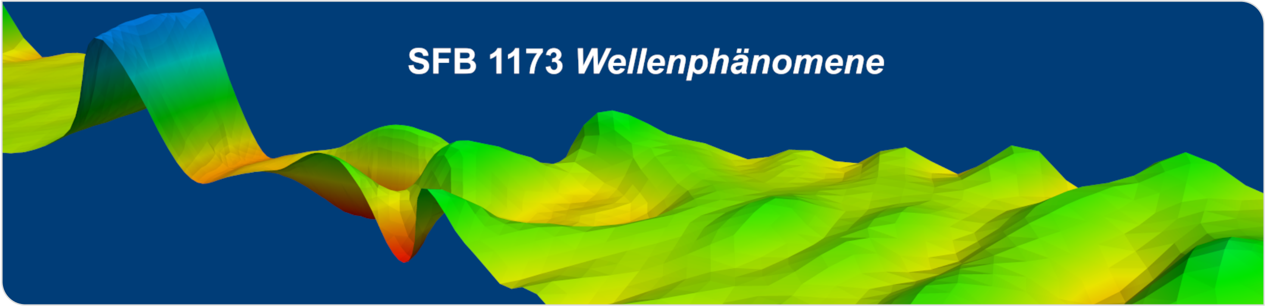
# Sobolev spaces

Rainer Mandel, Zois Moitier | Karlsruhe, 06.07.2021



# Recap

- **Trace operator**
  A bounded linear operator $\gamma : W^{1,p}(\Omega) \to L^q(\partial\Omega)$ is called trace operator if $\gamma u = u|_{\partial\Omega}$ for all $u \in C(\overline{\Omega}) \cap W^{1,p}(\Omega)$.

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- **Trace Theorem**
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  Then there is a trace operator $\gamma : W^{1,p}(\Omega) \to L^q(\partial\Omega)$ for $1 \le q \le \frac{(N-1)p}{N-p}$.
  It is compact provided that $1 \le q < \frac{(N-1)p}{N-p}$.

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# Today

- $u \in W_0^{1,p}(\Omega)$ iff $\gamma u = 0$
- Application to boundary value problems

## Proposition

*$\Omega \subset \mathbb{R}^N$ a bounded Lipschitz domain, $1 \leq p < \infty$. Then $\gamma(C^\infty(\overline{\Omega}))$ is dense in $L^p(\partial\Omega)$.*

**Proof:** Let $v \in L^p(\partial\Omega)$ and $\varepsilon > 0$.

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- Tietze's Extension Theorem: There is $W \in C(\mathbb{R}^N)$ such that $W|_{\partial\Omega} = w$. (W.l.o.g. $\mathrm{supp}(W)$ compact)

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$$
\begin{aligned}
\|\gamma(V_\varepsilon) - v\|_{L^p(\partial\Omega)} &\le \|V_\varepsilon - W\|_{L^p(\partial\Omega)} + \|w - v\|_{L^p(\partial\Omega)} \\
&\le \sup_{x\in\partial\Omega} |(\rho_\varepsilon * W)(x) - W(x)| \cdot |\partial\Omega|^{\frac{1}{p}} + \varepsilon \\
&\le \sup_{x\in\partial\Omega} \int_{\mathbb{R}^N} \rho_\varepsilon(x-y)|W(y) - W(x)|\,dy \cdot |\partial\Omega|^{\frac{1}{p}} + \varepsilon \\
&\le \sup_{x\in\partial\Omega, |y-x|\le\varepsilon} |W(y) - W(x)| \cdot |\partial\Omega|^{\frac{1}{p}} + \varepsilon \\
&\to 0 \qquad \text{as } \varepsilon \to 0.
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## Proposition

*Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain with unit outer normal vector field $\nu : \partial\Omega \to \mathbb{R}^N$ and surface measure $\sigma$. Then there is a smooth vector field $F \in C^\infty(\overline{\Omega}; R^N)$ such that $F(x) \cdot \nu(x) \geq 1$ for $\sigma$-almost all $x \in \partial\Omega$ and there is $t^* > 0$ such that $x + tF(x) \in \overline{\Omega}^c$ for $0 < t < t^*$, $x + tF(x) \in \Omega$ for $-t^* < t < 0$ for all $x \in \partial\Omega$.*

## Lemma

*$\Omega \subset \mathbb{R}^N$ a bounded Lipschitz domain, $1 \leq p < \infty$, $u \in W^{1,p}(\Omega)$. Equivalent are:*

(i) $u \in W_0^{1,p}(\Omega)$.

(ii) $u \in \ker(\gamma)$, *i.e.*, $\gamma u = 0$.

