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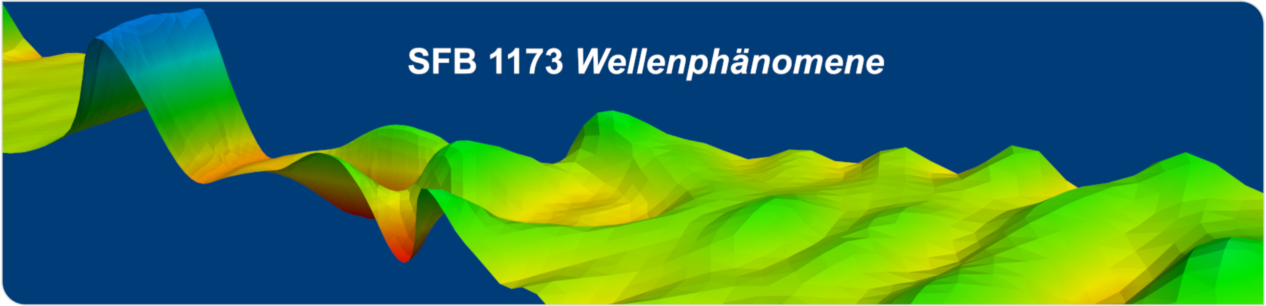
Sobolev spaces

Rainer Mandel, Zois Moitier | Karlsruhe, 13.07.2021



Today

- $W^{k,p}(\Omega)$ (and subspaces) are separable for $1 \leq p < \infty$
- $W^{k,\infty}(\Omega)$ is not separable

Definition

A Banach space X is called separable if it has a countable dense subset.

Idea: Take polynomials with rational coefficients.

Nonseparability criterion

Proposition

Let X be a Banach space with an uncountable set of open, non-empty and pairwise disjoint sets. Then X is not separable.

Proof:

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- Denote these sets by $(U_i)_{i\in I}$
- Assume that $M := \{x_n : n \in \mathbb{N}\} \subset X$ is dense.

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- $U_i \cap U_j = \emptyset$ for $i \neq j$, so $n(i) \neq n(j)$ for $i \neq j$.

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- $U_i \neq \emptyset$ open, M dense $\Rightarrow \; \exists n = n(i) \in \mathbb{N}$ such that $x_{n(i)} \in U_i$.
- $U_i \cap U_j = \emptyset$ for $i \neq j$, so $n(i) \neq n(j)$ for $i \neq j$.
- Conclusion: $n : I \to \mathbb{N}$ ist injective, i.e., I countable. $\lightning$ $\square$

Separability of $L^p(\Omega)$

Theorem

Let $\Omega \subset \mathbb{R}^N$ be open and non-empty.

(i) *$L^p(\Omega)$ is separable provided that $1 \leq p < \infty$.*

(ii) *$L^\infty(\Omega)$ is not separable.*

Proof:

- For $x \in \Omega$ choose $r_x > 0$ such that $B_{r_x}(x) \subset \Omega$. Set $U_x := \left\{ f \in L^\infty(\Omega) : \|f - \mathbb{1}_{B_{r_x}(x)}\|_\infty < \frac{1}{2} \right\}$.

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- Uncountable, pairwise disjoint, open and non-empty! Note:

$$f \in U_x \cap U_y \quad \Rightarrow \quad \|\mathbb{1}_{B_{r_x}(x)} - \mathbb{1}_{B_{r_y}(y)}\|_\infty < 1 \quad \Rightarrow \quad B_{r_x}(x) = B_{r_y}(y) \quad \Rightarrow \quad x = y.$$

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- So $L^\infty(\Omega)$ is not separable $\rightsquigarrow$ (ii).

