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1 Introduction

This course is about Sobolev spaces which are indispensable for a modern theory of Partial Differential Equations (PDEs). One example for such a Sobolev space is given by

$$H^1(\Omega) = \{u \in L^2(\Omega) : \partial_i u \in L^2(\Omega) \ \forall i \in \{1, \dots, N\}\}$$

where $\Omega \subset \mathbb{R}^N, N \in \mathbb{N}$ is an open set and $\partial_i u$ is the i-th weak (sometimes called distributional) partial derivative of u. We will clarify later how this is defined. For the moment it suffices to know that it is a generalization of the classical i-th partial derivative. The main reasons for using these spaces are the following:

- One can formulate PDEs in these spaces. Finding a solution u of some PDE is often equivalent to finding a function $\tilde{u} \in H^1(\Omega)$ solving the corresponding PDE in its “weak formulation”. These functions are then called weak solutions of the given problem.
- Since Sobolev spaces carry the structure of Banach spaces – the space $H^1(\Omega)$ above is even a Hilbert space – one can use powerful tools from functional analysis to prove the existence of weak solutions of PDEs.

In addition to that, Sobolev space allow to prove the existence of (weak) solutions even when classical solutions do not exist, for instance when coefficient functions are discontinuous. Furthermore, they are nowadays indispensable for numerical methods like Finite Element Methods or Galerkin Methods. This is our motivation to study these spaces in more detail. The plan of the lecture is the following:

(1) (1L) Introduction and Preliminaries
(2) (2L) Weak derivatives and Sobolev spaces
(3) (1L) Lax-Milgram Theorem and Riesz’ Representation Theorem
(4) (2L) Approximation by smooth functions
(5) (2L) Stein’s Extension Theorem
(6) (2L) Sobolev’s Embedding Theorem and Applications
(7) (2L) Morrey’s Embedding Theorem and Applications
(8) (2L) Compact Embeddings: The Rellich-Kondrachov Theorem and beyond
(9) (2L) Poincaré’s Inequality and Applications
(10) (2L) Trace Theorem and Applications
(11) (1L) Separability

(12) (1L) Reflexivity

I am going to illustrate the benefits of the theory with the aid of a running example that we will understand better and better during this course. This example is an elliptic boundary value problem of the form

$$-\Delta u(x) + c(x)u(x) = f(x) \quad (x \in \Omega), \qquad u(x) = g(x) \quad (x \in \partial\Omega) \tag{1.1}$$

where the functions $c, f, g : \overline{\Omega} \to \overline{\mathbb{R}}$ are given. We are looking for a solution u of this problem under as weak assumptions on c, f, g, Ω as possible. To use Sobolev space theory we first pass to its weak formulation. To this end we assume that $u \in C^2(\overline{\Omega})$ is a classical solution of this problem and multiply the PDE with some test function $\phi \in C_0^\infty(\Omega)$ the support of which is strictly contained in Ω. Integration over Ω and the Divergence Theorem imply[1]

$$\begin{aligned}
\int_\Omega f(x)\phi(x)\,dx &= \int_\Omega (-\Delta u(x) + c(x)u(x))\phi(x)\,dx \\
&= \int_\Omega -\operatorname{div}(\phi\nabla u)(x) + \nabla\phi(x)\cdot\nabla u(x) + c(x)u(x)\phi(x)\,dx \\
&= -\int_{\partial\Omega} \phi(x)\nabla u(x)\cdot\nu(x)\,d\sigma(x) + \int_\Omega \nabla\phi(x)\cdot\nabla u(x) + c(x)u(x)\phi(x)\,dx \\
&= \int_\Omega \nabla\phi(x)\cdot\nabla u(x) + c(x)u(x)\phi(x)\,dx
\end{aligned}$$

Here, $\nu : \partial\Omega \to \mathbb{R}^N$ denotes the outer unit normal vector field, σ is the surface measure of $\partial\Omega$. So our boundary value problem (1.1) takes the form

$$\int_\Omega \nabla\phi(x)\cdot\nabla u(x) + c(x)u(x)\phi(x)\,dx = \int_\Omega f(x)\phi(x)\,dx \quad \forall\phi \in C_0^\infty(\Omega), \quad u|_{\partial\Omega} = g.$$

For convenience we set $\gamma u := u|_{\partial\Omega}$ (the trace of u) and introduce

$$\begin{aligned}
a(u,v) &:= \int_\Omega \nabla u(x)\cdot\nabla v(x) + c(x)u(x)v(x)\,dx, \\
l(v) &:= \int_\Omega f(x)v(x)\,dx.
\end{aligned}$$

Then the so-called weak formulation of the boundary value problem (1.1) takes the form

$$a(u,\phi) = l(\phi) \quad \forall\phi \in C_0^\infty(\Omega), \qquad \gamma u = g. \tag{1.2}$$

In the following we shall develop the tools to get a rich existence theory for this problem. The interesting point is that classical solutions are not only necessarily solutions of (1.2), but even the converse holds in some cases (after modification on a null set).

Preliminaries:

[1] Recall $\operatorname{div}(\phi f) = \phi\operatorname{div}(f) + \nabla\phi\cdot f$ for scalar functions $\phi \in C^1(\Omega)$ and vector fields $f \in C^1(\Omega;\mathbb{R}^N)$. Moreover, $\operatorname{div}(\nabla u) = \Delta u$.

We collect some facts and fix the notation. All sets respectively functions considered in this lecture are assumed to be Lebesgue-measurable. Vector spaces come with the field $\mathbb{K} = \mathbb{R}$. In the following:

- $\Omega \subset \mathbb{R}^N$ denotes an open domain, i.e., an open connected set. It may be bounded or unbounded.
- **Lebesgue spaces:** For $1 \le p \le \infty$, $L^p(\Omega)$ is the vector space of (equivalence classes of) Lebesgue-measurable functions $f : \Omega \to \overline{\mathbb{R}}$ that are p-integrable with respect to the Lebesgue measure. This means that

$$\|f\|_p := \left(\int_\Omega |f(x)|^p \, dx \right)^{1/p} \quad (1 \le p < \infty), \qquad \|f\|_\infty := \operatorname{ess\,sup}_\Omega |f|.$$

  is finite for these functions and $\|\cdot\|_p$ defines a norm that turns $(L^p(\Omega), \|\cdot\|_p)$ into a Banach space. In the case $p = 2$ it is a Hilbert space endowed with the inner product

$$\langle f, g \rangle_2 := \int_\Omega f(x) g(x) \, dx.$$

  We write $L^p_{\text{loc}}(\Omega)$ consists of all measurable functions such that $f \cdot \mathbb{1}_K \in L^p(\Omega)$ for all compact subsets $K \subset \Omega$.
- **Minkowski's and Hölder's inequality:** We recall Minkowski's inequality $\|f + g\|_p \le \|f\|_p + \|g\|_p$, which is nothing but the triangle inequality in $L^p(\Omega)$. Very important: Hölder's inequality for $1 \le p, q \le \infty$ reads

$$\|fg\|_1 \le \|f\|_p \|g\|_q \qquad \text{if } \frac{1}{p} + \frac{1}{q} = 1.$$

  In that case we write $q = p' = \frac{p}{p-1}$.
- **Density arguments:** Many inequalities / properties of functions belonging to Sobolev spaces will be proved using the denseness of smooth functions over Ω. We call this vector space $C^\infty(\Omega)$. We will call test functions all functions $u \in C^\infty(\Omega)$ the support of which $\operatorname{supp}(u) := \overline{\{x \in \Omega : u(x) \neq 0\}}$ is a compact subset of Ω. In particular, these functions vanish in a neighbourhood of $\partial\Omega$. The space of test functions will be denoted by $C_0^\infty(\Omega)$.
- **Fundamental Lemma of the Calculus of Variations:** It reads that for any $f \in L^p(\Omega), 1 \le p \le \infty$

$$\int_\Omega f(x)\phi(x) \, dx = 0 \quad \text{for all } \phi \in C_0^\infty(\Omega) \quad \Rightarrow \quad f = 0 \text{ almost everywhere.}$$

  We provide an argument for this in the case $1 \le p < \infty$. It suffices to find a sequence $(\phi_n) \subset C_0^\infty(\Omega)$ such that $\phi_n \to |f|^{p-2} f$ in $L^{p'}(\Omega)$. We will see later why such a sequence exists. Then

$$\int_\Omega |f(x)|^p \, dx = \lim_{n\to\infty} \left(\int_\Omega f(x)\phi_n(x) \, dx + \int_\Omega f(x)(|f(x)|^{p-2} f(x) - \phi_n(x)) \, dx \right)$$

$$\le 0 + \limsup_{n\to\infty} \|f\|_p \|\phi_n - |f|^{p-2} f\|_{p'}$$
$$= 0,$$

hence $f = 0$ almost everywhere. A similar proof shows that the same conclusion is true[2] under the weaker assumption $f \in L^1_{\mathrm{loc}}(\Omega)$.

- **Differential operators:** The differential operators $\partial_i, \nabla, \Delta$ have the usual meaning: ∂_i is the i-the partial derivative of a function, $\nabla = (\partial_1, \dots, \partial_N)$ is the gradient and $\Delta = \mathrm{div}(\nabla) = \sum_{i=1}^N \partial_{ii}$ is the Laplacian, which plays a central role in many PDEs. Later on $\partial_i u, \nabla u$, etc. will denote the weak i-th partial derivative, weak gradient of u, etc. We will not make a notational distinction.
- **Bounded linear operators:** A bounded linear operator $T : X \to Y$ between some Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ is an $\mathbb{R}$-linear map satisfying $\|Tx\|_Y \le C\|x\|_X$ for all $x \in X$ with some positive number $C > 0$ independent of x. The least such number is the operator norm of T, namely

$$\|T\| := \|T\|_{X\to Y} = \sup_{x\neq 0} \frac{\|Tx\|_Y}{\|x\|_X} < \infty.$$

  We will only deal with linear operators (in contrast to nonlinear ones) in this lecture.
- **Compact operators:** A bounded linear operator between Banach spaces $T : X \to Y$ is compact if for each bounded sequence $(x_n) \subset X$ the image sequence $(Tx_n) \subset Y$ has a convergent subsequence. This definition is worth keeping in mind; compactness is a very important concept.
- **$C^{m,\alpha}$-Domains:** Consider a bounded domain $\Omega \subset \mathbb{R}^N$ as above. We say that it is a $C^{m,\alpha}$-domain if each point on its boundary $\partial\Omega$ has a neighbourhood U such that, "after some permutation of coordinates",

$$\partial\Omega \cap U = \{(x', x_N) \in U : x_N = \psi(x')\},$$
$$\Omega \cap U = \{(x', x_N) \in U : x_N > \psi(x')\}$$

  for some function $\psi \in C^{m,\alpha}(\mathbb{R}^{N-1})$. In that case the outer unit normal vector field at the boundary point $x := (x', x_N) \in \partial\Omega$ is given by[3]

$$\nu(x) = \frac{1}{\sqrt{1 + |\nabla\psi(x')|^2}} \begin{pmatrix} \nabla\psi(x') \\ -1 \end{pmatrix}. \tag{1.3}$$

- **Surface integrals:** We want to (rather: need to) integrate over the boundaries of $C^{m,\alpha}$-domains $\Omega \subset \mathbb{R}^N$. This is done via

$$\int_{\partial\Omega} g\, d\sigma := \sum_{i=1}^M \int_{\partial\Omega\cap U_i} g\, d\sigma_{U_i}$$

[2] Apply the previous reasoning to $f \cdot \mathbb{1}_K$ for all compact subsets $K \subset \Omega$.

[3] It is indeed the "outer" one because you can formally check via Taylor expansion that $x + t\nu(x) \in \overline{\Omega}^c$ for $0 < t < t_0$ and $x + t\nu(x) \in \Omega$ for $-t_0 < t < 0$ provided that $t_0 > 0$ is chosen sufficiently small.

where $\partial\Omega = \bigcup_{i=1}^{M} U_i$. Here, the U_i's are disjoint neighbourhoods (graphical pieces) as above with $\psi_i \in C^{m,\alpha}(\mathbb{R}^{N-1})$. For such neighbourhoods the latter integrals are defined according to

$$\int_{\partial\Omega\cap U_i} g\, d\sigma_{U_i} = \int_{\{x'\in\mathbb{R}^{N-1}:(x',\psi_i(x'))\in U_i\}} g(x',\psi_i(x'))\sqrt{1+|\nabla\psi_i(x')|^2}\, dx'.$$

- **Divergence Theorem:** Surface integrals are important in view of the Divergence Theorem, which is a higher-dimensional version of the Fundamental Theorem of Calculus. For vector fields $f \in C^1(\overline{\Omega};\mathbb{R}^N)$ and Lipschitz domains ($m = 0, \alpha = 1$) it reads
$$\int_\Omega \operatorname{div}(f)\, dx = \int_{\partial\Omega} f\cdot\nu\, d\sigma,$$
where the boundary integral is given by the previous definition. Notice that the outer unit normal vector field is defined "locally" in terms of the parametrizing function ψ as in (1.3). As a consequence one obtains the integration-by-parts formula for $u, v \in C^1(\overline{\Omega})$:
$$\int_\Omega \partial_i uv\, dx = \int_{\partial\Omega} uv\nu_i\, d\sigma - \int_\Omega u\partial_i v\, dx. \tag{1.4}$$
We shall use this for test functions $v = \phi \in C_0^\infty(\Omega)$ that vanish close to the boundary. In that case we obtain for all open sets $\Omega \subset \mathbb{R}^N$ and all $u \in C^1(\Omega)$ the equality[4]
$$\int_\Omega \partial_i u\phi\, dx = -\int_\Omega u\partial_i\phi\, dx.$$

End Lec 01

2 Weak derivatives and Sobolev spaces

We start with the definition of a weak derivative of a given function $u \in L^1_{\text{loc}}(\Omega)$ for some open subset $\Omega \subset \mathbb{R}^N, N \in \mathbb{N}$.

Definition 2.1. *Let $u \in L^1_{\text{loc}}(\Omega)$ and $i \in \{1,\dots,N\}$. A function $w \in L^1_{\text{loc}}(\Omega)$ is called i-th weak partial derivative of u if it satisfies*

$$\int_\Omega u(x)\partial_i\phi(x)\, dx = -\int_\Omega w(x)\phi(x)\, dx \qquad \text{for all } \phi \in C_0^\infty(\Omega).$$

In this case we will write $\partial_i u := w$.

[4] Establishing this rigorously is a bit technical, we skip this. Essentially, it is a consequence of (1.4) where Ω is replaced by some large enough ball where all boundary terms are well-defined.

In the one-dimensional case one replaces the i-th partial derivative by the usual derivative. We shall also use the symbols ∂_x, ∂_y etc. as for classical derivatives. The definition $\partial_i u := w$ makes sense because we now prove that two different weak partial derivatives coincide almost everywhere.

