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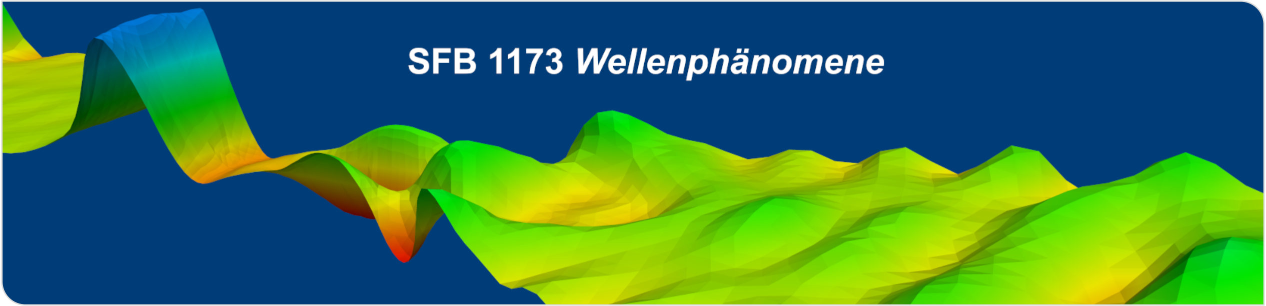
Sobolev spaces

Rainer Mandel, Zois Moitier | Karlsruhe, 15.07.2021



Today:

- $W^{k,p}(\Omega)$ are reflexive for $1 < p < \infty$
- So bounded sequences in these spaces have “weakly convergent” subsequences.

Reflexivity

Basics from Functional Analysis:

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  Those satisfy $|\phi(f)| \leq C\|f\|_X$ and

$$\|\phi\|_{X'} = \sup\{|\phi(f)| : f \in X, \|f\|_X = 1\}.$$

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  That’s a bounded linear injective operator, but in general not surjective.

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Definition

A Banach space X is called reflexive if $J : X \to X''$ is surjective.

Reflexivity

Examples:

- $L^p(\Omega)$ is reflexive if and only if $1 < p < \infty$
- $W^{k,p}(\Omega)$ is reflexive if and only if $1 < p < \infty$
- Hilbert spaces are reflexive
- The sequence spaces l^1, l^∞, c_0 are not reflexive, $C([0,1])$ is not reflexive.

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Most useful criterion: Check whether the norm is uniformly convex!

Theorem (Milman-Pettis)

Assume that $(X, \|\cdot\|_X)$ is a uniformly convex Banach space. Then X is reflexive.

Reflexivity

Definition (Uniform Convexity)

A normed vector space $(X, \|\cdot\|_X)$ *is called uniformly convex if*

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x, y \in X \qquad \left(\|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon \quad \Rightarrow \quad \|\frac{x+y}{2}\| \leq 1 - \delta \right)$$

Lemma

Let $1 < p < \infty$ *and* $\Omega \subset \mathbb{R}^N$. *Then* $(L^p(\Omega), \|\cdot\|_p)$ *is uniformly convex.*

- For a proof see the book by Adams
- $L^p(\Omega)^K$ is reflexive for $1 < p < \infty$, $K \in \mathbb{N}$ ...
- ... and all their closed subspaces

Reflexivity

Recall $\Psi : W^{k,p}(\Omega) \to L^p(\Omega)^K, \quad u \mapsto (\partial^\alpha u)_{0\le|\alpha|\le k}$.

Corollary

$(L^p(\Omega), \|\cdot\|_p)$ and $(W^{k,p}(\Omega), \|\cdot\|_{k,p})$ are reflexive for $1 < p < \infty$.

Proof:

- Only $W^{k,p}(\Omega)$: Consider $\Psi : W^{k,p}(\Omega) \to L^p(\Omega)^K$

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- So $W^{k,p}(\Omega)$ is uniformly convex, hence reflexive. $\square$

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- $\Psi(W^{k,p}(\Omega))$ is a subspace of $L^p(\Omega)^K$, hence uniformly convex
- So $W^{k,p}(\Omega)$ is uniformly convex, hence reflexive. □

Same for $W_0^{k,p}(\Omega)$.

Question: Why should we care about reflexivity?

Banach-Alaoglu Theorem

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Definition

Let X be a Banach space with dual space X'.

(i) *(Weak-$\star$-convergence) A sequence $(f_k)_{k\in\mathbb{N}} \subset X'$ is said to converge to $f \in X'$ in the weak-$\star$-sense (i.e., pointwise), written $f_k \rightharpoonup^\star f$, if $f_k(x) \to f(x)$ as $k \to \infty$ for all $x \in X$.*

(ii) *(Weak convergence) A sequence $(x_k)_{k\in\mathbb{N}} \subset X$ is said to converge weakly to $x \in X$, written $x_k \rightharpoonup x$, if $f(x_k) \to f(x)$ as $k \to \infty$ for all $f \in X'$.*

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Examples:

- Weak convergence in finite-dimensional spaces is equivalent to norm-convergence.
- Weak limits are uniquely determined if they exist.
- $x_n \to x$ implies $x_n \rightharpoonup x$, but in general not vice versa.

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Theorem (Banach-Alaoglu)

Let X be a separable Banach space. Then every bounded sequences in X' has a weak-*-convergent subsequence.

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- Define $f : M \to \mathbb{R},\ y \mapsto \lim_{k\to\infty} f_k(y)$.
- $|f(x_i) - f(x_j)| = |\lim_{k\to\infty} f_k(x_i - x_j)| \leq \|x_i - x_j\|$ implies

$$F \in X', \|F\|_{X'} \leq 1 \quad \text{where } F(x) = \lim_{\substack{y\to x,\\ y\in M}} f(y)$$

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- For every fixed $x \in X$ choose $x_i \approx x$ and obtain

$$|f_k(x) - F(x)| \leq |f_k(x - x_i)| + |f_k(x_i) - F(x_i)| + |F(x_i - x)| \leq 2\|x - x_i\| + \frac{\varepsilon}{3} \leq \varepsilon. \quad \square$$

Banach-Alaoglu Theorem

Corollary

Let X be a (separable and) reflexive Banach space, then every bounded sequence in X has a weakly convergent subsequence.

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- Banach-Alaoglu $\rightsquigarrow$ $J(x_{n_j}) \rightharpoonup^\star T$ in $(X')'$, $T = J(x)$ by reflexivity.

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- So for all $f \in X'$

$$f(x_{n_j}) = J(x_{n_j})(f) \to T(f) = J(x)(f) = f(x) \quad (j \to \infty).$$

Hence $x_{n_j} \rightharpoonup x$ as $j \to \infty$. □

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Quod erat demonstrandum. Finito. Basta!

End of 18th Lecture

(and of the whole course)

– IT WAS A PLEASURE –