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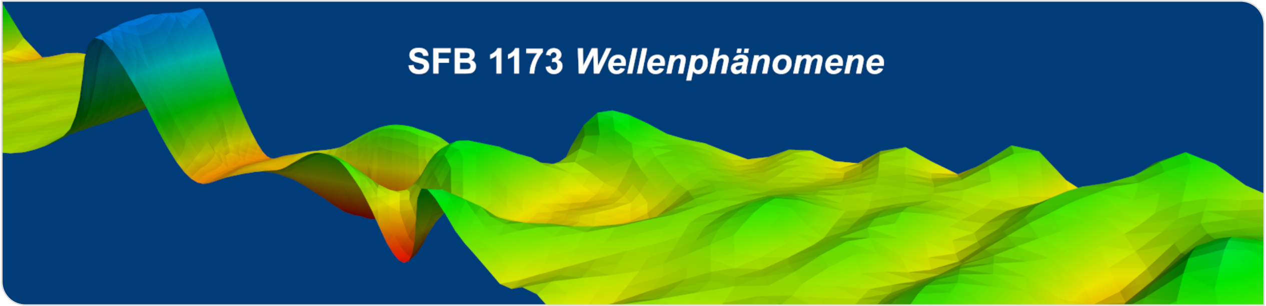
Sobolev spaces

Rainer Mandel, Zois Moitier | Karlsruhe, 01.07.2021



Recap

- Poincaré's Inequality for $W_0^{1,p}(\Omega)$
  Let $\Omega \subset \mathbb{R}^N$ be bounded in one direction. For $1 \leq p < \infty$ and $u \in W_0^{1,p}(\Omega)$ we have

$$||u||_p \leq C||\nabla u||_p.$$

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Today

- Trace operator
- Integration-by-parts formula

Trace Theorem

- Trace of u on $A \subset \overline{\Omega}$ gives a meaning to $u|_A$, in particular for $A = \partial\Omega$
- Definition is clear for continuous u, hence for $u \in W^{1,p}(\Omega)$, $p > N$
- Problem: What about $p < N$?

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Definition

*$\Omega \subset \mathbb{R}^N$ a bounded domain, $1 \leq p \leq N$.
A bounded linear operator $\gamma : W^{1,p}(\Omega) \to L^q(\partial\Omega)$ is called trace operator if $\gamma u = u|_{\partial\Omega}$ for all $u \in C(\overline{\Omega}) \cap W^{1,p}(\Omega)$.*

Aim: Lipschitz domains admit such trace operators.

Trace Theorem

Technical warm-up:

Proposition

$\Omega \subset \mathbb{R}^N$ a bounded Lipschitz domain, unit outer normal vector field $\nu : \partial\Omega \to \mathbb{R}^N$, surface measure σ. Then there is a smooth vector field $F \in C^\infty(\overline{\Omega}; \mathbb{R}^N)$ such that $F(x) \cdot \nu(x) \geq 1$ for σ-almost all $x \in \partial\Omega$.

Idea: Define F locally $\approx \nu : \partial\Omega \to \mathbb{R}^N$, but on the whole of Ω !

Theorem

$\Omega \subset \mathbb{R}^N$ a bounded Lipschitz domain, $1 \leq p < N$.
Then there is a trace operator $\gamma : W^{1,p}(\Omega) \to L^q(\partial\Omega)$ for $1 \leq q \leq \frac{(N-1)p}{N-p}$.
It is compact provided that $1 \leq q < \frac{(N-1)p}{N-p}$.

Proof:

(1) Prove the estimate for $u \in C_0^\infty(\mathbb{R}^N)$, only for $q > 1$.