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^N$ be an open set and $u \in L^1_{\text{loc}}(\Omega)$, $i \in \{1, \dots, N\}$. Assume that $w, \tilde{w} \in L^1_{\text{loc}}(\Omega)$ are an i-th weak derivative of u. Then $w = \tilde{w}$ (almost everywhere).*

Beweis:
Let $\phi \in C_0^\infty(\Omega)$ be arbitrary. By Definition 2.1,

$$-\int_\Omega w(x)\phi(x)\,dx = \int_\Omega u(x)\partial_i\phi(x)\,dx = -\int_\Omega \tilde{w}(x)\phi(x)\,dx.$$

So we infer

$$\int_\Omega (w(x) - \tilde{w}(x))\phi(x)\,dx = 0 \qquad \text{for all } \phi \in C_0^\infty(\Omega).$$

The Fundamental Lemma of the Calculus of Variations implies $w - \tilde{w} = 0$ almost everywhere, which is all we had to show. □

So we can speak of "the" weak partial derivative, "the" weak gradient (defined via $\nabla := (\partial_1, \dots, \partial_N)$) of a function $u \in L^1_{\text{loc}}(\Omega)$. A function $u \in L^1_{\text{loc}}(\Omega)$ may in general be discontinuous on Ω, but nevertheless admits weak derivatives. We will see some examples below. Furthermore, in contrast to the classical derivatives that are defined pointwise for each $x \in \Omega$, the weak derivative a priori depends on Ω as a whole. Higher weak derivatives are defined accordingly: For a given multi-index $\alpha \in \mathbb{N}_0^N$ the corresponding weak partial derivative is supposed to satisfy

$$\int_\Omega u(x)\partial^\alpha\phi(x)\,dx = (-1)^{|\alpha|}\int_\Omega w(x)\phi(x)\,dx \qquad \text{for all } \phi \in C_0^\infty(\Omega).$$

Here, for any given $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}_0^N$ the symbol ∂^α stands for $\partial_1^{\alpha_1} \dots \partial_N^{\alpha_N}$ and $|\alpha| := \alpha_1 + \dots + \alpha_N$. We first show that this differentiation concept generalizes the notion of a classical derivative. The following result tells us that the classical gradient is the only candidate for the weak derivative if it exists.

Proposition 2.3. *Let $\Omega \subset \mathbb{R}^N$ be open.*

(i) If $u \in C^1(\Omega)$ then the classical gradient of u is also a weak gradient of u.

(ii) If $u \in L^1_{\text{loc}}(\Omega)$ has a weak gradient $\nabla u \in L^1_{\text{loc}}(\Omega; \mathbb{R}^N)$ and $u \in C^1(\tilde{\Omega})$ for some open subset $\tilde{\Omega} \subset \Omega$, then the weak gradient coincides with the classical gradient on $\tilde{\Omega}$.

Beweis:
In this proof we denote the classical i-th partial derivative by $\frac{\partial}{\partial x_i}$. The classical integration-by-parts formula (1.4) yields for all $i = 1, \dots, N$

$$\int_{\tilde{\Omega}} \frac{\partial u}{\partial x_i}(x)\phi(x)\,dx = -\int_{\tilde{\Omega}} u(x)\partial_i\phi(x)\,dx \qquad \text{for all } \phi \in C_0^\infty(\tilde{\Omega}).$$

Using this fact for $\tilde{\Omega} = \Omega$ proves (i). In order to prove (ii) we assume that a weak gradient on Ω exists. Then each $\phi \in C_0^\infty(\tilde{\Omega})$ belongs to $\phi \in C_0^\infty(\Omega)$, so the definition of a weak derivative implies

$$\int_{\tilde{\Omega}} \partial_i u(x)\phi(x)\, dx = -\int_{\tilde{\Omega}} u(x)\partial_i \phi(x)\, dx \qquad \text{for all } \phi \in C_0^\infty(\tilde{\Omega}).$$

The Fundamental Lemma of the Calculus of Variations gives $\partial_i u = \frac{\partial u}{\partial x_i}$ (almost everywhere) on $\tilde{\Omega}$. □

One can check: $\partial_i(\beta_1 u + \beta_2 v) = \beta_1 \partial_i u + \beta_2 \partial_i v$ for all $\beta_1, \beta_2 \in \mathbb{R}$ provided that the weak derivatives on the right exist.

Example 2.4.

(a) Let $u(x) := |x|$ for $x \in \Omega := (-1,1) \subset \mathbb{R}$. Proposition 2.3 tells us that the function $v(x) := 1$ for $x > 0$ and $v(x) := -1$ for $x < 0$ is the only candidate for a weak derivative of u. We can check this by hand. For $\phi \in C_0^\infty(\Omega)$ we have

$$\begin{aligned}\int_{-1}^{1} u(x)\phi'(x)\, dx &= -\int_{-1}^{0} x\phi'(x)\, dx + \int_{0}^{1} x\phi'(x)\, dx \\ &= -[x\phi(x)]_{-1}^{0} + \int_{-1}^{0} \phi(x)\, dx + [x\phi(x)]_{0}^{1} - \int_{0}^{1} \phi(x)\, dx \\ &= \phi(-1) + \phi(1) - \int_{-1}^{1} v(x)\phi(x)\, dx \\ &= -\int_{-1}^{1} v(x)\phi(x)\, dx.\end{aligned}$$

So v is indeed the weak derivative of u.

(b) Consider the function $u : \Omega \to \mathbb{R}, (x,y) \mapsto \mathbb{1}_{x>0} + \mathbb{1}_{y>0}$ where $\Omega := (-1,1) \times (-1,1)$. We claim that it has a second weak derivative $\partial_{xy} u$ even though it does not have first order derivatives. We claim that $\partial_{xy} u = 0$ holds in the weak sense. Indeed, for $\phi \in C_0^\infty(\Omega)$,

$$\begin{aligned}&\int_{\Omega} u(x,y)\partial_{xy}\phi(x,y)\, d(x,y) \\ &= \int_{-1}^{1} \left(\int_{0}^{1} \partial_{xy}\phi(x,y)\, dx \right) dy + \int_{-1}^{1} \left(\int_{0}^{1} \partial_{xy}\phi(x,y)\, dy \right) dx \\ &= \int_{-1}^{1} \left(\partial_y \phi(1,y) - \partial_y \phi(0,y)\right) dy + \int_{-1}^{1} \left(\partial_x \phi(x,1) - \partial_x \phi(x,0)\right) dx \\ &= -\int_{-1}^{1} \partial_y \phi(0,y)\, dy - \int_{-1}^{1} \partial_x \phi(x,0)\, dx \\ &= -\phi(0,1) + \phi(0,-1) - \phi(1,0) - \phi(-1,0) \\ &= 0.\end{aligned}$$

The last equality holds because ϕ vanishes close to the boundary of Ω and

$$(0,1),(0,-1),(1,0),(0,1)\in\partial\Omega.$$

On the other hand, $\partial_x u$ does not exist:

$$\begin{aligned}\int_\Omega u(x,y)\partial_x\phi(x,y)\,d(x,y) &= \int_{-1}^1\left(\int_0^1 \partial_x\phi(x,y)\,dx\right)dy+\int_0^1\left(\int_{-1}^1 \partial_x\phi(x,y)\,dx\right)dy\\ &= \int_{-1}^1 (\phi(1,y)-\phi(0,y))\,dy+\int_0^1 (\phi(1,y)-\phi(-1,y))\,dy\\ &= -\int_{-1}^1 \phi(0,y)\,dy.\end{aligned}$$

This cannot be written as $-\int_\Omega w(x,y)\phi(x,y)\,dy$ for some $w\in L^1_{\text{loc}}(\Omega)$.

(c) Schwarz's Theorem is always true: $\partial_{xy}u$ exists as a weak derivative if and only if $\partial_{yx}u$ exists. On the other hand, (b) shows that $\partial_x(\partial_y u)$ may not be a meaningful equivalent expression.

Further examples will be given below. We now introduce the Sobolev spaces.

Definition 2.5 (Sobolev spaces). *Let $k\in\mathbb{N}$ and $1\le p\le\infty$.*

$$W^{k,p}(\Omega) := \left\{u\in L^p(\Omega) : \partial^\alpha u\in L^p(\Omega) \text{ for all } \alpha\in\mathbb{N}_0^N, 0\le|\alpha|\le k\right\}$$

$$\|u\|_{W^{k,p}(\Omega)} := \left(\sum_{|\alpha|\le k}\|\partial^\alpha u\|_p^p\right)^{\frac{1}{p}} \quad \text{if } 1\le p<\infty, \qquad \|u\|_{W^{k,\infty}(\Omega)} := \max_{|\alpha|\le k}\|\partial^\alpha u\|_\infty.$$

We also define $H^k(\Omega) := W^{k,2}(\Omega)$.

Here, $\partial^\alpha u\in L^p(\Omega)$ stands for the statement that the weak derivative $\partial^\alpha u$ exists and that it lies in $L^p(\Omega)$ (not only in $L^1_{\text{loc}}(\Omega)$). We remark that other equivalent norms can be taken without changing the theory. For instance, for $1\le p\le\infty$ one may also take

$$u\mapsto\sum_{|\alpha|\le k}\|\partial^\alpha u\|_p.$$

The definition given above has the pleasant feature that the most important spaces $H^k(\Omega)$ are generated by the inner product

$$\langle u,v\rangle_{k,2} := \langle u,v\rangle_{H^k(\Omega)} := \int_\Omega\sum_{|\alpha|\le k}\partial^\alpha u(x)\partial^\alpha v(x)\,dx.$$

In the special case $k=1$, which is the most important one for us,

$$\langle u,v\rangle_{1,2} = \int_\Omega\sum_{i=1}^N \partial_i u(x)\partial_i v(x)+u(x)v(x)\,dx = \int_\Omega \nabla u(x)\cdot\nabla v(x)+u(x)v(x)\,dx.$$

Satz 2.6. *Let $k \in \mathbb{N}, 1 \le p \le \infty$. Then $(W^{k,p}(\Omega), \|\cdot\|_{W^{k,p}(\Omega)})$ is a Banach space and $(H^k(\Omega), \langle\cdot,\cdot\rangle_{k,2})$ is a Hilbert space.*

Beweis:
We use that $\|\cdot\|_{W^{k,p}(\Omega)}, \langle\cdot,\cdot\rangle_{k,2}$ are norms respectively inner products. The proof of this fact is straightforward and therefore omitted. So it remains to show that the spaces $W^{k,p}(\Omega)$ are complete with respect to these norms. To show this, we use that the spaces $(L^p(\Omega), \|\cdot\|_p)$ are complete.

Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $W^{k,p}(\Omega)$, i.e., for all $\varepsilon > 0$ there is $m_0 \in \mathbb{N}$ such that

$$\|u_m - u_n\|_{W^{k,p}(\Omega)} \le \varepsilon \qquad \text{for all } m, n \ge m_0.$$

By definition of the norm we conclude that for each fixed $\alpha \in \mathbb{N}_0^N, |\alpha| \le k$ we have

$$\|\partial^\alpha u_m - \partial^\alpha u_n\|_p \le \|u_m - u_n\|_{W^{k,p}(\Omega)} \le \varepsilon \qquad \text{for all } m, n \ge m_0.$$

So $(\partial^\alpha u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\Omega)$. Completeness of $L^p(\Omega)$ yields $v_\alpha \in L^p(\Omega)$ such that

$$\partial^\alpha u_n \to v_\alpha \qquad \text{in } L^p(\Omega). \tag{2.1}$$

Define $v := v_{(0,\dots,0)} \in L^p(\Omega)$. We claim $\partial^\alpha v = v_\alpha$. Indeed, for all test functions $\phi \in C_0^\infty(\Omega)$, we have

$$\begin{aligned}
\int_\Omega v(x)\partial^\alpha \phi(x)\,dx &= \lim_{n\to\infty} \int_\Omega u_n(x)\partial^\alpha\phi(x)\,dx \\
&= \lim_{n\to\infty} (-1)^{|\alpha|} \int_\Omega \partial^\alpha u_n(x)\phi(x)\,dx \\
&= (-1)^{|\alpha|} \int_\Omega v_\alpha(x)\phi(x)\,dx.
\end{aligned}$$

This implies that v_α is the α-th weak derivative of v. Since $v_\alpha \in L^p(\Omega)$, we infer $\partial^\alpha v = v_\alpha$ for all $\alpha \in \mathbb{N}_0^N$ such that $|\alpha| \le k$, hence $v \in W^{k,p}(\Omega)$. Hence,

$$\|u_n - v\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha|\le k} \|\partial^\alpha(u_n - v)\|_p^p\right)^{1/p} = \left(\sum_{|\alpha|\le k} \|\partial^\alpha u_n - v_\alpha\|_p^p\right)^{1/p} \overset{(2.1)}{\to} 0 \quad \text{as } n \to \infty.$$

We have thus proved that (u_n) converges in $W^{k,p}(\Omega)$, which finishes the proof. □

Definition 2.7. $W_0^{k,p}(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{W^{k,p}(\Omega)}}$ *and* $H_0^k(\Omega) := W_0^{k,2}(\Omega)$.

As a closed subspace of $W^{k,p}(\Omega)$ the space $W_0^{k,p}(\Omega)$ is a Banach space (equipped with the same norm as $W^{k,p}(\Omega)$).

Example 2.8.

(a) Consider $u(x) := |x|^\gamma$ for $x \in \Omega := \{y \in \mathbb{R}^N : |y| < 1\}$ and $\gamma \in \mathbb{R} \setminus \{0\}, N \in \mathbb{N}, N \geq 2$. By Proposition 2.3, the only candidate for the weak gradient is $\nabla u := \gamma x |x|^{\gamma-2}$. One may show that this function is indeed the weak partial derivative of u provided that $\gamma > 1 - N$ (which ensures that ∇u is locally integrable). We compute using polar coordinates[5]

$$\int_\Omega |u(x)|^p = \int_{|x|<1} |x|^{\gamma p}\, dx = \int_0^1 r^{N-1} \cdot |\mathbb{S}^{N-1}| r^{\gamma p}\, dr = |\mathbb{S}^{N-1}| \int_0^1 r^{N+\gamma p-1}\, dr = \frac{|\mathbb{S}^{N-1}|}{N+\gamma p}$$

if and only if $N + \gamma p > 0$, otherwise $+\infty$. The same way we get precisely for $N + (\gamma-1)p > 0$

$$\int_\Omega |\nabla u(x)|^p = \int_{|x|<1} |\gamma x |x|^{\gamma-2}|^p\, dx = |\gamma|^p \int_0^1 r^{N-1} \cdot |\mathbb{S}^{N-1}| r^{(\gamma-1)p}\, dr = \frac{|\gamma||\mathbb{S}^{N-1}|}{N+(\gamma-1)p}.$$

We conclude:

$$u \in W^{1,p}(\Omega) \quad \Leftrightarrow \quad N + \gamma p > 0,\ N + (\gamma-1)p > 0 \quad \Leftrightarrow \quad \gamma > 1 - \frac{N}{p}.$$

(b) Set $I := (0,1)$. Assume that $g \in L^p(I)$ and define

$$G(x) := \int_0^x g(t)\, dt.$$

We claim $G \in W^{1,p}(I)$ and $G' = g$ in the weak sense. So let $\phi \in C_0^\infty(I)$ be a test function. Using Fubini's Theorem we get

$$\begin{aligned}
\int_0^1 G(x)\phi'(x)\, dx &= \int_0^1 \left(\int_0^x g(t)\, dt \right) \phi'(x)\, dx \\
&= \int_0^1 \int_0^1 \mathbb{1}_{t \leq x \leq 1}\, g(t)\phi'(x)\, dt\, dx \\
&= \int_0^1 \left(\int_0^1 \mathbb{1}_{t \leq x \leq 1}\, g(t)\phi'(x)\, dx \right) dt \\
&= \int_0^1 g(t) \left(\int_t^1 \phi'(x)\, dx \right) dt \\
&= \int_0^1 g(t) \left(\phi(1) - \phi(t) \right) dt \\
&= -\int_0^1 g(t)\phi(t)\, dt.
\end{aligned}$$

[5]For integrable functions we have

$$\int_{\mathbb{R}^N} u(x)\, dx = \int_0^\infty r^{N-1} \left(\int_{\mathbb{S}^{N-1}} u(r\omega)\, d\sigma(\omega) \right) dr,$$

where $\mathbb{S}^{N-1} = \{\omega \in \mathbb{R}^N : |\omega| = 1\}$ denotes the unit sphere.

