p}(\Omega)$ so by the Rellich-Kondrachov theorem, there exists a converging subsequence $u_{\sigma(n)} \to u$ in $L^p(\Omega)$. Furthermore, $\lim_{n \to +\infty} \|u_{\sigma(n)}\|_a = 0$ implies that $\lim_{n \to +\infty} \|\nabla u_{\sigma(n)}\|_p = 0$ and, for $\phi \in C_0^\infty(\Omega)$ and $i \in \{1, \ldots, N\}$, we compute

$$0 = \lim_{n \to +\infty} \int_{\Omega} \partial_i u_{\sigma(n)}(x) \phi(x) \, dx = - \lim_{n \to +\infty} \int_{\Omega} u_{\sigma(n)}(x) \partial_i \phi(x) \, dx = - \int_{\Omega} u(x) \partial_i \phi(x) \, dx$$

therefore $u \equiv c$ a constant. From $\lim_{n \to +\infty} \|u_n\|_a = 0$, we deduce

$$0 = \lim_{n \to +\infty} \int_{\Omega} a(x) |u_{\sigma(n)}|^p \, dx = \int_{\Omega} a(x) |c|^p \, dx$$

which give $u \equiv c = 0$ and, from $\|u_n\|_{1,p} = 1$, we have the contradiction

$$1 = \|\nabla u_{\sigma(n)}\|_p + \|u_{\sigma(n)}\|_p \to 0$$

So by contradiction, there exists a constant $C_a > 0$ such that $\|u\|_{1,p} \leq C_a \|u\|_a$ for all $u \in W^{1,p}(\Omega)$ and we have

$$C_a^{-1} \|u\|_{1,p} \leq \|u\|_a \leq \max \left( 1, \max_{\Omega} \frac{1}{a} \right) \|u\|_{1,p}.$$ 

Question 2. For the definiteness: if $\|u\|_b = 0$ then $\|\nabla u\|_p = 0$ which implies that $u$ is constant and $\int_{\partial \Omega} b |\gamma u|^p \sigma(x) \, dx = 0$ implies that the constant is zero. Now we show that the norm $\|\cdot\|_b$ is equivalent to $\|\cdot\|_{1,p}$. We directly have

$$\|u\|_b \leq \left( 1 + \|\gamma\|_{W^{1,p}(\Omega) \to L^p(\partial \Omega)} \max_{\partial \Omega} b \right)^{\frac{1}{p}} \|u\|_{1,p}.$$ 

We show the converse inequality $\|u\|_{1,p} \leq C_b \|u\|_b$ by contradiction. We assume that there exists a sequence $(u_n)_{n \in \mathbb{N}} \in W^{1,p}(\Omega)^N$ such that

$$\|u_n\|_{1,p} = 1 \quad \text{and} \quad \lim_{n \to +\infty} \|u_n\|_b = 0.$$
The sequence \((u_n)_{n\in\mathbb{N}}\) is bounded in \(W^{1,p}(\Omega)\) so by the Rellich-Kondrachov theorem, there exists a converging subsequence \(u_{\sigma(n)} \to u\) in \(L^p(\Omega)\). Furthermore, \(\lim_{n\to+\infty} \|u_{\sigma(n)}\|_b = 0\) implies that \(\lim_{n\to+\infty} \|\nabla u_{\sigma(n)}\|_p = 0\) and, for \(\phi \in C_0^\infty(\Omega)\) and \(i \in \{1, \ldots, N\}\), we compute

\[
0 = \lim_{n\to+\infty} \int_\Omega \partial_i u_{\sigma(n)}(x)\phi(x) \, dx = -\lim_{n\to+\infty} \int_\Omega u_{\sigma(n)}(x)\partial_i \phi(x) \, dx = -\int_\Omega u(x)\partial_i \phi(x) \, dx
\]

to therefore \(u \equiv c\) a constant. From \(\lim_{n\to+\infty} \|u_n\|_b = 0\), we deduce

\[
0 = \lim_{n\to+\infty} \int_\Omega b(x)|\gamma u_{\sigma(n)}|^p \, d\sigma(x) = \int_\Omega b(x)|c|^p \, dx
\]

which give \(u \equiv c = 0\) and, from \(\|u_n\|_{1,p} = 1\), we have a contradiction

\[
1 = \left( \|\nabla u_{\sigma(n)}\|_p + \|u_{\sigma(n)}\|_p \right) \to 0.
\]