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- Approximation by $C_0^\infty(\Omega)$ $\checkmark$

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Proof:

- Approximation by $C_0^\infty(\Omega)$ ✓
- Define the countable set ($\mathbb{1}_{\Omega \cap B_M(0)}$ is only needed for unbounded domains)

$$\mathcal{P} := \left\{ p \cdot \mathbb{1}_{\Omega \cap B_M(0)} : \ p \text{ is a polynomial with rational coefficients and } M \in \mathbb{N} \right\}.$$

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- So let $\phi \in C_0^\infty(\Omega)$ and $\varepsilon > 0$.
- Choose $M \in \mathbb{N}$ with $\mathrm{supp}(\phi) \subset B_M(0)$. Weierstrass' Approximation Theorem $\rightsquigarrow$ polynomial $\tilde{p}$ with

$$\|\tilde{p} - \phi\|_{C(\overline{B_M(0)})} \cdot |B_M(0)|^{\frac{1}{p}} < \frac{\varepsilon}{2}.$$

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Proof:

$$||\tilde{p} \cdot \mathbb{1}_{\Omega \cap B_M(0)} - \phi||_{L^p(\Omega)} = ||\tilde{p} - \phi||_{L^p(\Omega \cap B_M(0))} \leq ||\tilde{p} - \phi||_{C(\overline{B_M(0)})} \cdot |B_M(0)|^{\frac{1}{p}} < \frac{\varepsilon}{2}. \tag{1}$$

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If $\tilde{p}(x) = \sum_{|\alpha| \leq n} \tilde{a}_\alpha x^\alpha$ then $p(x) := \sum_{|\alpha| \leq n} a_\alpha x^\alpha$ with $a_\alpha \in \mathbb{Q}$, $a_\alpha \approx \tilde{a}_\alpha$.

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$$\begin{aligned} \|\tilde{p} \cdot \mathbb{1}_{\Omega \cap B_M(0)} - p \cdot \mathbb{1}_{\Omega \cap B_M(0)}\|_{L^p(\Omega)} &\le \|\tilde{p} - p\|_{C(\overline{B_M(0)})} \cdot |B_M(0)|^{\frac{1}{p}} \\ &\le \sum_{|\alpha| \le N} |\tilde{a}_\alpha - a_\alpha| \|x^\alpha\|_{C(\overline{B_M(0)})} |B_M(0)|^{\frac{1}{p}} < \frac{\varepsilon}{2}. \end{aligned} \quad (2)$$

$$\Rightarrow \qquad \inf_{q \in \mathcal{P}} \|q - \phi\|_{L^p(\Omega)} \le \|p \cdot \mathbb{1}_{\Omega \cap B_M(0)} - \phi\|_{L^p(\Omega)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \square$$

Separability of $W^{k,p}(\Omega)$

Question: Consequences for Sobolev spaces?

- (Finite) product spaces $L^p(\Omega) \times \ldots \times L^p(\Omega)$ are also separable for $1 \leq p < \infty$.
- $\Psi : W^{k,p}(\Omega) \to L^p(\Omega)^K, \quad u \mapsto (\partial^\alpha u)_{0\leq|\alpha|\leq k}$ is an isometry (not surjective).
  $\Psi(W^{k,p}(\Omega))$ is a closed subspace.

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Let X be a separable Banach space and $\emptyset \neq M \subset X$. Then M is separable.

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- $\{x_n : n \in \mathbb{N}\}$ be dense in X, $x^* \in M$.
- Define $\quad y_{n,m} = x^*$ if $B_{1/m}(x_n) \cap M = \emptyset \quad$ and $\quad y_{n,m} \in B_{1/m}(x_n) \cap M$ (arbitrary) otherwise.

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- Claim: $Y := \{y_{n,m} : n, m \in \mathbb{N}\} \subset M$ is dense.

Separability of $W^{k,p}(\Omega)$

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Corollary

Let $\Omega \subset \mathbb{R}^N$ be open and non-empty, $k \in \mathbb{N}$.

(i) *$W^{k,p}(\Omega)$ is separable provided that $1 \leq p < \infty$.*

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Proof:

(i) $\Psi(W^{k,p}(\Omega))$ is separable as a closed subspace of $L^p(\Omega)^M$.