So the weak derivative of G is g, i.e., $G' = g$ in the weak sense. Moreover, Hölder's inequality gives

$$\begin{aligned}\int_0^1 |G(x)|^p + |G'(x)|^p \, dx &= \int_0^1 \left| \int_0^x g(t) \, dt \right|^p + |g(x)|^p \, dx \\ &\le \int_0^1 \left(\int_0^x 1 \, dt \right)^{\frac{1}{p'}} \|g\|_p^p + |g(x)|^p \, dx \\ &\le 2\|g\|_p^p < \infty.\end{aligned}$$

We conclude $G \in W^{1,p}(I)$.
Question: Why is this false for $I = [0, \infty)$?

End Lec 02

To prove further elementary properties of Sobolev functions we anticipate the following approximation result (Meyers-Serrin Theorem). Let $k \in \mathbb{N}, 1 \le p < \infty$. Then, for any given $u \in W^{k,p}(\Omega)$, there is a sequence $(u_n)_{n\in\mathbb{N}} \subset C^\infty(\Omega) \cap W^{k,p}(\Omega)$ such that $u_n \to u$ in $W^{k,p}(\Omega)$ and almost everywhere. We will call such sequences "approximating sequences". In particular,

$$\overline{C^\infty(\Omega) \cap W^{k,p}(\Omega)}^{\|\cdot\|_{W^{k,p}(\Omega)}} = W^{k,p}(\Omega) \qquad (k \in \mathbb{N}, 1 \le p < \infty).$$

This is not true for $p = \infty$. To see this choose $\Omega = \{y \in \mathbb{R}^N : |y| < 1\}$ and $u(x) = |x|$. Then $u \in W^{1,\infty}(\Omega)$ and its weak gradient is given by $\nabla u(x) = \frac{x}{|x|}$. If a sequence (u_n) as above existed, then $\partial_1 u$ would be the $L^\infty(\Omega)$-limit of continuous (even smooth) functions. But a uniform limit of continuous functions is continuous, so $x \mapsto \frac{x_1}{|x|}$ would have to be continuous, which is false. Nevertheless, the sequences can be chosen to satisfy

$$\|u_n\|_p \le \|u\|_p \qquad \text{for all } 1 \le p \le \infty, \ n \in \mathbb{N}. \tag{2.2}$$

Proposition 2.9. *Let $\Omega \subset \mathbb{R}^N$ be open and $1 \le p < \infty$.*

(i) ***(Product rule)***[6] *Assume $u, v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Then $uv \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $\partial_i(uv) = v\partial_i u + u\partial_i v$.*

[6] In the case $v \in W^{1,p}(\Omega) \cap C^\infty(\Omega)$ an easier proof without density argument is possible. The observation is that for any test function $\phi \in C_0^\infty(\Omega)$ the function $v\phi$ is again a test function. So if $\partial_i u$ denotes the weak partial derivative, we get

$$\begin{aligned}\int_\Omega uv\partial_i\phi \, dx &= \int_\Omega u\big(\partial_i(v\phi) - \phi\partial_i v\big) \, dx \\ &= -\int_\Omega \partial_i u(v\phi) \, dx - \int_\Omega u\phi\partial_i v \, dx \\ &= -\int_\Omega (v\partial_i u + u\partial_i v)\phi \, dx.\end{aligned}$$

This proves $\partial_i(uv) = v\partial_i u + u\partial_i v$ in the weak sense as claimed.

(ii) ***(Chain rule)*** *Assume $u \in W^{1,p}(\Omega)$ and that $G \in C^1(\mathbb{R})$ has a bounded derivative. Then $G \circ u \in W^{1,p}(\Omega)$ with $\partial_i(G \circ u) = G'(u)\partial_i u$.*

(iii) *Assume $u \in W^{1,p}(\Omega)$. Then $u^+ := \max\{u, 0\}, u^- := \max\{-u, 0\}, |u| \in W^{1,p}(\Omega)$ with*[7]

$$\partial_i u^+ = \partial_i u \cdot \mathbb{1}_{\{u>0\}}, \quad \partial_i u^- = -\partial_i u \cdot \mathbb{1}_{\{u<0\}}, \quad \partial_i |u| = \operatorname{sign}(u)\partial_i u.$$

Beweis:
We first prove (i). Choose approximating sequences $(u_n), (v_n)$. The classical chain rule implies $\partial_i(u_n v_n) = u_n \partial_i v_n + v_n \partial_i u_n$. Then

$$\begin{aligned}
\int_\Omega u(x)v(x)\partial_i\phi(x)\,dx &\overset{?}{=} \lim_{n\to\infty} \int_\Omega u_n(x)v_n(x)\partial_i\phi(x)\,dx \\
&= -\lim_{n\to\infty} \int_\Omega \partial_i(u_n(x)v_n(x))\phi(x)\,dx \\
&= -\lim_{n\to\infty} \int_\Omega \big[(\partial_i u_n)(x)v_n(x) + (\partial_i v_n)(x)u_n(x)\big]\phi(x)\,dx \\
&\overset{?}{=} -\int_\Omega \big[(\partial_i u)(x)v(x) + (\partial_i v)(x)u(x)\big]\phi(x)\,dx
\end{aligned}$$

We justify the equalities with ?. Applying Hölder's inequality a couple of times we get

$$\begin{aligned}
\left|\int_\Omega (u(x)v(x) - u_n(x)v_n(x))\partial_i\phi(x)\,dx\right| &\le \|uv - u_n v_n\|_p \|\partial_i\phi\|_{p'} \\
&\le (\|u(v - v_n)\|_p + \|v_n(u - u_n)\|_p)\|\partial_i\phi\|_{p'} \\
&\le (\|u\|_\infty \|v - v_n\|_p + \|v_n\|_\infty \|u - u_n\|_p)\|\partial_i\phi\|_{p'} \\
&\overset{(2.2)}{\le} (\|u\|_\infty + \|v\|_\infty)(\|v - v_n\|_p + \|u - u_n\|_p)\|\partial_i\phi\|_{p'} \\
&\to 0 \quad (n \to \infty).
\end{aligned}$$

The second equality is a consequence of the Dominated Convergence Theorem[8]: After passing to subsequences still denoted by $(u_n), (v_n)$, we know $\|u_n\|_\infty + \|v_n\|_\infty \le \|u\|_\infty + \|v\|_\infty$ and $|\nabla u_n| + |\nabla v_n| \le w$ for some $w \in L^p(\Omega)$. Hence,

$$|(\partial_i u_n)(x)v_n(x) + (\partial_i v_n)(x)u_n(x)| \le |w(x)|(\|v\|_\infty + \|u\|_\infty) \in L^p(\Omega).$$

So the pointwise almost everyhwere convergence of $(\partial_i u_n)(x)v_n(x) + (\partial_i v_n)(x)u_n(x) \to (\partial_i u)(x)v(x) + (\partial_i v)(x)u(x)$ gives the claim.

We now prove (ii). Since the assumptions imply $G'(u)\partial_i u \in L^p(\Omega)$, it suffices to prove that the i-th weak partial derivative is given by $\partial_i(G \circ u) = G'(u)\partial_i u$. So let $\phi \in C_0^\infty(\Omega)$

[7] $\operatorname{sign}(z) = 1$ if $z > 0$, $\operatorname{sign}(0) = 0$, $\operatorname{sign}(z) = -1$ if $z < 0$.

[8] The Riesz-Fischer Theorem, which establishes the completeness of $L^p(\Omega)$, tells you that $u_n \to u$ in $L^p(\Omega)$ implies $u_n \to u$ almost everywhere and that there is subsequence (u_{n_k}) satisfying $|u_{n_k}| \le w$ for some $w \in L^p(\Omega)$. So $u_n \to u, v_n \to v$ in $W^{1,p}(\Omega)$ implies $u_{n_k} \to u, v_{n_k} \to v, \nabla u_{n_k} \to \nabla u, \nabla v_{n_k} \to \nabla v$ almost everywhere and $|u_{n_k}| + |v_{n_k}| + |\nabla u_{n_k}| + |\nabla v_{n_k}| \le w$ for some $w \in L^p(\Omega)$.
Example: $u_n(x) := \sum_{n\in\mathbb{N}} \mathbb{1}_{[n,1/n]}(x)$ coverges to the trivial function in $L^p(\mathbb{R})$ for $1 \le p < \infty$. In the case $1 < p < \infty$ we can take $w(x) := \sum_{n\in\mathbb{N}} |u_n(x)| \in L^p(\mathbb{R})$. In the case $p = 1$ this is not true, but we may take $w(x) := \sum_{n\in\mathbb{N}} |u_{n^2}(x)| \in L^1(\mathbb{R})$, which is a bound for the subsequence $(u_{n^2})_{n\in\mathbb{N}}$. Notice $\|w\|_1 = \sum_{n\in\mathbb{N}} \frac{1}{n^2} < \infty$.

be given and choose an approximating sequence (u_n) for u. The classical chain rule gives $\partial_i(G\circ u_n) = G'(u_n)\partial_i u_n$ for all $n\in\mathbb{N}$ and hence

$$\begin{aligned}\int_\Omega G(u(x))\partial_i\phi(x)\,dx &\overset{?}{=} \lim_{n\to\infty}\int_\Omega G(u_n(x))\partial_i\phi(x)\,dx\\ &= -\lim_{n\to\infty}\int_\Omega \partial_i\big(G(u_n(x))\big)\phi(x)\,dx\\ &= -\lim_{n\to\infty}\int_\Omega G'(u_n(x))\partial_i u_n(x)\phi(x)\,dx\\ &\overset{?}{=} -\int_\Omega G'(u(x))\partial_i u(x)\phi(x)\,dx\end{aligned}$$

The claim is proved once we have justified the equalities with ?. The first one is a consequence of

$$\begin{aligned}&\left|\int_\Omega (G(u(x)) - G(u_n(x)))\partial_i\phi(x)\,dx\right|\\ &\le \|G(u) - G(u_n)\|_p\|\partial_i\phi\|_{p'}\\ &\le \|G'\|_\infty\|u-u_n\|_p\|\partial_i\phi\|_{p'} \to 0 \quad (n\to\infty).\end{aligned}$$

The second one follows again by the Dominated Convergence Theorem. Notice that $G'(u_n)\to G'(u)$ holds pointwise almost everywhere because G' is continuous.

We prove (iii). Set $G_\varepsilon(z) := \sqrt{z^2+\varepsilon^2} - \varepsilon$ for $\varepsilon>0$. Then

$$|G_\varepsilon(z) - |z|| = \frac{2\varepsilon|z|}{\sqrt{z^2+\varepsilon^2}+\varepsilon+|z|} \le \varepsilon$$

Part (ii) gives $\partial_i(G_\varepsilon(u)) = G_\varepsilon'(u)\partial_i u$ in the weak sense. We thus obtain from the Dominated Convergence Theorem

$$\begin{aligned}\int_\Omega |u(x)|\partial_i\phi(x)\,dx &= \lim_{\varepsilon\to0^+}\int_\Omega G_\varepsilon(u(x))\partial_i\phi(x)\,dx\\ &= -\lim_{\varepsilon\to0^+}\int_\Omega G_\varepsilon'(u(x))\partial_i u(x)\phi(x)\,dx\\ &= -\lim_{\varepsilon\to0^+}\int_\Omega \frac{u(x)}{\sqrt{u(x)^2+\varepsilon^2}}\partial_i u(x)\phi(x)\,dx\\ &= -\int_\Omega \operatorname{sign}(u(x))\partial_i u(x)\phi(x)\,dx.\end{aligned}$$

This proves the claim for $|u|$. The remaining statements are a consequence of $u^+ = \frac12(|u|+u)$ and $u^- = \frac12(|u|-u)$ and the linearity of weak derivatives. □

Similarly, one can prove further elementary properties of Sobolev functions by exploiting the denseness of smooth functions.

3 Lax-Milgram Theorem and Riesz' Representation Theorem

We now show how Sobolev spaces may be used to solve Partial Differential Equations. To this end we go back to (1.1) and study the elliptic boundary value problem

$$-\Delta u(x) + c(x)u(x) = f(x) \quad (x \in \Omega), \qquad u(x) = g(x) \quad (x \in \partial\Omega).$$

As announced earlier, we gradually weaken the hypotheses on c, g, f in our considerations related to this problem. Accordingly, our assumptions on the data are by no means "optimal". We start assuming $g = 0$, $c = 1$ and $f \in L^2(\Omega)$. In this case the above problem takes the form

$$-\Delta u(x) + u(x) = f(x) \quad (x \in \Omega), \qquad u(x) = 0 \quad (x \in \partial\Omega). \tag{3.1}$$

We have shown in Section 1 that the corresponding weak formulation is given by

$$\int_\Omega \nabla\phi(x)\cdot\nabla u(x) + u(x)\phi(x)\,dx = \int_\Omega f(x)\phi(x)\,dx \quad \forall\phi \in C_0^\infty(\Omega), \quad u|_{\partial\Omega} = 0.$$

The boundary conditions are encoded in the solution space. We are thus looking for a function $u \in H_0^1(\Omega)$ satisfying

$$\int_\Omega \nabla\phi(x)\cdot\nabla u(x) + u(x)\phi(x)\,dx = \int_\Omega f(x)\phi(x)\,dx \quad \forall\phi \in C_0^\infty(\Omega).$$

Satz 3.1 (Riesz' Representation Theorem [21]). *Let H be a Hilbert space*[9] *and $l : H \to \mathbb{R}$ a bounded linear functional. Then there is a unique function $u \in H$ such that*

$$\langle u, \phi\rangle = l(\phi) \qquad \text{for all } \phi \in H.$$

Beweis. If l is the trivial functional, we may take $u = 0$ (and that's the only possible choice). So assume l is nontrivial. In that case, we have $\ker(l) \subsetneq H$, so there is $v \in \ker(l)^\perp$ such that $l(v) = 1$. This implies

$$l\big(\phi - l(\phi)v\big) = l(\phi) - l(\phi)l(v) = 0 \qquad \text{for all } \phi \in H,$$

[9] One indeed needs the completeness: $l(f) := \int_0^{1/2} f(x)\,dx$ defines a bounded linear functional on $C([0,1])$ equipped with the L^2-inner product, but $l(\cdot) \neq \langle v, \cdot\rangle$ for any $v \in C([0,1])$ because $\mathbb{1}_{[0,1/2]}$ is not continuous. By extending l to the completion of H one however gets $l(\cdot) = \langle v, \cdot\rangle$ for some v in the completion of $C([0,1])$, namely $v = \mathbb{1}_{[0,1/2]} \in L^2([0,1])$.
Where does the proof fail? It is the existence of $v \in \ker(l)^\perp$ such that $l(v) = 1$. For that, one needs that the kernel (more generally: a closed subspace) admits an orthogonal complement, i.e., $H = \ker(l) \oplus_\perp \ker(l)^\perp$. Recall that the construction of the orthogonal complement uses that Cauchy sequences converge: For $u \in H$ one defines its projection $\pi(u) \in \ker(l)$ onto $\ker(l)$ via $\|\pi(u) - u\| = \inf\{\|v - u\| : v \in \ker(l)\} = \min\{\|v - u\| : v \in \ker(l)\}$, so $u = \pi(u) + (u - \pi(u))$. From this construction: $u - \pi(u) \perp \ker(l)$. The existence of a minimizer is due to the fact that the minimizing sequence (which is a Cauchy sequence) converges. So here is the point where the completeness of H is used.