So by contradiction, there exists a constant \(C_b > 0\) such that \(\|u\|_{1,p} \leq C_b \|u\|_b\) for all \(u \in W^{1,p}(\Omega)\) and we have

\[
C_b^{-1} \|u\|_{1,p} \leq \|u\|_b \leq \left(1 + \|\gamma\|_{W^{1,p}(\Omega)\to L^p(\partial\Omega)} \max_{\partial\Omega} b \right)^\frac{1}{p} \|u\|_{1,p}.
\]

**Exercise 3.**

Let \(\Omega \subset \mathbb{R}^N\) be a bounded Lipschitz domain, \(f \in L^2(\Omega)\) a function, and \(\alpha \in C(\partial\Omega)\) a nontrivial function with \(\alpha(x) \geq 0\) for almost every \(x \in \partial\Omega\). We denote \(a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}\) a bilinear form, and \(\ell : L^2(\Omega) \to \mathbb{R}\) a linear form define by

\[
a(u, v) := \int_\Omega \nabla u(x) \cdot \nabla v(x) \, dx + \int_{\partial\Omega} \alpha(x) \gamma u(x) \gamma v(x) \, d\sigma(x),
\]

\[
\ell(v) := \int_\Omega f(x) \, v(x) \, dx.
\]

1. Show that there exists \(u \in H^1(\Omega)\) such that \(a(u, v) = \ell(v)\), for all \(v \in H^1(\Omega)\).

2. Assume that the solution \(u\) of **Question 1** is in \(H^2(\Omega)\). Find the PDE that \(u\) satisfies.

**Solution 3.**

**Question 1.** From previous exercise \(a\) is coercive. From the trace theorem \(a\) and \(\ell\) are continues. By the Lax-Milgram theorem, there exists a unique \(u \in H^1(\Omega)\) such that \(a(u, v) = \ell(v)\), for all \(v \in H^1(\Omega)\).

**Question 2.** For \(v \in H^1(\Omega)\), we have \(a(u, v) = \ell(v)\) which give

\[
\int_\Omega \nabla u(x) \cdot \nabla v(x) \, dx + \int_{\partial\Omega} \alpha(x) \gamma u(x) \gamma v(x) \, d\sigma(x) = \int_\Omega f(x) v(x) \, dx
\]

we deduce the expression

\[
\int_\Omega [-\Delta u(x) - f(x)] v(x) \, dx + \int_{\partial\Omega} \left( \partial_n u(x) + \alpha(x) \gamma u(x) \right) \gamma v(x) \, d\sigma(x) = 0 \quad (1)
\]
with \( \partial_n \) the exterior normal derivative. First take \( v \in C^\infty_0(\Omega) \), Eq. (1) become

\[
\int_\Omega \left[ -\Delta u(x) - f(x) \right] v(x) \, dx = 0
\]

and \(-\Delta u(x) - f(x) \in L^2(\Omega)\), therefore, we obtain \(-\Delta u(x) = f(x)\), for almost everywhere \( x \in \Omega \). Now, taking \( w \in C^\infty(\partial\Omega) \) and \( v \in H^1(\Omega) \) such that \( w = \gamma v \), Eq. (1) reduce to

\[
\int_{\partial\Omega} \left[ \partial_n u(x) + \gamma u(x) \right] w(x) \, d\sigma(x) = 0
\]

and \( \partial_n u(x) + \gamma u(x) \in L^2(\partial\Omega) \) therefore \( \partial_n u(x) + \gamma u(x) = 0 \), for almost everywhere \( x \in \partial\Omega \). So the PDE is

\[
\begin{cases}
-\Delta u = f & \text{in } \Omega \\
\partial_n u + \alpha u = 0 & \text{in } \partial\Omega.
\end{cases}
\]

Remark 1. A boundary condition of the form \( \beta \partial_n u + \alpha u = g \) is usually referred as a Robin boundary condition or impedance boundary conditions.