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Proof:

(1) Prove the estimate for $u \in C_0^\infty(\mathbb{R}^N)$, only for $q > 1$. In fact

$$
\begin{aligned}
\|\gamma u\|^q_{L^q(\partial\Omega)} &= \int_{\partial\Omega} |u|^q d\sigma \;\le\; \int_{\partial\Omega} |u|^q F \cdot \nu d\sigma \;=\; \int_\Omega \operatorname{div}(|u|^q F)\, dx \\
&= \int_\Omega \Big(q|u|^{q-2} u\langle \nabla u, F\rangle + |u|^q \operatorname{div}(F)\Big)\, dx \\
&\le C\|F\|_{C^1(\overline{\Omega};\mathbb{R}^N)} \int_\Omega (q|u|^{q-1}|\nabla u| + |u|^q)\, dx \\
&\le C\|F\|_{C^1(\overline{\Omega};\mathbb{R}^N)} \left(q\|u\|^{q-1}_{\frac{(q-1)p}{p-1}} \|\nabla u\|_p + \|u\|_q^q \right) \\
&\le C'\|F\|_{C^1(\overline{\Omega};\mathbb{R}^N)} \|u\|^q_{W^{1,p}(\Omega)}.
\end{aligned}
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Proof:

(2) Define the trace operator $\gamma : W^{1,p}(\Omega) \to L^q(\Omega)$ by density:

$$\gamma(u) := \lim_{\|\tilde{u}-u\|_{W^{1,p}(\Omega)} \to 0, \tilde{u} \in C_0^\infty(\mathbb{R}^N)} \tilde{u}|_{\partial\Omega}.$$

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(3) Compactness. Use $\|\gamma u\|^q_{L^q(\partial\Omega)} \le \|F\|_{C^1(\overline{\Omega};\mathbb{R}^N)} \left(q\|u\|^{q-1}_{\frac{(q-1)p}{p-1}} \|\nabla u\|_p + \|u\|^q_q \right)$.

$(u_n)_{n\in\mathbb{N}}$ is bounded in $W^{1,p}(\Omega)$

$\Rightarrow \quad (u_n)_{n\in\mathbb{N}}$ subconverges in $L^{\frac{(q-1)p}{p-1}}(\Omega)$ and in $L^q(\Omega)$. (because $\max\{q, \frac{(q-1)p}{p-1}\} < \frac{Np}{N-p}$)

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$\Rightarrow$ $(\gamma(u_n))$ is a Cauchy sequence because

$$\|\gamma(u_n) - \gamma(u_m)\|_{L^q(\partial\Omega)} \le \|F\|_{C^1(\overline{\Omega};\mathbb{R}^N)} \left(q\|u_n - u_m\|^{q-1}_{\frac{(q-1)p}{p-1}} \|\nabla(u_n - u_m)\|_p + \|u_n - u_m\|_q^q \right). \quad \square$$

Integration by parts

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Proposition (Integration by parts)

Let $\Omega \subset \mathbb{R}^N$ *be a bounded Lipschitz domain and* $u \in W^{1,p}(\Omega), v \in W^{1,q}(\Omega)$ *for* $1 \leq p, q < N$ *such that* $\frac{1}{p} + \frac{1}{q} \leq \frac{N+1}{N}$*. Then, for all* $j = 1, \dots, N$*,*

$$\int_\Omega \partial_j u v \, dx = \int_{\partial\Omega} \gamma(u)\gamma(v)\nu_i \, d\sigma - \int_\Omega u \partial_j v \, dx$$

Integration by parts

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Proposition (Integration by parts)

Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain and $u \in W^{1,p}(\Omega)$, $v \in W^{1,q}(\Omega)$ for $1 \leq p, q < N$ such that $\frac{1}{p} + \frac{1}{q} \leq \frac{N+1}{N}$. Then, for all $j = 1, \ldots, N$,

$$\int_\Omega \partial_j uv \, dx = \int_{\partial\Omega} \gamma(u)\gamma(v)\nu_i \, d\sigma - \int_\Omega u\partial_j v \, dx$$

Why $\frac{1}{p} + \frac{1}{q} \leq \frac{N+1}{N}$??

By Hölder's Inequality:

- $\partial_j uv \in L^1(\Omega)$ if $\frac{Nq}{N-q} \geq p'$
- $u\partial_j v \in L^1(\Omega)$ if $\frac{Np}{N-p} \geq q'$
- $\gamma(u)\gamma(v) \in L^1(\partial\Omega)$ if $\frac{N(p-1)}{N-p} \geq (\frac{N(q-1)}{N-q})'$

Integration by parts

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Proof:

- Choose $u_n, v_n \in C_0^\infty(\mathbb{R}^N)$ s.t. $u_n \to u$ in $W^{1,p}(\Omega)$ and $v_n \to v$ in $W^{1,q}(\Omega)$.