so $\phi - l(\phi)v \in \ker(l)$ for all $\phi \in H$. Since v is orthogonal to the kernel, we obtain

$$0 = \langle v, \phi - l(\phi)v\rangle = \langle v, \phi\rangle - l(\phi)\langle v, v\rangle \qquad \text{for all } \phi \in H.$$

So the claim follows for $u := \langle v, v\rangle^{-1} v$. (Uniqueness: clear.) $\square$

Korollar 3.2. *Assume* $f \in L^2(\Omega)$. *Then* (3.1) *has a unique weak solution* $u \in H_0^1(\Omega)$ *that satisfies*

$$\|u\|_{1,2} \leq \|f\|_2.$$

Beweis. We apply Riesz' Representation Theorem to the Hilbert space $H_0^1(\Omega)$, equipped with inner product $\langle\cdot,\cdot\rangle_{1,2}$, and the linear functional $l : H_0^1(\Omega) \to \mathbb{R}$ given by $l(\phi) = \int_\Omega f(x)\phi(x)\,dx$. This linear functional is bounded because of

$$|l(\phi)| \leq \|f\|_2\|\phi\|_2 \leq \|f\|_2\|\phi\|_{1,2}.$$

So Riesz' Representation Theorem shows that there is precisely one $u \in H_0^1(\Omega)$ satisfying

$$\int_\Omega \nabla u(x)\cdot\nabla\phi(x) + u(x)\phi(x)\,dx = \langle u, \phi\rangle_{1,2} = l(\phi) = \int_\Omega f(x)\phi(x)\,dx \quad \text{for all } \phi \in H_0^1(\Omega).$$

So (3.1) has precisely one weak solution. It satisfies

$$\|u\|_{1,2}^2 = \langle u, u\rangle_{1,2} = \int_\Omega f(x)u(x)\,dx \leq \|f\|_2\|u\|_{1,2}$$

and the claim follows. $\square$

End Lec 03

In principle, one may apply Riesz' Representation Theorem not only to the standard inner product, but any other equivalent one may be taken. So in fact this result allows to solve a whole family of boundary problems and not only the particular one from (3.1). Anyway, there is a more general result, which is called the Lax-Milgram Lemma. It essentially tells us that the symmetry requirement of an inner product (i.e. $\langle u, v\rangle = \langle v, u\rangle$ $\forall u, v \in H$) is not needed for a solution theory for problems like

$$a(u, v) = l(v) \qquad \forall v \in H. \tag{3.2}$$

Satz 3.3 (Lax-Milgram Lemma [13])**.** *Let* $(H, \langle\cdot,\cdot\rangle)$ *be a (real) Hilbert space, let* $a(\cdot,\cdot) : H \times H \to \mathbb{R}$ *be a bilinear form and* $l : H \to \mathbb{R}$ *a linear functional such that:*

(i) a *is bounded, i.e., there is* $C > 0$ *such that* $|a(u, v)| \leq C\|u\|\|v\|$ *for all* $u, v \in H$,

(ii) a *is coercive, i.e., there is* $c > 0$ *such that* $a(u, u) \geq c\|u\|^2$ *for all* $u \in H$,

(iii) l *is bounded, i.e., there is* $M > 0$ *such that* $|l(v)| \leq M\|v\|$ *for all* $v \in H$.

Then (3.2) *has a unique solution* $u \in H$ *satisfying* $\|u\| \le c^{-1}M$.

Beweis:
For any given $u \in H$, the maps $v \mapsto a(u,v)$ and $v \mapsto l(v)$ are bounded linear functionals by assumption (i) and (iii). So Riesz' Representation Theorem yields uniquely determined elements $w_u, r \in H$ such that

$$a(u,v) = \langle w_u, v\rangle, \qquad l(v) = \langle r, v\rangle.$$

Define $A : H \to H, u \mapsto w_u$. Then we have the following equivalence:

$$a(u,v) = l(v) \quad \forall v \in H \quad \Leftrightarrow \quad \langle Au, v\rangle = \langle r, v\rangle \quad \forall v \in H \quad \Leftrightarrow \quad Au = r.$$

To find a unique solution to this problem we apply Banach's Fixed Point Theorem to

$$T : H \to H, \quad u \mapsto u - \varrho \cdot (Au - r)$$

where $\varrho \neq 0$ will be chosen suitably.

We first show that A is linear and bounded. For any given $u_1, u_2, v \in H$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ we have

$$\begin{aligned}
\langle A(\alpha_1 u_1 + \alpha_2 u_2), v\rangle &= \langle w_{\alpha_1 u_1 + \alpha_2 u_2}, v\rangle \\
&= a(\alpha_1 u_1 + \alpha_2 u_2, v) \\
&= \alpha_1 a(u_1, v) + \alpha_2 a(u_2, v) \\
&= \alpha_1 \langle w_{u_1}, v\rangle + \alpha_2 \langle w_{u_2}, v\rangle \\
&= \langle \alpha_1 w_{u_1} + \alpha_2 w_{u_2}, v\rangle \\
&= \langle \alpha_1 A u_1 + \alpha_2 A u_2, v\rangle.
\end{aligned}$$

This proves the linearity. Moreover, for all $u \in H$,

$$\|Au\|^2 = \langle Au, Au\rangle = a(u, Au) \le C\|u\|\|Au\|.$$

This proves $\|Au\| \le C\|u\|$ for all $u \in H$.

Using these facts we now show that T is a contraction for suitable $\varrho \neq 0$. Using that T is linear, we get for all $u_1, u_2 \in H$

$$\begin{aligned}
\|Tu_1 - Tu_2\|^2 &= \|T(u_1 - u_2)\|^2 \\
&= \|u_1 - u_2 - \varrho \cdot A(u_1 - u_2)\|^2 \\
&= \|u_1 - u_2\|^2 - 2\varrho\langle A(u_1 - u_2), u_1 - u_2\rangle + \varrho^2\|A(u_1 - u_2)\|^2 \\
&= \|u_1 - u_2\|^2 - 2\varrho a(u_1 - u_2, u_1 - u_2) + \varrho^2\|A(u_1 - u_2)\|^2 \\
&\le \|u_1 - u_2\|^2 - 2\varrho c\|u_1 - u_2\|^2 + \varrho^2 C^2\|u_1 - u_2\|^2
\end{aligned}$$

Choosing $\varrho = cC^{-2}$ we thus obtain

$$\|Tu_1 - Tu_2\| \le \sqrt{1 - c^2C^{-2}}\|u_1 - u_2\|.$$

So T is a contraction and hence posses precisely one fixed point. As we have seen above, this implies that (3.2) has a unique solution. This solution, call it u, satisfies $c\|u\|^2 \le a(u,u) = l(u) \le M\|u\|$ so that $\|u\| \le c^{-1}M$ is proved, too. □

We apply this result to problems of the form

$$-\Delta u(x) + c(x)u(x) = f(x) \quad (x \in \Omega), \qquad u(x) = 0 \quad (x \in \partial\Omega). \tag{3.3}$$

A weak solution to this problem $u \in H_0^1(\Omega)$ satisfies $a(u,v) = l(v)$ for all $v \in H$ where

$$\begin{aligned} a(u,v) &:= \int_\Omega \nabla u(x) \cdot \nabla v(x) + c(x)u(x)v(x)\,dx, \\ l(v) &:= \int_\Omega f(x)v(x)\,dx. \end{aligned}$$

Korollar 3.4. *Assume $f \in L^2(\Omega), c \in L^\infty(\Omega)$ with $c(x) \ge \mu > 0$ almost everywhere. Then* (3.3) *has a unique weak solution $u \in H_0^1(\Omega)$ that satisfies*

$$\|u\|_{1,2} \le \min\{1,\mu\}^{-1}\|f\|_2.$$

Beweis. We verify the assumptions of the Lax-Milgram Lemma. (Bi-)Linearity is clear, (i) follows from

$$\begin{aligned} |a(u,v)| &\le \int_\Omega |\nabla u(x)||\nabla v(x)| + |c(x)||u(x)||v(x)|\,dx \\ &\le \max\{1,\|c\|_\infty\} \int_\Omega |\nabla u(x)||\nabla v(x)| + |u(x)||v(x)|\,dx \\ &\le \max\{1,\|c\|_\infty\}\|u\|_{1,2}\|v\|_{1,2}. \end{aligned}$$

Moreover, $|l(v)| \le \|f\|_2\|v\|_2$ as before and

$$\begin{aligned} a(u,u) &= \int_\Omega |\nabla u(x)|^2 + c(x)|u(x)|^2\,dx \\ &\ge \min\{1,\mu\} \int_\Omega |\nabla u(x)|^2 + |u(x)|^2\,dx \\ &= \min\{1,\mu\}\|u\|_{1,2}^2. \end{aligned}$$

So the Lax-Milgram Lemma proves the claim. □

4 Approximation by smooth functions

In this section we want to show that smooth functions approximate Sobolev functions $u \in W^{k,p}(\Omega)$ for $k \in \mathbb{N}, 1 \le p < \infty$. In particular we will prove the existence of approximating sequences $(u_n) \subset C^\infty(\Omega) \cap W^{k,p}(\Omega)$ satisfying (2.2). We start with some preliminaries about test functions.

4.1 Test functions

We first need to establish the mere existence of test functions. The starting point is the following fact about

$$\zeta(x) := \begin{cases} e^{-1/x} & \text{if } x > 0, \\ 0 & \text{if } x \le 0. \end{cases}$$

Proposition 4.1. $\zeta \in C^\infty(\mathbb{R})$.

The main difficulty is to inductively prove $\zeta^{(n)}(x) = p_n(x) x^{-2n} e^{-1/x}$ for all $x \in (0,\infty)$ where p_n is a polynomial (of degree $\le n$). Using this and $e^{-z} z^m \to 0$ as $z \to \infty$ for all $m \in \mathbb{N}$ one gets the result. Notice that this counterexample shows that there are $C^\infty(\mathbb{R})$-functions that are not real-analytic[10]. The following result establishes the existence of cut-off (or bump) functions, which are a special kind of test functions from $C_0^\infty(\Omega)$.

Proposition 4.2. *Let $\Omega \subset \mathbb{R}^N$ be open, $x_0 \in \Omega$ and $0 < r < R < \operatorname{dist}(x_0, \partial\Omega)$. Then there is $\psi \in C_0^\infty(\Omega)$ such that*

$$0 \le \psi(x) \le 1, \qquad \psi(x) = 1 \textit{ for } x \in \overline{B_r(x_0)}, \qquad \psi(x) = 0 \textit{ for } x \in B_R(x_0)^c$$

as well as $|\nabla\psi(x)| \le C|R-r|^{-1}$ for all $x \in B_R(x_0) \setminus B_r(x_0)$ and some $C > 0$. In particular, $C_0^\infty(\Omega) \supsetneq \{0\}$.

Beweis:
Choose ζ as in Proposition 4.1 and define $\psi_1 \in C^\infty(\mathbb{R})$ via $\psi_1(t) := \zeta(1-t)\zeta(t)$, in particular $\psi_1 \ge 0, \operatorname{supp}(\psi_1) = [0,1]$. As a consequence,

$$0 \le \psi_2 \le 1, \quad \psi_2|_{(-\infty,0]} \equiv 1, \quad \psi_2|_{[1,\infty)} \equiv 0 \quad \text{where } \psi_2(t) := \frac{\int_t^\infty \psi_1(s)\,ds}{\int_{\mathbb{R}} \psi_1(s)\,ds}.$$

Then $\psi(x) := \psi_2(\frac{|x-x_0|-r}{R-r})$ has all the desired properties. □

[10]Notice that real-analytic functions have isolated zeros whereas the zero 0 is not an isolated one of ζ.

They are the building blocks for the following more general result that allows to "localize" the considerations. We will see an example for this in the proof of the Meyers-Serrin Theorem.

Satz 4.3 (Partition of Unity). *Let I be a set and $(O_i)_{i\in I}$ a family of open subsets of $\mathbb{R}^N$, $\Omega := \bigcup_{i\in I} O_i$. Then there is a sequence $(\phi_j)_{j\in\mathbb{N}} \subset C_0^\infty(\Omega)$ with the following properties:*

(i) $0 \le \phi_j(x) \le 1$ *for all* $x \in \Omega$ *for all* $j \in \mathbb{N}$,

(ii) $\operatorname{supp}(\phi_j) \subset O_{i(j)}$ *for some* $i(j) \in I$ *for all* $j \in \mathbb{N}$,

(iii) $\sum_{j=1}^\infty \phi_j(x) = 1$ *for all* $x \in \Omega$,

(iv) For each compact set $K \subset \Omega$ *there is an* $m \in \mathbb{N}$ *and an open set* W *such that*

$$K \subset W \subset \Omega \quad \text{and} \quad \phi_1(x) + \ldots + \phi_m(x) = 1 \quad \text{for all } x \in W.$$

Beweis:
We define the set of open balls[11]

$$\mathfrak{B} = \left\{ B_r(q) \,:\, q \in \mathbb{Q}^n, r \in \mathbb{Q} \text{ such that } \overline{B_r(q)} \subset O_i \text{ for some } i \in I \right\}.$$

Since $\mathfrak{B}$ is bijective to a subset of $\mathbb{Q}^n \times \mathbb{Q}$, it is countable. So we may write $\mathfrak{B} = \{B_{r_j}(q_j) : j \in \mathbb{N}\}$. Proposition 4.2 provides functions $\psi_j \in C_0^\infty(\Omega)$ satisfying

$$0 \le \psi_j \le 1, \qquad \psi_j = 1 \text{ on } \overline{B_{r_j/2}(q_j)}, \qquad \psi_j = 0 \text{ on } B_{r_j}(q_j)^c. \tag{4.1}$$

Then define

$$\phi_1 := \psi_1, \qquad \phi_j := (1-\psi_1)\cdot\ldots\cdot(1-\psi_{j-1})\psi_j \quad (j \in \mathbb{N}, j \ge 2).$$

Then (i) and (ii) are clear and it remains to prove (iii),(iv).

One inductively proves

$$\phi_1 + \ldots + \phi_j = 1 - (1-\psi_1)\cdot\ldots\cdot(1-\psi_j) \qquad \text{for all } j \in \mathbb{N}.$$

For any given compact subset $K \subset \Omega$ we have[12] $K \subset \bigcup_{j=1}^m B_{r_j/2}(q_j) =: W$ for some $m \in \mathbb{N}$. Hence $(1-\psi_1(x))\cdot\ldots\cdot(1-\psi_m(x)) = 0$ for $x \in W$. We obtain for all $n \in \mathbb{N}, n \ge m$

$$(\phi_1 + \ldots + \phi_n)(x) = 1 - \underbrace{(1-\psi_1(x))\cdot\ldots\cdot(1-\psi_m(x))}_{=0}\cdot\ldots\cdot(1-\psi_n(x)) = 1$$

This proves (iii) and (iv). □

[11] We call them $B_r(q) := \{x \in \mathbb{R}^N : |x-q| < r\}$.
[12] Here we use that $K \subset \Omega \subset \{B_{r_j/2}(q) : j \in \mathbb{N}\}$. Prove this!