If $\tilde{\mathcal{P}} \subset \Psi(W^{k,p}(\Omega))$ is countable and dense

$\Rightarrow \quad \mathcal{P} := \{\Psi^{-1}(p) : p \in \tilde{\mathcal{P}}\}$ is countable and dense in $W^{k,p}(\Omega)$.

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Proof:

(For simplicity $\Omega = \Omega' \times (0,1), \Omega' \subset \mathbb{R}^{N-1}$ bounded.)

- $0 < z < 1$ choose $r_z > 0$ such that $I_z := (z - r_z, z + r_z) \subset (0,1)$ and

$$F_z(x', x_N) := \int_0^{x_N} \int_0^{t_1} \dots \int_0^{t_{k-1}} \mathbb{1}_{I_z}(s)\, ds \dots dt_{k-1} \quad \text{where } x = (x', x_N) \in \Omega = \Omega' \times (0,1).$$

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- Then $F_z \in W^{k,\infty}(\Omega)$ and set $U_z := \left\{ f \in W^{k,\infty}(\Omega) : \|f - F_z\|_{W^{k,\infty}(\Omega)} < \frac{1}{2} \right\} \qquad (z \in I).$

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Corollary

Let $\Omega \subset \mathbb{R}^N$ *be open and non-empty,* $k \in \mathbb{N}$.

(i) $W^{k,p}(\Omega)$ *is separable provided that* $1 \leq p < \infty$.

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Proof:

- $(U_z)_{z \in I}$: uncountable, pairwise disjoint, open and non-empty subsets of $W^{k,\infty}(\Omega)$.

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$$
\begin{aligned}
f \in U_{z_1} \cap U_{z_2} \quad &\Rightarrow \quad \|F_{z_1} - F_{z_2}\|_{W^{k,\infty}(\Omega)} < 1 \\
&\Rightarrow \quad \|\partial_N^k(F_{z_1} - F_{z_2})\|_{L^\infty(\Omega)} < 1 \\
&\Rightarrow \quad \|\mathbb{1}_{I_{z_1}} - \mathbb{1}_{I_{z_2}}\|_{L^\infty(0,1)} < 1 \\
&\Rightarrow \quad z_1 = z_2.
\end{aligned}
$$

So $W^{k,\infty}(\Omega)$ is not separable.

Reflexivity

Next time:

- $W^{k,p}(\Omega)$ (and closed subspaces) are reflexive for $1 < p < \infty$

Motivation: Bounded sequences in reflexive spaces have **weakly convergent** subsequences.

Reflexivity

Basics from Functional Analysis:

- $(X, \|\cdot\|_X)$ be a real Banach space.

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- $(X, \|\cdot\|_X)$ be a real Banach space.
- $(X', \|\cdot\|_{X'})$ its dual space consisting of bounded linear functionals $\phi : X \to \mathbb{R}$.
  Those satisfy $|\phi(f)| \leq C\|f\|_X$ and

$$\|\phi\|_{X'} = \sup\{|\phi(f)| : f \in X, \|f\|_X = 1\}.$$

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- $X'' := (X')'$ = the dual space of the dual space (the “bidual”).

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- $X'' := (X')'$ = the dual space of the dual space (the “bidual”).
- Consider $J : X \to X''$ via $(Jf)(g) := g(f)$ for $g \in X'$.
  That’s a bounded linear injective operator, but in general not surjective.

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Definition

A Banach space X is called reflexive if $J : X \to X''$ is surjective.

Reflexivity

Examples:

- $L^p(\Omega)$ is reflexive if and only if $1 < p < \infty$
- $W^{k,p}(\Omega)$ is reflexive if and only if $1 < p < \infty$
- Hilbert spaces are reflexive
- The sequence spaces l^1, l^∞, c_0 are not reflexive, $C([0,1])$ is not reflexive.

End of 17th Lecture