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Proof:

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- Classical integration-by-parts rule:

$$\begin{aligned}\int_\Omega \partial_j u_n v_n \, dx &= \int_{\partial\Omega} u_n v_n \nu_j \, d\sigma - \int_\Omega u_n \partial_j v_n \, dx \\ &= \int_{\partial\Omega} \gamma(u_n)\gamma(v_n)\nu_j \, d\sigma - \int_\Omega u_n \partial_j v_n \, dx\end{aligned}$$

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- $$\gamma(u_n) \to \gamma(u) \text{ in } L^{\frac{(N-1)p}{N-p}}(\partial\Omega), \qquad \gamma(v_n) \to \gamma(v) \text{ in } L^{\frac{(N-1)q}{N-q}}(\partial\Omega).$$

  and

$$\partial_j u_n \to \partial_j u \text{ in } L^p(\Omega), \qquad u_n \to u \text{ in } L^{\frac{Np}{N-p}}(\Omega),$$

$$\partial_j v_n \to \partial_j v \text{ in } L^q(\Omega), \qquad v_n \to v \text{ in } L^{\frac{Nq}{N-q}}(\Omega)$$

Integration by parts

Proof:
Choose $r \in [1, \infty]$ via $\frac{1}{r} := \frac{N+1}{N} - \frac{1}{p} - \frac{1}{q}$. Then

$$\begin{aligned}\left|\int_\Omega \partial_j u_n v_n \, dx - \int_\Omega \partial_j u v \, dx\right| &\leq \int_\Omega |\partial_j u_n - \partial_j u||v_n| + |\partial_j u||v_n - v| \, dx \\ &\leq \|\partial_j u_n - \partial_j u\|_p \|v_n\|_{\frac{Nq}{N-q}} \|1\|_r + \|\partial_j u\|_p \|v_n - v\|_{\frac{Nq}{N-q}} \|1\|_r \\ &\to 0 \qquad (n \to \infty).\end{aligned}$$

Similarly,

$$\left|\int_\Omega u_n \partial_j v_n \, dx - \int_\Omega u \partial_j v \, dx\right| \to 0 \qquad (n \to \infty).$$

Integration by parts

Proof:
Moreover,

$$
\begin{aligned}
&\left|\int_{\partial\Omega} \gamma(u_n)\gamma(v_n)\nu_j \, d\sigma - \int_{\partial\Omega} \gamma(u)\gamma(v)\nu_j \, d\sigma\right| \\
&\leq \int_{\partial\Omega} |\gamma(u_n - u)||\gamma(v_n)| + |\gamma(u)||\gamma(v_n - v)| \, d\sigma \\
&\leq \|\gamma(u_n - u)\|_{L^{\frac{(N-1)p}{N-p}}(\partial\Omega)} \|\gamma(v_n)\|_{L^{\frac{(N-1)q}{N-q}}(\partial\Omega)} + \|\gamma(u)\|_{L^{\frac{(N-1)p}{N-p}}(\partial\Omega)} \|\gamma(v_n - v)\|_{L^{\frac{(N-1)p}{N-p}}(\partial\Omega)} \\
&\leq C\big(\|u_n - u\|_{W^{1,p}(\Omega)}\|v_n\|_{W^{1,q}(\Omega)} + \|v_n - v\|_{W^{1,q}(\Omega)}\|u\|_{W^{1,p}(\Omega)}\big) \\
&\to 0 \qquad (n \to \infty).
\end{aligned}
$$

Integration by parts

Proof:
Conclusion:

$$\begin{aligned}\int_\Omega \partial_j u v \, dx &= \lim_{n\to\infty} \int_\Omega \partial_j u_n v_n \, dx \\ &= \lim_{n\to\infty} \int_{\partial\Omega} \gamma(u_n)\gamma(v_n)\nu_j \, d\sigma - \lim_{n\to\infty} \int_\Omega u_n \partial_j v_n \, dx \\ &= \int_{\partial\Omega} \gamma(u)\gamma(v)\nu_j \, d\sigma - \int_\Omega u \partial_j v \, dx \quad \square\end{aligned}$$

Plan for next lecture

- For $u \in W^{1,p}(\Omega)$: $\quad \gamma u = 0$ iff $u \in W_0^{1,p}(\Omega)$
- Applications to BVP

End of 15th Lecture