A particularly important role is played by so-called "mollifiers". These are test functions $\varphi \in C_0^\infty(\mathbb{R}^N)$ with $\varphi \geq 0, \operatorname{supp}(\varphi) \subset \overline{B_1(0)}$ and $\int_{\mathbb{R}^N} \varphi(x)\,dx = 1$. Proposition 4.2 tells us that such functions exist. Considering

$$\varphi_\varepsilon(x) := \frac{1}{\varepsilon^N}\varphi(\frac{x}{\varepsilon})$$

we obtain a mollifying sequence satisfying

$$\varphi_\varepsilon \geq 0, \qquad \operatorname{supp}(\varphi_\varepsilon) \subset \overline{B_\varepsilon(0)}, \qquad \int_{\mathbb{R}^N} \varphi_\varepsilon(x)\,dx = 1.$$

4.2 Convolution with mollifiers

We first define the convolution of two nonnegative measurable functions $f, g : \mathbb{R}^N \to [0, \infty]$.

$$(f * g)(x) = \int_{\mathbb{R}^N} f(x-y)g(y)\,dy \qquad (x \in \mathbb{R}^N).$$

Proposition 4.4 (Young [27]). *Assume* $1 \leq p, q, r \leq \infty$ *and* $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. *Then:*

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

Beweis:
We only consider $1 \leq p, q, r < \infty$. This is a consequence of the following application of Hölder's inequality (notice $r > p, r > q$ and $1 = \frac{1}{r} + \frac{1}{\frac{pr}{r-p}} + \frac{1}{\frac{qr}{r-q}}$) and Tonelli's Theorem:

$$\begin{aligned}
\|f * g\|_r^r &= \int_{\mathbb{R}^N} |f * g|^r\,dx \\
&\leq \int_{\mathbb{R}^N} \Big(\int_{\mathbb{R}^N} |f(y)||g(x-y)|\,dy \Big)^r dx \\
&= \int_{\mathbb{R}^N} \Big(\int_{\mathbb{R}^N} \big(|f(y)|^{\frac{p}{r}} |g(x-y)|^{\frac{q}{r}}\big) \cdot |f(y)|^{\frac{r-p}{r}} \cdot |g(x-y)|^{\frac{r-q}{r}}\,dy \Big)^r\,dx \\
&\leq \int_{\mathbb{R}^N} \Big(\int_{\mathbb{R}^N} |f(y)|^p |g(x-y)|^q\,dy \Big) \Big(\int_{\mathbb{R}^N} |f(y)|^p\,dy \Big)^{\frac{r-p}{p}} \Big(\int_{\mathbb{R}^N} |g(x-t)|^q\,dt \Big)^{\frac{r-q}{q}}\,dx \\
&= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |f(y)|^p |g(x-y)|^q\,dy\,dx \cdot \|f\|_p^{r-p} \|g\|_q^{r-q} \\
&= \int_{\mathbb{R}^N} |f(y)|^p\,dy \|g\|_q^q \cdot \|f\|_p^{r-p} \|g\|_q^{r-q} \\
&= \|f\|_p^r \|g\|_q^r.
\end{aligned}$$

The claim for $p = \infty$ or $q = \infty$ or $r = \infty$ is proved analogously. □

In particular, convolution is a well-defined operation $L^p(\mathbb{R}^N) * L^q(\mathbb{R}^N) \subset L^r(\mathbb{R}^N)$ with the corresponding inequality. One checks that the following rules hold for $f, g, h \in C_0^\infty(\mathbb{R}^N)$:

(i) $f * g = g * f$,

(ii) $(f * g) * h = f * (g * h)$,

(iii) $\operatorname{supp}(f * g) \subset \overline{\operatorname{supp}(f) + \operatorname{supp}(g)} = \overline{\{x + y : x \in \operatorname{supp}(f), y \in \operatorname{supp}(g)\}}$,

(iv) $\partial^\alpha (f * g) = \partial^\alpha f * g = f * \partial^\alpha g$,

(v) $\int_{\mathbb{R}^N} (f * g) h \, dx = \int_{\mathbb{R}^N} f (g * h) \, dx$.

We will prove these identities in the exercise sessions. The strong hypothesis $f, g, h \in C_0^\infty(\mathbb{R}^N)$ is chosen here for simplicity. Each item (i)-(v) actually holds for a much more general class of functions.

End Lec 04

4.3 Approximation of $L^p(\Omega)$-functions

For a given function $u \in L^p(\Omega)$, i.e., $u \mathbb{1}_\Omega \in L^p(\mathbb{R}^N)$, we consider the convolution products

$$u_\varepsilon(x) := (\varphi_\varepsilon * u)(x) = \int_\Omega \varphi_\varepsilon(x - y) u(y) \, dy.$$

Our aim is to prove $u_\varepsilon \to u$ in $L^p(\mathbb{R}^N)$. To this end, we first subsequently approximate u by the classes of functions:

- step functions,
- step functions with compact support inside Ω,
- continuous functions with compact support inside Ω,
- smooth functions with compact support inside Ω.

In the last step we will use convolution for mollification. In order to pass from step functions to continuous functions, we need to approximate indicator function $\mathbb{1}_A$ of measurable subsets $A \subset \mathbb{R}^N$ with finite measure $|A|$. We use the fact that the Lebesgue measure is regular.

Lemma 4.5. *Let $A \subset \mathbb{R}^N$ measurable, $|A| < \infty$. Then, for every $\varepsilon > 0$, there is a compact set $K \subset \mathbb{R}^N$ and an open set $O \subset \mathbb{R}^N$ such that*

$$K \subset A \subset O, \qquad |O \setminus K| < \varepsilon.$$

This can be used as follows. In the situation of the Lemma, consider the continuous[13] function

$$\phi(x) := \frac{\operatorname{dist}(x, O^c)}{\operatorname{dist}(x, O^c) + \operatorname{dist}(x, K)} \qquad (x \in \mathbb{R}^N).$$

[13] Prove this!

We want to show that it is a good L^p-approximation for the indicator function whenever $1 \le p < \infty$. It satisfies $\phi(x) = \mathbb{1}_A(x) = 0$ for $x \in O^c$ as well as $\phi(x) = \mathbb{1}_A(x) = 1$ for $x \in K$. Moreover, $0 \le \phi - \mathbb{1}_A \le 1$ on $\mathbb{R}^N$. Hence,

$$\|\phi - \mathbb{1}_A\|_{L^p(\mathbb{R}^N)}^p = \|\phi - \mathbb{1}_A\|_{L^p(O \setminus K)}^p \le \|1\|_{L^p(O \setminus K)}^p = |O \setminus K| < \varepsilon.$$

We now generalize this idea as follows.

Proposition 4.6. *Let $\Omega \subset \mathbb{R}^N$ be open and $1 \le p < \infty$. Then $C_0(\Omega)$ is dense in $L^p(\Omega)$.*

Beweis:
Let $u \in L^p(\Omega)$. By construction of the Lebesgue measure there is a step function $s = \sum_{j=1}^M a_j \mathbb{1}_{A_j} \in L^p(\Omega)$ with

$$\|u - s\|_{L^p(\Omega)} \le \frac{\delta}{4}.$$

By the Dominated Convergence Theorem[14] there is a compact subset $K \subset \Omega$ such that

$$\|s\|_{L^p(\Omega \setminus K)} \le \frac{\delta}{4}.$$

Following the ideas from above, we find continuous functions $\phi_1, \dots, \phi_M \in C_0(\Omega)$ as above with

$$\|\mathbb{1}_{A_j \cap K} - \phi_j\|_{L^p(\Omega)} \le \frac{\delta}{2M(|a_j| + 1)} \quad (j = 1, \dots, M).$$

Supports inside Ω can be achieved because $A_j \cap K$ has compact support[15] inside Ω. We define

$$v := \sum_{j=1}^M a_j \phi_j \in C_0(\Omega).$$

This function satisfies

$$\begin{aligned}
\|u - v\|_{L^p(\Omega)} &\le \|u - s\|_{L^p(\Omega)} + \|s - s\mathbb{1}_K\|_{L^p(\Omega)} + \|s\mathbb{1}_K - v\|_{L^p(\Omega)} \\
&\le \frac{\delta}{4} + \|s\|_{L^p(\Omega \setminus K)} + \|s\mathbb{1}_K - v\|_{L^p(\Omega)} \\
&\le \frac{\delta}{4} + \frac{\delta}{4} + \left\| \sum_{j=1}^M a_j \mathbb{1}_{A_j \cap K} - \sum_{j=1}^M a_j \phi_j \right\|_{L^p(\Omega)} \\
&\le \frac{\delta}{2} + \sum_{j=1}^M |a_j| \|\mathbb{1}_{A_j \cap K} - \phi_j\|_{L^p(\Omega)} \\
&\le \frac{\delta}{2} + \sum_{j=1}^M |a_j| \frac{\delta}{2M(|a_j| + 1)} \\
&\le \delta.
\end{aligned}$$

[14] Apply this for instance to the sequence $v_n := s\mathbb{1}_{K_n}$ where $K_n := \{x \in \Omega : |x| \le n, \operatorname{dist}(x, \partial\Omega) \ge \frac{1}{n}\}$.
[15] Indeed: If not you may consider $\phi_j \chi$ instead, where $\chi \in C_0^\infty$ satisfies $0 \le \chi \le 1$ and $\chi(x) = 1$ on K.

This proves the claim. $\square$

Proposition 4.7. *Assume $u \in L^p(\mathbb{R}^N)$. Then $u_\varepsilon \in L^p(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$ with*

$$\begin{aligned} &\|u_\varepsilon - u\|_{L^p(\mathbb{R}^N)} \to 0 && \text{as } \varepsilon \to 0,\ 1 \le p < \infty,\\ &\|u_\varepsilon\|_{L^p(\mathbb{R}^N)} \le \|u\|_{L^p(\mathbb{R}^N)} && \text{for } \varepsilon > 0,\ 1 \le p \le \infty. \end{aligned}$$

Beweis:
$u_\varepsilon \in C^\infty(\mathbb{R}^N)$ follows from (iv). Young's inequality gives for $1 \le p \le \infty$ and $\varepsilon > 0$

$$\|u_\varepsilon\|_{L^p(\mathbb{R}^N)} = \|\varphi_\varepsilon * u\|_{L^p(\mathbb{R}^N)} \le \|\varphi_\varepsilon\|_{L^1(\mathbb{R}^N)} \|u\|_{L^p(\mathbb{R}^N)} = \|u\|_{L^p(\mathbb{R}^N)}.$$

It remains to prove $u_\varepsilon \to u$ in $L^p(\mathbb{R}^N)$ for $1 \le p < \infty$ as $\varepsilon \to 0$. So let $\delta > 0$ be arbitrary. Proposition 4.6 yields $v \in C_0(\mathbb{R}^N)$ with

$$\|u - v\|_{L^p(\mathbb{R}^N)} \le \frac{\delta}{4}.$$

We then have

$$\begin{aligned} \|u_\varepsilon - u\|_{L^p(\mathbb{R}^N)} &\le \|u_\varepsilon - v_\varepsilon\|_{L^p(\mathbb{R}^N)} + \|v_\varepsilon - v\|_{L^p(\mathbb{R}^N)} + \|v - u\|_{L^p(\mathbb{R}^N)} \\ &\le \|(u - v)_\varepsilon\|_{L^p(\mathbb{R}^N)} + \|v_\varepsilon - v\|_{L^p(\mathbb{R}^N)} + \|u - v\|_{L^p(\mathbb{R}^N)} \\ &\le 2\|u - v\|_{L^p(\mathbb{R}^N)} + \|v_\varepsilon - v\|_{L^p(\mathbb{R}^N)} \\ &\le \frac{\delta}{2} + \|v_\varepsilon - v\|_{L^p(\mathbb{R}^N)}. \end{aligned}$$

So it remains to show that the latter term tends to zero as $\varepsilon \to 0$. Choose $K \subset \mathbb{R}^N$ a compact superset of $\operatorname{supp}(v) + \overline{B_1(0)}$. Then $\operatorname{supp}(v_\varepsilon), \operatorname{supp}(v) \subset K$ for $0 < \varepsilon < 1$ and we obtain

$$\begin{aligned} \|v_\varepsilon - v\|_{L^\infty(K)} &= \max_{x \in K} \left| \int_{\mathbb{R}^N} \varphi_\varepsilon(x - y) v(y)\, dy - v(x) \right| \\ &= \max_{x \in K} \left| \int_K \varphi_\varepsilon(x - y)(v(y) - v(x))\, dy \right| \\ &\le \sup_{\substack{|x-y| \le \varepsilon,\\ x,y \in K}} |v(y) - v(x)| \int_K \varphi_\varepsilon(x - y)\, dy \\ &\le \sup_{\substack{|x-y| \le \varepsilon,\\ x,y \in K}} |v(y) - v(x)| \\ &\to 0 \quad \text{as } \varepsilon \to 0. \end{aligned}$$

Since v is uniformly continuous, this last expression tends to zero as $\varepsilon \to 0$. So we can choose $\varepsilon > 0$ so small that $0 < \varepsilon < \varepsilon_0$ implies

$$\|v_\varepsilon - v\|_{L^p(\mathbb{R}^N)} = \|1 \cdot (v_\varepsilon - v)\|_{L^p(K)} = \underbrace{\|1\|_{L^p(K)}}_{< \infty} \|v_\varepsilon - v\|_{L^\infty(K)} \le \delta.$$

This proves the claim. □

Satz 4.8. *Let $\Omega \subset \mathbb{R}^N$ be open. Then $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$.*

Beweis:
Let $u \in L^p(\Omega)$ and $\delta > 0$. As above, the Dominated Convergence Theorem yields a compact subset $K \subset \Omega$ such that $\|u\|_{L^p(\Omega\setminus K)} \le \frac{\delta}{2}$. For $0 < \varepsilon < \operatorname{dist}(K, \partial\Omega)$ set $v_\varepsilon := (u\mathbb{1}_K)_\varepsilon \in C_0^\infty(\Omega)$. Proposition 4.7 shows $\|u\mathbb{1}_K - (u\mathbb{1}_K)_\varepsilon\|_{L^p(\mathbb{R}^N)} \le \frac{\delta}{2}$ for small enough $\varepsilon > 0$, so

$$\|u - v_\varepsilon\|_{L^p(\Omega)} \le \|u\mathbb{1}_K - (u\mathbb{1}_K)_\varepsilon\|_{L^p(\Omega)} + \frac{\delta}{2} = \|u\mathbb{1}_K - (u\mathbb{1}_K)_\varepsilon\|_{L^p(\mathbb{R}^N)} + \frac{\delta}{2} \le \delta,$$

which proves the claim. □

NB: This approximation by smooth functions with compact support is possible in $L^p(\Omega)$, but in most cases not for $W^{k,p}(\Omega)$ with $k \ge 1$. The reason is that "cutting away the regions close to the boundary" produces large derivatives. Later on, we will prove Poincaré's inequality for functions $W_0^{k,p}(\Omega)$ that obviously does not hold for functions from $W^{k,p}(\Omega)$ for reasonable $\Omega \subset \mathbb{R}^N$. This will provide an indirect proof of $W_0^{k,p}(\Omega) \subsetneq W^{k,p}(\Omega)$.

