Karlsruhe Institut für Technologie

Sobolev spaces

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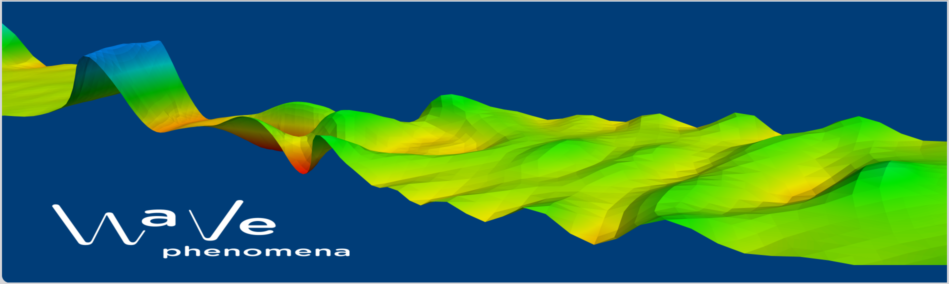
KIT, Institut for Analysis



Plan for today

- Sobolev's Embedding Theorem for $W^{1,p}(\mathbb{R}^N), 1 \leq p < N$
- Morrey's Embedding Theorem for $W^{1,p}(\mathbb{R}^N), p > N$
- Consequences for $W^{k,p}(\Omega)$
- General overview
- Outlook

Embedding Theorems

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Which properties do functions $u \in W^{1,p}(\mathbb{R}^N)$ have?

$$W^{1,p}(\mathbb{R}^N) \hookrightarrow X?? \qquad \|u\|_X \leq C\|u\|_{W^{1,p}(\mathbb{R}^N)} ??$$

- $1 \leq p < N$: $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ for $p \leq q \leq \frac{Np}{N-p}$ (Sobolev's Embedding Theorem)
- $p > N$: $W^{1,p}(\mathbb{R}^N) \hookrightarrow C^{0,\alpha}(\mathbb{R}^N)$ for $\alpha = 1 - \frac{N}{p}$ (Morrey's Embedding Theorem)

Sobolev's Embedding Theorem

Theorem (Gagliardo-Nirenberg-Sobolev)

Assume $N \in \mathbb{N}, N \geq 2$ *and* $1 \leq p < N$. *Then*

$$\|u\|_{L^{\frac{Np}{N-p}}(\mathbb{R}^N)} \leq C \|\nabla u\|_{L^p(\mathbb{R}^N)}.$$

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- W.l.o.g. $u \in C_0^\infty(\mathbb{R}^N)$ + "density argument"
- For $p = 1$ use (non-standard) induction based on

$$|u(x)| \leq \int_{\mathbb{R}} |\partial_j u(x_1, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_N)| \, dt.$$

- For $p > 1$ apply the $p = 1$-result to $v := |u|^\alpha u$

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Corollary (Sobolev's Embedding Theorem)

Assume $N \in \mathbb{N}, N \geq 2$ and $1 \leq p < N$ and $p^* := \frac{Np}{N-p}$. Then there is a continuous embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ precisely for $p \leq q \leq p^*$.

Morrey's Embedding Theorem

Recall: $C^{0,\alpha}(\overline{\Omega})$ = Banach space of α–Hölder-continuous functions

$$||u||_{C^{0,\alpha}(\overline{\Omega})} := \sup_{\Omega}|u| + \sup_{x,y\in\Omega, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}}$$

$W^{1,p}(\mathbb{R}^N) \hookrightarrow C^{0,\alpha}(\mathbb{R}^N)$ means:

There is $\tilde{u} \in C^{0,\alpha}(\overline{\Omega})$ such that $\tilde{u} = u$ almost everywhere.

Morrey's Embedding Theorem

Theorem (Morrey)

Let $N < p < \infty$, $u \in W^{1,p}(\mathbb{R}^N)$ and $\alpha := 1 - \frac{N}{p} \in (0,1)$. Then a.e.

$$|u(x)| \leq C\|u\|_{W^{1,p}(\mathbb{R}^N)}, \qquad \frac{|u(x) - u(y)|}{|x-y|^\alpha} \leq C\|\nabla u\|_{L^p(\mathbb{R}^N)}.$$

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- W.l.o.g. $u \in C_0^\infty(\mathbb{R}^N)$ + "density argument"
- Use for $m = (x + y)/2$ the MVT in integral form and

$$\omega_N \rho^N |u(x) - u(y)| \leq \int_{B_\rho(m)} |u(x) - u(z)| + |u(y) - u(z)|\, dz$$

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Corollary

Let $N \in \mathbb{N}$ and $N < p < \infty$. Then there is a continuous embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow C^{0,\alpha}(\mathbb{R}^N)$ where $\alpha = 1 - \frac{N}{p}$.