(iii) *The trivial extension $U$ of $u$ belongs to $W^{1,p}(\mathbb{R}^N)$ with $\partial_j U = \partial_j u \cdot \mathbb{1}_\Omega$.*

(iv) *There is $C > 0$ such that $|\int_\Omega u \partial_i \phi \, dx| \leq C \|\phi\|_{L^{p'}(\Omega)}$ for alle $\phi \in C_0^\infty(\mathbb{R}^N)$.*

(i)→ (ii) Choose $(u_n) \subset C_0^\infty(\Omega)$ s.t. $u_n \to u$ in $W^{1,p}(\Omega)$. Then $\gamma u = 0$ because

$$\|\gamma u\|_{L^p(\partial\Omega)} = \|\gamma(u - u_n)\|_{L^p(\partial\Omega)} \leq C\|u - u_n\|_{W^{1,p}(\Omega)} \to 0.$$

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(ii)→ (iii) For all $\phi \in C_0^\infty(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} U\partial_j\phi\, dx = \int_\Omega u\partial_j\phi\, dx = \int_{\partial\Omega} \underbrace{\gamma(u)}_{=0} \phi\nu_j\, d\sigma - \int_\Omega (\partial_j u)\phi\, dx = -\int_{\mathbb{R}^N} (\partial_j u \cdot \mathbb{1}_\Omega)\phi\, dx$$

This shows that $U$ has a $j$-th weak derivative on $\mathbb{R}^N$ given by $\partial_j U = \partial_j u \cdot \mathbb{1}_\Omega$. In particular, $U \in W^{1,p}(\mathbb{R}^N)$.

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- Shrinking $\delta_\varepsilon$ further if necessary: $\quad \|u_\varepsilon - U_\varepsilon \cdot \mathbb{1}_\Omega\|_{W^{1,p}(\mathbb{R}^N)} \le \varepsilon$
- Conclusion:

$$\|u_\varepsilon \cdot \mathbb{1}_\Omega - u\|_{W^{1,p}(\Omega)} \le \|u_\varepsilon - U\|_{W^{1,p}(\Omega)} \le \|u_\varepsilon - U_\varepsilon\|_{W^{1,p}(\Omega)} + \|U_\varepsilon - U\|_{W^{1,p}(\Omega)} \to 0 \quad \text{as } \varepsilon \to 0.$$

(iii) $\to$ (iv)

For all $\phi \in C_0^\infty(\mathbb{R}^N)$

$$\begin{aligned} \left|\int_\Omega u\partial_j\phi\,dx\right| &= \left|\int_{\mathbb{R}^N} U\partial_j\phi\,dx\right| \\ &= \left|\int_{\mathbb{R}^N} (\partial_j U)\phi\,dx\right| \\ &= \left|\int_{\mathbb{R}^N} (\partial_j u \mathbb{1}_\Omega)\phi\,dx\right| \\ &\leq \int_\Omega |\partial_j u||\phi|\,dx \\ &\leq \|\partial_j u\|_{L^p(\Omega)}\|\phi\|_{L^{p'}(\Omega)}. \end{aligned}$$

(iv) → (ii)

For all $\phi \in C_0^\infty(\mathbb{R}^N) \subset C^\infty(\overline{\Omega})$

$$C\|\phi\|_{L^{p'}(\Omega)} \geq \left|\int_\Omega u\partial_j\phi \, dx\right| = \left|\int_{\partial\Omega} \gamma(u)\phi\nu_j \, d\sigma - \int_\Omega \partial_j u\phi \, dx\right|$$

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Replacing $\phi$ by $\phi \cdot \chi_j$ with $\chi_j \searrow \mathbb{1}_{\partial\Omega}$ pointwise:

$$0 \geq \left| \int_{\partial\Omega} \gamma(u) \phi \nu_j \, d\sigma \right|.$$

Hence, $\gamma(u) = 0$.

# Application I: Refined Poincaré Inequality

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## Corollary

*$\Omega \subset \mathbb{R}^N$ a bounded Lipschitz domain, $\Gamma \subset \partial\Omega$ with positive surface measure, $p \in (1, \infty)$. Then*

$$\left\| u - \frac{1}{|\Gamma|} \int_\Gamma \gamma(u)\, d\sigma \right\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)} \qquad \text{for all } u \in W^{1,p}(\Omega)$$

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**Proof:** Apply the “Abstract Poincaré Theorem” to

$$V = \left\{ v \in W^{1,p}(\Omega) : \int_\Gamma \gamma(v)\, d\sigma = 0 \right\}.$$

($V$ is closed and the only constant function in $V$ is the trivial one.) □

# Application II: Boundary Value Problems

Karlsruher Institut für Technologie

**Question:** How to deal with nontrivial boundary conditions?

Define

$$a(u,v) := \int_\Omega \nabla u(x) \cdot \nabla v(x) + c(x)u(x)v(x)\,dx + \int_{\partial\Omega} (\gamma u)(x)(\gamma v)(x)\,d\sigma(x),$$
$$l(v) := \int_\Omega f(x)v(x)\,dx + \int_{\partial\Omega} \kappa(x)(\gamma v)(x)\,d\sigma(x).$$

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Lax-Milgram applies on $H^1(\Omega)$ under "reasonable conditions on $c, \kappa, f$".

**Consequence:** Existence and uniqueness of weak solutions to

$$-\Delta u + c(x)u = f(x) \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = \kappa \quad \text{on } \partial\Omega.$$

End of 16th Lecture