4.4 Approximation of $W^{k,p}(\Omega)$-functions

Proposition 4.9. *Let $\Omega \subset \mathbb{R}^N$ be an open set and $u \in W^{k,p}(\Omega), k \in \mathbb{N}, 1 \le p < \infty$. Let $\chi \in C_0^\infty(\Omega)$. Then $(u\chi)_\varepsilon \to u\chi$ in $W^{k,p}(\Omega)$.*

Beweis:
The product rule[16] from Proposition 2.9 shows $v := u\chi \in W^{k,p}(\Omega)$ and, as a function trivially extended to $\mathbb{R}^N$, $v \in W^{k,p}(\mathbb{R}^N)$ because χ (and all its derivatives) has compact support inside Ω. So property (iv) of the convolution product for functions from $W^{k,p}(\mathbb{R}^N)$ (see the Exercise sheet) implies

$$\partial^\alpha v_\varepsilon = \partial^\alpha(\varphi_\varepsilon * v) = \varphi_\varepsilon * \partial^\alpha v = (\partial^\alpha v)_\varepsilon.$$

Proposition 4.7 then gives for $0 < \varepsilon < \varepsilon_0$ sufficiently small

$$\begin{aligned}\|v_\varepsilon - v\|_{W^{k,p}(\Omega)}^p &= \sum_{|\alpha|\le k} \left\|\partial^\alpha\big(v_\varepsilon - v\big)\right\|_{L^p(\Omega)}^p \\ &= \sum_{|\alpha|\le k} \left\|\partial^\alpha\big(v_\varepsilon - v\big)\right\|_{L^p(\mathbb{R}^N)}^p\end{aligned}$$

[16]Notice that $\chi \in C_0^\infty(\Omega)$ implies that the proof of the product rule actually does not rely on the approximation result that we are about to prove. So no danger of circular reasoning!

$$= \sum_{|\alpha|\le k} \|(\partial^\alpha v)_\varepsilon - \partial^\alpha v\|^p_{L^p(\mathbb{R}^N)}$$
$$\le \delta^p.$$

This proves the claim. □

Satz 4.10 (Meyers,Serrin (1964) [14]). *Let $\Omega \subset \mathbb{R}^N$ be open and $1 \le p < \infty$. Then $W^{k,p}(\Omega) = \overline{C^\infty(\Omega) \cap W^{k,p}(\Omega)}^{\|\cdot\|_{W^{k,p}(\Omega)}}$.*

Beweis:
For $k \in \mathbb{N}$ define the open sets

$$\Omega_j := \left\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \frac{1}{j}\right\}, \qquad U_j := \Omega_j \setminus \overline{\Omega_{j-2}},$$

where $\Omega_{-1} := \Omega_0 := \varnothing$. Then $(U_j)_{j\in\mathbb{N}}$ is an open covering of Ω. Choose some subordinate partition of unity $(\psi_j)_{j\in\mathbb{N}}$, see Theorem 4.3.

Let $u \in W^{k,p}(\Omega)$ and $\varepsilon > 0$ be arbitrary. Since $\operatorname{supp}(u\psi_j) \subset U_j \setminus \overline{U_{j-2}} \subset \Omega$, Proposition 4.9 yields a mollifier $\varphi_{\varepsilon_j} \in C_0^\infty(\mathbb{R}^n)$ such that $v_j := (u\psi_j)_{\varepsilon_j} = \varphi_{\varepsilon_j} * (u\psi_j)$ satisfies

$$\operatorname{supp}(v_j) \subset U_{j+1} \setminus \overline{U_{j-3}}, \qquad \|v_j - u\psi_j\|_{W^{k,p}(\Omega)} \le \varepsilon 2^{-j}.$$

Set $v := \sum_{j=1}^\infty v_j$. Then $v \in C^\infty(\Omega)$ since v is a locally finite sum, see Theorem 4.3 (iv). Moreover,

$$\|v - u\|_{W^{k,p}(\Omega)} = \left\| \sum_{j=1}^\infty v_j - \sum_{j=1}^\infty u\psi_j \right\|_{W^{k,p}(\Omega)} \le \sum_{j=1}^\infty \|v_j - u\psi_j\|_{W^{k,p}(\Omega)} \le \sum_{j=1}^\infty \varepsilon 2^{-j} \le \varepsilon.$$

This is all we had to show. □

This holds regardless of any regularity assumptions on the boundary of Ω. The situation is different if we require the approximating sequence (u_n) to be an element of $C^\infty(\mathbb{R}^N) \cap W^{k,p}(\mathbb{R}^N)$ or even $C_0^\infty(\mathbb{R}^N)|_\Omega := \{u|_\Omega : u \in C_0^\infty(\mathbb{R}^N)\}$. For the proof of the following result we refer to [1, Theorem 3.22, §4.11].

Satz 4.11. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain and $1 \le p < \infty$. Then*

$$W^{k,p}(\Omega) = \overline{C_0^\infty(\mathbb{R}^N)|_\Omega}^{\|\cdot\|_{W^{k,p}(\Omega)}}.$$

This result extends to many important unbounded "uniform" Lipschitz domains where, essentially, the boundary of Ω can be written as a graph of Lipschitz functions with uniformly bounded Lipschitz constants. Notice that for "generic"[17] open sets $\Omega \neq \mathbb{R}^N$ we have that the closure of $C_0^\infty(\mathbb{R}^N)|_\Omega$ is a strict superset of the closure of $C_0^\infty(\Omega)$. The case $\Omega = \mathbb{R}^N$ (no boundary at all) is the only important exception.

Lemma 4.12. *Let $k \in \mathbb{N}, 1 \leq p < \infty$. Then $W_0^{k,p}(\mathbb{R}^N) = W^{k,p}(\mathbb{R}^N)$.*

Beweis. We only prove the result for $k = 1$ to avoid technicalities (i.e. the product rule for higher derivatives). Let $u \in W^{1,p}(\mathbb{R}^N)$ and choose a cut-off function $\phi \in C_0^\infty(\mathbb{R}^N)$ as in Proposition 4.2 with $0 \leq \phi \leq 1$, $\phi(x) = 1$ for $|x| \leq 1$ and $\phi(x) = 0$ for $|x| \geq 2$. Set $\phi_R(x) := \phi(x/R)$. We claim $u\phi_R \to u$ in $W^{1,p}(\Omega)$. Indeed,

$$\begin{aligned}
\|\partial_i(u\phi_R) - \partial_i u\|_{L^p(\mathbb{R}^N)} &= \|\partial_i u(\phi_R - 1) + u\partial_i\phi_R\|_{L^p(\mathbb{R}^N)} \\
&\leq \|\partial_i u(\phi_R - 1)\|_{L^p(\mathbb{R}^N)} + \|u\partial_i\phi_R\|_{L^p(\mathbb{R}^N)} \\
&\leq \|\partial_i u(\phi_R - 1)\|_{L^p(\mathbb{R}^N)} + \frac{1}{R}\|\partial_i\phi\|_\infty \|u\|_{L^p(\mathbb{R}^N)}
\end{aligned}$$

The first term converges to zero because of the Dominated Convergence Theorem because of $|\partial_i u(\phi_R - 1)| \leq |\partial_i u| \in L^p(\mathbb{R}^N)$ and $\phi_R \to 1$ pointwise almost everywhere. So we get

$$\lim_{R\to\infty} \|\partial_i(u\phi_R) - \partial_i u\|_{L^p(\mathbb{R}^N)} = 0 \qquad (i = 1, \ldots, N).$$

Similarly, the Dominated Convergence Theorem gives

$$\lim_{R\to\infty} \|u\phi_R - u\|_{L^p(\mathbb{R}^N)} = 0.$$

This proves $u\phi_R \to u \in W^{1,p}(\mathbb{R}^N)$. So Proposition 4.9 (with $\Omega = \mathbb{R}^N, \chi = \phi_R$) shows that the function $(u\phi_R)_\varepsilon \in C_0^\infty(\mathbb{R}^N)$ converges to $u\phi_R$ as $\varepsilon \to 0$. Hence, $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N)$, i.e.,

$$W^{1,p}(\mathbb{R}^N) \subset \overline{C_0^\infty(\mathbb{R}^N)}^{\|\cdot\|_{W^{1,p}(\mathbb{R}^N)}} = W_0^{1,p}(\mathbb{R}^N) \subset W^{1,p}(\mathbb{R}^N).$$

This proves the claim (for $k = 1$). □

Korollar 4.13. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain and $u \in W^{k,p}(\Omega), k \in \mathbb{N}, 1 \leq p < \infty$. Then $u_\varepsilon \to u$ in $W^{k,p}(\Omega)$ as $\varepsilon \to 0$.*

Beweis:
Let $(u_n) \subset C_0^\infty(\mathbb{R}^N)$ be an approximating sequence given by Theorem 4.11. Then

$$\|u_\varepsilon - u\|_{W^{k,p}(\Omega)} \leq \|u_\varepsilon - (u_n)_\varepsilon\|_{W^{k,p}(\Omega)} + \|(u_n)_\varepsilon - u_n\|_{W^{k,p}(\Omega)} + \|u_n - u\|_{W^{k,p}(\Omega)}$$

[17] $\Omega = \mathbb{R}^N \setminus \{0\}$ is not such a generic open set.

$$\leq 2\|u_n - u\|_{W^{k,p}(\Omega)} + \|(u_n)_\varepsilon - u_n\|_{W^{k,p}(\Omega)}.$$

So, for any given $\delta > 0$ we may choose $n \in \mathbb{N}$ such that

$$\|u_n - u\|_{W^{k,p}(\Omega)} \leq \frac{\delta}{4}.$$

On the other hand, by Proposition 4.9 for $\Omega = \mathbb{R}^N$ and $\chi \in C_0^\infty(\mathbb{R}^N)$ satisfying $\chi = 1$ on the support of u_n, we have $u_n = u_n\chi$ and hence

$$\|(u_n)_\varepsilon - u_n\|_{W^{k,p}(\Omega)} = \|(u_n\chi)_\varepsilon - u_n\chi\|_{W^{k,p}(\Omega)} \leq \frac{\delta}{2} \quad \text{for } 0 < \varepsilon < \varepsilon_0.$$

Taking these two estimates together, we obtain

$$\|u_\varepsilon - u\|_{W^{k,p}(\Omega)} \leq 2 \cdot \frac{\delta}{4} + \frac{\delta}{2} = \delta \qquad \text{for all } \varepsilon \in (0, \varepsilon_0).$$

This means $u_\varepsilon \to u$ in $W^{k,p}(\Omega)$ as $\varepsilon \to 0$, which is all we had to show. □

End Lec 05

5 Stein's Extension Theorem

In this section we want to prove Stein's Extension Theorem [23]. It states that for bounded Lipschitz domains[18] $\Omega \subset \mathbb{R}^N$ each function $u \in W^{k,p}(\Omega)$ with $k \in \mathbb{N}, p \in [1, \infty]$ admits an extension $Eu \in W^{k,p}(\mathbb{R}^N)$ such that[19]

$$(Eu)|_\Omega = u \qquad \text{and} \qquad \|Eu\|_{W^{k,p}(\mathbb{R}^N)} \leq C^* \|u\|_{W^{k,p}(\Omega)}.$$

We will show (indirectly) that this requirement on the boundary regularity of Ω is close to optimal. In fact, the result is not true for mere $C^{0,\alpha}$-domains with $0 < \alpha < 1$ such as $\Omega := \{(x,y) \in (0,1) : 0 < x < 1, 0 < y < x^{1+\delta}\}$ with $\delta > 0$. We recall that a bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$ is such that $\partial\Omega \subset \bigcup_{j=1}^M U_j \subset \mathbb{R}^N$ for open sets $U_1, \ldots, U_M$ such that, after "permutation of coordinates",

$$\begin{aligned} \partial\Omega \cap U_j &= \{(x', x_N) \in U_j : x_N = \psi_j(x')\}, \\ \Omega \cap U_j &= \{(x', x_N) \in U_j : x_N > \psi_j(x')\} \end{aligned}$$

for Lipschitz-continuous functions $\psi_j : \mathbb{R}^{N-1} \to \mathbb{R}$.

[18]This actually also holds for unbounded domains with the "strong local Lipschitz property". For typical unbounded domains (half-spaces, paraboloids, etc.) this is satsified. Since the technicalities are even larger, we do not insist on this generalization.

[19]Such an extension operator is not uniquely determined since what happens away from Ω is not so important. For instance, instead of $u \mapsto Eu$ one may consider $u \mapsto (Eu)\chi$ where $\chi \in C_0^\infty(\mathbb{R}^N)$ satisfies $\chi|_\Omega \equiv 1$.

Why is it interesting and of practical relevance to have such an operator? Assume you want to prove some estimate of the form

$$\|Tu\|_{L^q(\Omega)} \le C(\Omega)\|u\|_{W^{k,p}(\Omega)}$$

for some linear operator T. We will assume this operator to satisfy

$$\|T(U \cdot \mathbb{1}_\Omega)\|_{L^q(\Omega)} \le D\|T(U)\|_{L^q(\mathbb{R}^N)} \qquad (U \in W^{k,p}(\mathbb{R}^N))$$

for some positive constant D. This is for instance satisfied for the identity operator $T = \mathrm{id}$ or integral operators $TU(x) = \int_{\mathbb{R}^N} K(x,y)U(y)\,dy$ with nonnegative kernels K. We show that estimates for such operators can be obtained with the aid of the corresponding estimates on $\mathbb{R}^N$ that are sometimes easier to prove. Having proved the latter and having an extension operator E as above at our disposal, one obtains the desired estimate on Ω for free. Indeed,

$$\begin{aligned}\|Tu\|_{L^q(\Omega)} &= \|T(Eu \cdot \mathbb{1}_\Omega)\|_{L^q(\Omega)} \\ &\le D\|T(Eu)\|_{L^q(\mathbb{R}^N)} \\ &\le D\,C(\mathbb{R}^N)\|Eu\|_{W^{k,p}(\mathbb{R}^N)} \\ &\le D\,C(\mathbb{R}^N)C^*\|u\|_{W^{k,p}(\Omega)}.\end{aligned}$$

We start with a technical tool known as Whitney Decomposition Theorem (or Whitney's Covering Lemma [26, pp.67-69]). We say that W is a closed dyadic cube if

$$W = \{2^{-k}(z+w) : w \in [0,1]^N\} \qquad \text{for some } z \in \mathbb{Z}^N, k \in \mathbb{Z}.$$

Two such dyadic cubes W, W' are called almost disjoint if $\overline{W} \cap \overline{W'}$ is a null set. So they intersect at some corner or along parts of their faces, but their interiors are disjoint. For example, when $N = 2$, set

$$W_1 := [0,1]\times[0,1], \quad W_2 := [0,1]\times[1,2], \quad W_3 := [1,\frac{3}{2}]\times[1,\frac{3}{2}], \quad W_4 := [\frac{3}{4},1]\times[\frac{3}{4},1].$$

Each of these cubes is dyadic, W_1, W_2, W_3 are mutually almost disjoint, W_3, W_4 and W_2, W_4 are almost disjoint, too, but W_1, W_4 are not. The following preliminary results are essentially due to Caldéron and Zygmund [4, Section 3].

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^N$ be open and $\emptyset \subsetneq \Omega \subsetneq \mathbb{R}^N$. Then there are closed almost disjoint dyadic cubes $W_1, W_2, \ldots$ with the following properties*

(I) $\bigcup_{j\in\mathbb{N}} W_j = \Omega$,

(II) $\mathrm{diam}(W_j) \le \mathrm{dist}(W_j, \Omega^c) \le 4\,\mathrm{diam}(W_j)$ *for all* $j \in \mathbb{N}$.