Embeddings: A summary

First order Sobolev spaces:

(i) If $1 < p < N$: $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ for $p \leq q \leq \frac{Np}{N-p}$

(ii) If $p = N$: $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ for $p \leq q < \infty$

(iii) If $p > N$: $W^{1,p}(\mathbb{R}^N) \hookrightarrow C^{0,\alpha}(\mathbb{R}^N)$ for $\alpha = 1 - \frac{N}{p}$

Regularity gain from “weak differentiability”

What about $W^{k,p}(\mathbb{R}^N)$ with $k \geq 2$?

Embeddings: A summary

Assume $u \in W^{2,p}(\mathbb{R}^N)$ where $1 \le p < \frac{N}{2}$.

- $\partial_i u \in W^{1,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ with $p \le q \le \frac{Np}{N-p}$
- $u \in W^{1,q}(\mathbb{R}^N)$ with $p \le q \le \frac{Np}{N-p} < N$
- $u \in L^r(\mathbb{R}^N)$ with $q \le r \le \frac{Nq}{N-q}$
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Consequence: For $1 \leq p < \frac{N}{2}$

$$W^{2,p}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N) \qquad \text{for } p \leq r \leq \frac{Np}{N-2p}$$

Embeddings: A summary

Assume $u \in W^{2,p}(\mathbb{R}^N)$ where $\frac{N}{2} < p < N$.

- $u \in W^{1,q}(\mathbb{R}^N)$ with $p \leq q \leq \frac{Np}{N-p}$ and $\frac{Np}{N-p} > N$
- $u \in C^{0,\alpha}(\mathbb{R}^N)$ with $\alpha = 1 - \frac{N}{q}$
- $u \in C^{0,\alpha}(\mathbb{R}^N)$ with $\alpha = 2 - \frac{N}{p}$ choosing $q = \frac{Np}{N-p}$

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Consequence: For $\frac{N}{2} < p < N$

$$W^{2,p}(\mathbb{R}^N) \hookrightarrow C^{0,\alpha}(\mathbb{R}^N) \qquad \text{for } \alpha = 2 - \frac{N}{p}$$

Embeddings: A summary

Assume $u \in W^{2,p}(\mathbb{R}^N)$ where $p > N$.

- $\partial_1 u, \ldots, \partial_N u \in W^{1,p}(\mathbb{R}^N)$
- $\partial_1 u, \ldots, \partial_N u \in C^{0,\alpha}(\mathbb{R}^N)$ with $\alpha = 1 - \frac{N}{p}$
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Consequence: For $p > N$

$$W^{2,p}(\mathbb{R}^N) \hookrightarrow C^{1,\alpha}(\mathbb{R}^N) \qquad \text{for } \alpha = 1 - \frac{N}{p}$$

Embeddings: A summary

In this way one obtains for general $k \in \mathbb{N}$:

(A) If $1 \leq p < \frac{N}{k}$: $W^{k,p}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ for $p \leq r \leq \frac{Np}{N-kp}$.

(B) If $p = \frac{N}{k}$: $W^{k,p}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ for $p \leq r < \infty$.

(C) If $p > \frac{N}{k}$, $\frac{N}{p} \notin \mathbb{N}$: $W^{k,p}(\mathbb{R}^N) \hookrightarrow C^{l,\alpha}(\mathbb{R}^N)$ where $l = k - \lfloor \frac{N}{p} \rfloor - 1, \alpha := 1 + \lfloor \frac{N}{p} \rfloor - \frac{N}{p}$

Further topics

General recap of the Lecture

- Weak solutions vs. classical solutions? Physical interpretation
- Motivation of “our BVP”
- Extension operator and what it does
- …

Plan for the next lectures

- (2L) Compact Embeddings: The Rellich-Kondrachov Theorem + Applications
- (2L) Poincaré's Inequality + Applications
- (2L) Trace Theorem + Applications
- ...

Options:

- $W^{1,\infty}(\Omega) = Lip(\Omega)$ (Rademacher's Theorem)
- Fourier characterization of $H^k(\mathbb{R}^N)$
- Reflexivity and separability of Sobolev spaces
- $H(curl), H(div)$ etc