(III) $\overline{W}_i \cap \overline{W}_j \neq \emptyset$ *implies* $\frac{1}{4}\mathrm{diam}(W_i) \le \mathrm{diam}(W_j) \le 4\,\mathrm{diam}(W_i)$,

(IV) $\#\{i \in \mathbb{N} : \overline{W_i} \cap \overline{W_j} \neq \emptyset\} \le 12^N$ *for all* $j \in \mathbb{N}$.

Furthermore, for any fixed $\kappa \in (0, \frac{1}{4})$ there are $\phi_1, \phi_2, \ldots \in C_0^\infty(\mathbb{R}^N)$ such that

(V) $0 \le \phi_j \le 1$, *$\phi_j(x) = 1$ for $x \in W_j$ and $\phi_j(x) = 0$ for* $\mathrm{dist}(x, W_j) \ge \kappa \,\mathrm{diam}(W_j)$.
(In particular[20], $\phi_j(x) \ne 0$ and $x \in W_i$ implies $\overline{W_i} \cap \overline{W_j} \ne \varnothing$.)

(VI) $|\partial^\alpha \phi_j(x)| \le C_\alpha \,\mathrm{diam}(W_j)^{-|\alpha|}$ *for all $\alpha \in \mathbb{N}_0^N$.*

The proof of this result is quite technical. The interested reader may find it in the Appendix for completeness. We need this result in order to prove the existence of a smooth version of the distance function. We study

$$\delta(x) := \mathrm{dist}(x, \Omega^c) = \inf\{|x - z| : z \in \Omega^c\}.$$

Proposition 5.2. *Let $\Omega \subset \mathbb{R}^N$ be open, $\varnothing \subsetneq \Omega \subsetneq \mathbb{R}^N$. Then there is a nonnegative function $d_\Omega \in C^\infty(\Omega)$ and positive numbers $C_\alpha > 0$ such that*

(i) $\frac{1}{5}\delta(x) \le d_\Omega(x) \le 4 \cdot 12^N \delta(x)$,

(ii) $|\partial^\alpha d_\Omega(x)| \le \tilde{C}_\alpha \delta(x)^{1-|\alpha|}$ *for* $\alpha \in \mathbb{N}_0^N, |\alpha| \ge 1$

The function d_Ω is called "regularized distance function".

Beweis:
We choose dyadic cubes W_j and $\phi_j \in C_0^\infty(\mathbb{R}^N)$ as in Lemma 5.1, define

$$d_\Omega(x) := d(x) := \sum_{k=1}^\infty \mathrm{diam}(W_k)\phi_k(x).$$

In view of property (IV) this sum is actually a finite sum[21] Then cover a given compact set $K \subset \Omega$ by finitely many $O_{x_1}, \ldots, O_{x_m}$, so $\phi_j = 0$ on $O_{x_1} \cup \ldots O_{x_m} \supset K$ whenever $j \in \mathbb{N} \setminus (I_{x_1} \cup \ldots \cup I_{x_m})$. So only finitely many ϕ_j are non-zero on each compact subset of Ω. So $\phi_j \in C_0^\infty(\Omega)$ for all $j \in \mathbb{N}$ implies $d \in C^\infty(\Omega)$.

We start by proving (i). Let $x \in \Omega$, choose $k \in \mathbb{N}$ with $x \in W_k$, which is possible by (I). Then

$$\delta(x) \le \mathrm{dist}(W_k, \Omega^c) + \mathrm{diam}(W_k) \overset{(II)}{\le} 5\,\mathrm{diam}(W_k). \tag{5.1}$$

So the lower bound from (i) follows from $\phi_k(x) = 1$ by (V) and

$$d(x) \ge \mathrm{diam}(W_k)\phi_k(x) = \mathrm{diam}(W_k) \overset{(5.1)}{\ge} \frac{1}{5}\delta(x).$$

To prove the upper bound we use

$$\delta(x) \ge \mathrm{dist}(W_k, \Omega^c) \overset{(II)}{\ge} \mathrm{diam}(W_k) \overset{(III)}{\ge} \frac{1}{4}\mathrm{diam}(W_j) \qquad \text{if } \overline{W_j} \cap \overline{W_k} \ne \varnothing. \tag{5.2}$$

[20] This is a consequence of (III)

[21] Indeed, for any given point $x \in \Omega$ has an open neighbourhood O_x and a finite index set $I_x \subset \mathbb{N}$ such that $j \in \mathbb{N} \setminus I_x$ implies $\phi_j|_{O_x} = 0$. This follows from (I),(V).

This implies

$$d(x) \overset{(V)}{=} \sum_{\overline{W}_j \cap \overline{W}_k \neq \varnothing} \operatorname{diam}(W_j)\phi_j(x) \overset{(5.2),(V)}{\leq} \sum_{\overline{W}_j \cap \overline{W}_k \neq \varnothing} 4\delta(x) \overset{(IV)}{\leq} 4 \cdot 12^N \delta(x).$$

It remains to prove (ii). Assume once again $x \in W_k$ and $\alpha \in \mathbb{N}_0^N, |\alpha| \geq 1$. Then

$$\begin{aligned}
|\partial^\alpha d(x)| &\overset{(VI)}{\leq} \sum_{\overline{W}_j \cap \overline{W}_k \neq \varnothing} \operatorname{diam}(W_j) \cdot C_\alpha \operatorname{diam}(W_j)^{-|\alpha|} \\
&= C_\alpha \sum_{\overline{W}_j \cap \overline{W}_k \neq \varnothing} \operatorname{diam}(W_j)^{1-|\alpha|} \\
&\overset{(5.1)}{\leq} C_\alpha \sum_{\overline{W}_j \cap \overline{W}_k \neq \varnothing} \left(\frac{1}{5}\delta(x)\right)^{1-|\alpha|} \\
&\overset{(IV)}{\leq} \underbrace{C_\alpha 5^{|\alpha|-1} 12^N}_{\tilde{C}_\alpha} \delta(x)^{1-|\alpha|}.
\end{aligned}$$

□

End Lec 06

Another technical tool is the following[22].

Proposition 5.3. *There are $c, C > 0$ and a continuous function $\phi : [1, \infty) \to \mathbb{R}$ satisfying*

(i) $\int_1^\infty \phi(t)\, dt = 1$,

(ii) $\int_1^\infty t^k \phi(t)\, dt = 0$ *for all* $k \in \mathbb{N}$,

(iii) $|\phi(t)| \leq Ce^{-ct}$ *for all* $t \in [1, \infty)$.

Beweis:
The basic idea is to use the residue theorem for (i) and Cauchy's integral formula for (ii). We consider

$$\psi(z) := \frac{e}{\pi z} \exp\big(- \omega(z-1)^{1/4}\big) \qquad (z \in \mathbb{C} \setminus [1, \infty))$$

where $\omega := e^{-i\pi/4} = \frac{1-i}{\sqrt{2}}$. One can check[23] that the function

$$z = 1 + re^{i\phi} \mapsto (z-1)^{1/4} = r^{1/4} e^{i\phi/4} \qquad (r \geq 0, 0 < \phi < 2\pi)$$

[22] The full strength of this construction cannot be seen from the proof of the extension theorem for first order Sobolev spaces. For higher order Sobolev spaces, property (ii) is used for more k.

[23] The Cauchy-Riemann equations for the real part $u(r,\theta) := r^{1/4}\cos(\phi/4)$ and the imaginary part $v(r,\theta) := r^{1/4}\sin(\phi/4)$ read as follows:

$$\partial_r u = \frac{1}{r}\partial_\theta v, \qquad \partial_r v = -\frac{1}{r}\partial_\theta u.$$

is holomorphic in $\mathbb{C} \setminus [1,\infty)$. Hence, ψ is meromorphic and $z \mapsto \psi(z)z^k$ is holomorphic in $\mathbb{C} \setminus [1,\infty)$ for any given $k \in \mathbb{N}$. So the integrals along piecewise smooth closed curves γ encircling $z = 0$ may be computed as follows:

$$\frac{1}{2\pi i}\int_\gamma \psi = \lim_{z\to 0} \psi(z)z = \frac{e}{\pi}\exp\big(-\omega(-1)^{1/4}\big) = \frac{e}{\pi}\exp\big(-\omega\bar{\omega}\big) = \frac{1}{\pi}.$$

Moreover,

$$\int_\gamma (\cdot)^k \psi(\cdot) = 0 \qquad \text{for all } k \in \mathbb{N}.$$

Still, this is a statement about line integrals ("Kurvenintegrale") in the complex plane for complex-valued integrands and not about integrals along the real interval $[1,\infty)$ for real-valued integrands. So we approximate such an integral by suitable line integrals in the complex plane.

We define the following curve[24]

$$\gamma_\varepsilon := \gamma_\varepsilon^1 \oplus \gamma_\varepsilon^2 \oplus \gamma_\varepsilon^3 \oplus \gamma_\varepsilon^4 \qquad (\varepsilon > 0)$$

via

- (A part of the parallel to the right half-axis at height $\mathfrak{I}(z) = \varepsilon$)
  $\gamma_\varepsilon^1(t) = t + \varepsilon i$ for $t \in [1, \varepsilon^{-1}]$.
- (Almost full large circle from $\varepsilon^{-1} + \varepsilon i$ to $\varepsilon^{-1} - \varepsilon i$, counterclockwise)
  $\gamma_\varepsilon^2(t) = (\varepsilon^2 + \varepsilon^{-2})^{1/2} e^{it}$ for $t \in [\theta, 2\pi - \theta]$, $\theta := \arctan(\varepsilon^2)$.
- (A part of the parallel to the right half-axis at height $\mathfrak{I}(z) = -\varepsilon$)
  $\gamma_\varepsilon^3(t) = 1 + \varepsilon^{-1} - \varepsilon i - t$ for $t \in [1, \varepsilon^{-1}]$.
- (Small half-circle around 1)
  $\gamma_\varepsilon^4(t) = 1 + \varepsilon e^{-it}$ for $t \in [\pi/2, 3\pi/2]$.

Then we use[25] $(z \in \mathbb{C}, t > 1)$

$$\begin{aligned} |\psi(z)| &= \frac{e}{\pi|z|}\exp\big(-\mathfrak{R}(\omega(z-1)^{1/4})\big) \le \frac{e}{\pi|z|}\exp\big(-|z-1|^{1/4}/\sqrt{2}\big), \\ \psi(t+i0) &= \frac{e}{\pi t}\exp\big(-\omega(t-1)^{1/4}\big), \\ \psi(t-i0) &= \frac{e}{\pi t}\exp\big(-\omega(t-1)^{1/4}e^{i\pi/2}\big) = \overline{\psi(t+i0)}. \end{aligned} \tag{5.3}$$

[24] $\oplus$ means concatenation here, so "one after the other". If γ, η are w.l.o.g. continuous curves on $[0,1] \to \mathbb{C}$ with $\gamma(1) = \eta(0)$, then the continuous curve $\gamma \oplus \eta$ is given by

$$(\gamma \oplus \eta)(t) = \begin{cases} \gamma(2t) & \text{if } 0 \le t \le \frac{1}{2}, \\ \eta(2t-1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

[25] For $z = 1 + re^{i\phi}$ with $0 < \phi < 2\pi$ we have

$$\mathfrak{R}(\omega(z-1)^{1/4}) = \mathfrak{R}(e^{-i\pi/4} \cdot r^{1/4}e^{i\phi/4}) = r^{1/4}\cos(\frac{\phi-\pi}{4}) \ge r^{1/4}\frac{1}{\sqrt{2}} = \frac{|z-1|^{1/4}}{\sqrt{2}}.$$

From (5.3) and the Dominated Convergence Theorem we get

$$\int_{\gamma_\varepsilon^2} (\cdot)^k \psi = o(1), \qquad \int_{\gamma_\varepsilon^4} (\cdot)^k \psi = o(1) \qquad \text{as } \varepsilon \to 0.$$

and thus

$$\begin{aligned} 2\pi i \cdot \frac{\delta_{k0}}{\pi} &= \int_{\gamma_\varepsilon} (\cdot)^k \psi \\ &= \int_{\gamma_\varepsilon^1} (\cdot)^k \psi + \int_{\gamma_\varepsilon^3} (\cdot)^k \psi + o(1) \\ &= \int_1^{1/\varepsilon} (t+\varepsilon i)^k \psi(t+\varepsilon i) - (t-\varepsilon i)^k \psi(t-\varepsilon i)\, dt + o(1) \\ &= \int_1^\infty t^k \cdot \big(\psi(t+i\cdot 0) - \psi(t-i\cdot 0)\big)\, dt + o(1) \\ &\overset{(5.3)}{=} 2i \int_1^\infty t^k \cdot \underbrace{\mathrm{Im}(\psi(t+i\cdot 0))}_{=:\phi(t)}\, dt + o(1), \end{aligned}$$

where

$$\phi(t) = \frac{e}{\pi t} \exp\left(-\frac{(t-1)^{1/4}}{\sqrt{2}}\right) \sin\left(\frac{(t-1)^{1/4}}{\sqrt{2}}\right).$$

□

Until now we have not seen any reason why Lipschitz domains, or Lipschitz-continuous functions, play a particular role. The basic link between Lipschitz domains and the regularized distance function $d := d_{\mathbb{R}^N \setminus \overline{\Omega}}$ is the following.

Proposition 5.4. *Let $\psi : \mathbb{R}^{N-1} \to \mathbb{R}$ be Lipschitz-continuous and $\Omega = \{x \in \mathbb{R}^N : x_N > \psi(x')\}$. Then there is a $c > 0$ such that $0 \le c^{-1} d(x) \le \psi(x') - x_N \le c\, d(x)$ for all $x \in \mathbb{R}^N \setminus \overline{\Omega}$.*

Beweis:
We have by Proposition 5.2 (i) for $x \in \mathbb{R}^N \setminus \overline{\Omega}$

$$\begin{aligned} d(x) &\le 4 \cdot 15^N \delta(x) \\ &= 4 \cdot 15^N \inf\{|x - z| : z \in \mathbb{R}^N \setminus \overline{\Omega}\} \\ &\le 4 \cdot 15^N |(x', x_N) - (x', \psi(x'))| \\ &= 4 \cdot 15^N (\psi(x') - x_N). \end{aligned}$$

To prove the lower bound for d let L denote the Lipschitz constant of ψ. Then Proposition 5.2 (i) gives[26]

$$d(x) \ge \frac{1}{5} \delta(x)$$

[26] Here we distinguish between the Euclidean norm $|\cdot|_2$ on $\mathbb{R}^N$ and $|\cdot|_1$; they satisfy $|v|_2 \le |v|_1 \le \sqrt{N}|v|_2$ for all $v \in \mathbb{R}^N$. In the fourth line of this chain of inequalities $|x' - y'|_1 = \sum_{i=1}^{N-1} |x_i - y_i|$.

$$
\begin{aligned}
&= \inf_{y'\in\mathbb{R}^{N-1}} \frac{1}{5}|(x',x_N)-(y',\psi(y'))|_2 \\
&\geq \frac{1}{5\sqrt{N}} \inf_{y'\in\mathbb{R}^{N-1}} |(x',x_N)-(y',\psi(y'))|_1 \\
&= \frac{1}{5\sqrt{N}} \inf_{y'\in\mathbb{R}^{N-1}} \Big[|x'-y'|_1 + |x_N-\psi(y')|\Big] \\
&\geq \frac{1}{5\sqrt{N}} \inf_{y'\in\mathbb{R}^{N-1}} \Big[|x'-y'|_1 + \min\{1,L^{-1}\}\Big|(x_N-\psi(x'))+(\psi(x')-\psi(y'))\Big|\Big] \\
&\geq \frac{1}{5\sqrt{N}} \inf_{y'\in\mathbb{R}^{N-1}} \Big[|x'-y'|_2 + \min\{1,L^{-1}\}|x_N-\psi(x')| - \min\{1,L^{-1}\}\underbrace{|\psi(x')-\psi(y')|}_{\leq L|x'-y'|_2}\Big] \\
&\geq \frac{\min\{1,L^{-1}\}}{5\sqrt{N}}|x_N-\psi(x')| \\
&= \frac{\min\{1,L^{-1}\}}{5\sqrt{N}}(\psi(x')-x_N).
\end{aligned}
$$

This proves the claim. □

Define

$$
Tv(t) := t\int_t^\infty v(s)s^{-2}\,ds.
$$

Lemma 5.5 (Hardy's Inequality)**.** *Let* $p\in[1,\infty]$*. Then* $\|Tv\|_{L^p([0,\infty))} \leq \frac{p}{p+1}\|v\|_{L^p([0,\infty))}$.

Beweis:
The case $p=\infty$ results from

$$
|Tv(t)| \leq |t|\|v\|_\infty \int_t^\infty s^{-2}\,ds = \|v\|_\infty.
$$

So we may assume $p\in[1,\infty)$ from now on. Also, it suffices to prove the estimate for nontrivial $v\in C_0^\infty(\mathbb{R}^N)$ in view of Theorem 4.8. The idea is to use integration by parts. We have

$$
\begin{aligned}
\|Tv\|^p_{L^p([0,\infty))} &= \int_0^\infty t^p\left(\int_t^\infty v(s)s^{-2}\right)^p dt \\
&= \Big[\frac{t^{p+1}}{p+1}\left(\int_t^\infty v(s)s^{-2}\right)^p\Big]_0^\infty - \int_0^\infty \frac{t^{p+1}}{p+1}\cdot p\left(\int_t^\infty v(s)s^{-2}\right)^{p-1}\cdot(-v(t)t^{-2})\,dt \\
&= 0 + \frac{p}{p+1}\int_0^\infty\left(t\int_t^\infty v(s)s^{-2}\right)^{p-1}\cdot v(t)\,dt \\
&\leq \frac{p}{p+1}\left\|\left(t\int_t^\infty v(s)s^{-2}\right)^{p-1}\right\|_{L^{p'}([0,\infty))}\|v\|_{L^p([0,\infty))} \\
&= \frac{p}{p+1}\||Tv(t)|^{p-1}\|_{L^{p'}([0,\infty))}\|v\|_{L^p([0,\infty))}
\end{aligned}
$$

$$= \frac{p}{p+1} \|Tv\|^{p-1}_{L^p([0,\infty))} \|v\|_{L^p([0,\infty))}.$$

Since $\|Tv\|_{L^p([0,\infty))}$ is positive and finite, we may divide by $\|Tv\|^{p-1}_{L^p([0,\infty))}$ and obtain the result. □

Satz 5.6 (Stein)**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain and $k \in \mathbb{N}, 1 \le p \le \infty$. Then Ω has a bounded extension operator $E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^N)$.*

Beweis:
The proof is very advanced: we focus on $k = 1$, $1 \le p < \infty$ and do not provide all details, only the main ideas – Sorry!

The strategy is the following: We consider "special Lipschitz domains" (as in Proposition 5.4) first and prove the existence of an extension operator for those. This is the main intellectual challenge of the proof. Afterwards, we generalize this to a general bounded Lipschitz domain. Here one uses that the boundary of such a general Lipschitz domain is a finite union of "special" Lipschitz domains, for which we have already constructed an extension operator. So it remains to combine these finitely many extension operators to some extension operator for the whole domain Ω using a partition of unity[27], see Theorem 4.3.
It is again sufficient to prove the estimates for smooth functions $u \in C_0^\infty(\mathbb{R}^N)$, see Theorem 4.11. We fix a function ϕ as in Proposition 5.3 and define $d := d_{\mathbb{R}^N \setminus \overline{\Omega}}$ to be the regularized distance function of the complement of Ω.

Step 1: Special Lipschitz domains
We start our analysis with the construction of an extension operator for

$$\Omega_\psi := \{x = (x', x_N) \in \mathbb{R}^N : x_N > \psi(x')\}$$

where $\psi : \mathbb{R}^{N-1} \to \mathbb{R}$ is Lipschitz-continuous. By Proposition 5.4 there is are $c, C > 0$ such that

$$C(\psi(x') - x_N) \ge c\, d(x) \ge \psi(x') - x_N \ge 0 \qquad (x \in \mathbb{R}^N \setminus \overline{\Omega}). \tag{5.4}$$

We define an extension operator E as follows:

$$(Eu)(x) := \begin{cases} u(x) & \text{, if } x_N \ge \psi(x'), \text{i.e., for } x \in \overline{\Omega}, \\ \int_1^\infty u(x', x_N + 2cd(x)t)\phi(t)\, dt & \text{, if } x_N < \psi(x'), \text{i.e., for } x \in \mathbb{R}^N \setminus \overline{\Omega}. \end{cases}$$

This operator obviously satisfies $Eu|_\Omega = u|_\Omega$. We need to show $u \in W^{1,p}(\mathbb{R}^N)$ and

$$\|Eu\|_{W^{1,p}(\mathbb{R}^N)} \le C^* \|u\|_{W^{1,p}(\Omega)}.$$

[27] This should be seen as a technical step, so of minor theoretical importance that does not provide any new idea.

Step 1(a): L^p-bound for Eu.
Choose $A > 0$ such that $|\phi(t)| \le At^{-2}$ for all $t \ge 1$. This is possible in view of Proposition 5.3 (iii). We have

$$|(Eu)(x)| \le A \int_1^\infty \frac{|u(x', x_N + 2cd(x)t)|}{t^2}\, dt \qquad (x_N < \psi(x'))$$

The change of coordinates $2cd(x)t = \psi(x') - x_N + s$ gives for $v(s) := u(x', \psi(x') + s)$

$$\begin{aligned}|(Eu)(x', x_N)| &\le 2Acd(x) \int_{x_N - \psi(x') + 2cd(x)}^\infty \frac{|u(x', \psi(x') + s)|}{(s + \psi(x') - x_N)^2}\, ds \\ &\overset{(5.4)}{\le} 2AC(\psi(x') - x_N) \int_{\psi(x') - x_N}^\infty \frac{|v(s)|}{s^2}\, ds.\end{aligned}$$

So Hardy's Inequality gives

$$\begin{aligned}\int_{-\infty}^{\psi(x')} |(Eu)(x', x_N)|^p\, dx_N &\le (2AC)^p \int_{-\infty}^{\psi(x')} (\psi(x') - x_N)^p \left(\int_{\psi(x') - x_N}^\infty \frac{|v(s)|}{s^2}\, ds \right)^p dx_N \\ &= (2AC)^p \int_0^\infty \left(t \int_t^\infty \frac{|v(s)|}{s^2}\, ds \right)^p dt \\ &\le (2AC)^p \left(\frac{p}{p+1} \right)^p \int_0^\infty |v(s)|^p\, ds \\ &= (2AC)^p \left(\frac{p}{p+1} \right)^p \int_{\psi(x')}^\infty |u(x', x_N)|^p\, dx_N.\end{aligned}$$

Integrating now with respect to x' over $\mathbb{R}^{N-1}$ gives

$$\begin{aligned}\|Eu\|_{L^p(\mathbb{R}^N \setminus \Omega)}^p &= \int_{\mathbb{R}^{N-1}} \left(\int_{-\infty}^{\psi(x')} |(Eu)(x', x_N)|^p\, dx_N \right) dx' \\ &\le (2AC)^p \left(\frac{p}{p+1} \right)^p \int_{\mathbb{R}^{N-1}} \int_{\psi(x')}^\infty |u(x', x_N)|^p\, dx_N\, dx' \\ &= (2AC)^p \left(\frac{p}{p+1} \right)^p \|u\|_{L^p(\Omega)}^p.\end{aligned}$$

As a consequence,

$$\|Eu\|_{L^p(\mathbb{R}^N)} \le \|u\|_{L^p(\Omega)} + \|Eu\|_{L^p(\mathbb{R}^N \setminus \Omega)} \le \left(1 + 2AC \frac{p}{p+1} \right) \|u\|_{L^p(\Omega)}. \tag{5.5}$$

Step 1(b): L^p-bound for $\partial_i(Eu)$.
Next we have to prove the corresponding inequality for the derivatives. We will use[28] $Eu \in C^1(\mathbb{R}^N)$. We don't prove this in detail, but only present one computation related to this fact. We check $\nabla(Eu)(y) = \nabla u(y)$ for all $y \in \partial\Omega$. In fact we find

$$(Eu)(x) - (Eu)(y) - \nabla u(y) \cdot (x - y) = u(x) - u(y) - \nabla u(y) \cdot (x - y)$$

[28] More effort gives $u \in C^\infty(\mathbb{R}^N)$.

$$= O(|x-y|^2) \qquad \text{as } x \to y, x \in \overline{\Omega}.$$

Do we have the same for $x \to y, x \in \mathbb{R}^N \setminus \overline{\Omega}$ to conclude $\nabla(Eu)(y) = \nabla u(y)$? Using $d(x) \le C_1 \delta(x) \le C_2 |x-y|$ for some $C_1, C_2 > 0$ we get for some constants $C > 0$

$$\begin{aligned}
&|Eu(x) - u(y) - \nabla u(y) \cdot (x-y)| \\
&= \left| \int_1^\infty \big(u(x', x_N + 2cd(x)t) - u(y', y_N)\big)\, \phi(t)\, dt - \nabla u(y) \cdot (x-y) \right| \\
&= \Big| \int_1^\infty \int_0^1 \Big[\nabla u\big(y' + s(x'-y'), y_N + s(x_N + 2cd(x)t - y_N)\big) - \nabla u(y) \Big] \\
&\qquad \cdot (x'-y', x_N + 2cd(x)t - y_N)\, ds\, \phi(t)\, dt \Big| \\
&\le \|u\|_{C^2} \int_1^\infty \left[\int_0^1 \big(s|x'-y'| + 2cd(x)t + s|x_N + 2cd(x)t - y_N|\big) \left(|y-x| + |d(x)|t\right)\, ds \right] |\phi(t)|\, dt \\
&\le C \|u\|_{C^2} \int_1^\infty \int_0^1 |x-y|(1+t) \cdot |x-y|(1+t)\, ds\, |\phi(t)|\, dt \\
&\le C \|u\|_{C^2} |x-y|^2 \int_1^\infty (1+t)^2 |\phi(t)|\, dt \\
&= O(|x-y|^2).
\end{aligned}$$

Here we used:

- u is smooth and defined on $\mathbb{R}^N$. This is important because we apply the Mean Value Theorem in integral form: $u(z_1) - u(z_2) = \int_0^1 \nabla u(z_2 + s(z_1 - z_2))\, dx \cdot (z_1 - z_2)$. It requires that u is continuously differentiable in a neighbourhood of the segment joining z_1, z_2. It is a priori unclear to ensure this with a function only defined on Ω. Notice that we do not assume Ω to be convex.
- The first equality holds by definition of Eu and $\int_1^\infty \phi(t)\, dt = 1$.
- The second equality uses the Mean Value Theorem and $\int_1^\infty t\phi(t)\, dt = 0$.

Differentiation under the integral sign gives for $x \in \mathbb{R}^N \setminus \overline{\Omega}$, i.e., for $x_N < \psi(x')$,

$$\partial_i (Eu)(x) = \int_1^\infty \big(\partial_i u(x', x_N + 2cd(x)t) + 2ct\partial_i d(x) \partial_N u(x', x_N + 2cd(x)t)\big)\, \phi(t)\, dt.$$

We now estimate the gradient in a similar way as above. The estimate

$$\begin{aligned}
\left\| \int_1^\infty \partial_i u(x', x_N + 2cd(x)t)\, dt \right\|_{L^p(\mathbb{R}^N \setminus \Omega)} &\le 2AC \frac{p}{p+1} \|\partial_i u\|_{L^p(\Omega)} \\
&\le 2AC \frac{p}{p+1} \|u\|_{W^{1,p}(\Omega)}
\end{aligned}$$

follows as above: it suffices to replace u by $\partial_i u$. The second term is estimated similarly. Choose $B > 0$ such that $|\phi(t)| \le Bt^{-3}$ for all $t \in [1, \infty)$. Then we find for $x_N < \psi(x')$

$$\left| \int_1^\infty 2ct\partial_i d(x) \partial_N u(x', x_N + 2cd(x)t) \cdot \phi(t)\, dt \right|$$

$$\leq 2c\int_1^\infty |\partial_i d(x)||\partial_N u(x', x_N + 2cd(x)t)||t\phi(t)|\,dt$$
$$\leq 2cB\|\partial_i d\|_\infty \int_1^\infty \frac{|\partial_N u(x', x_N + 2cd(x)t)|}{t^2}\,dt.$$

Again, one obtains via Hardy's Inequality

$$\|\partial_i(Eu)\|_{L^p(\mathbb{R}^N)} \leq \left(1 + N\cdot 2(A+B)C\frac{p}{p+1}\right)\|u\|_{W^{1,p}(\Omega)}. \tag{5.6}$$

So (5.5),(5.6) yield the claim for special Lipschitz domains.

Step 2: General bounded Lipschitz domains.
We consider a covering $\partial\Omega \subset \bigcup_{j=1}^M U_j$ where the open sets U_j are given by

$$\partial\Omega \cap U_j = \{(x', x_N) \in U_j : x_N = \psi_j(x')\} \subset \partial\Omega_{\psi_j},$$
$$\Omega \cap U_j = \{(x', x_N) \in U_j : x_N > \psi_j(x')\} \subset \Omega_{\psi_j}$$

after some permutation of coordinates. Here, the $\psi_j : \mathbb{R}^{N-1} \to \mathbb{R}$ are Lipschitz-continuous functions. Step 1 provides extension operators $E_j : W^{1,p}(\Omega_{\psi_j}) \to W^{1,p}(\mathbb{R}^N)$. (This tells us how to extend a given function across the boundary.) Next choose an open set $U_0 \subset \Omega$ such that $\Omega \subset \bigcup_{j=0}^M U_j$ and define $E_0 : W^{1,p}(\Omega), u \mapsto u$. As in Theorem 4.3 choose finitely many test functions $(\phi_i)_{i\in I}$ with $\operatorname{supp}(\phi_i) \subset U_{j(i)}$ and $\sum_{i\in I}\phi_i(x) = 1$ for all $x \in \overline{\Omega}$. Set

$$Eu(x) := \chi(x)\cdot\frac{\sum_{i\in I}\phi_i(x)\big(E_{j(i)}(\phi_i u)\big)(x)}{\sum_{i\in I}\phi_i(x)^2}.$$

where $\chi \in C_0^\infty(\mathbb{R}^N)$ is an arbitrary function satisfying $\chi(x) = 1$ on $\overline{\Omega}$ and $\sum_{i\in I}\phi_i(x)^2 \neq 0$ on $\operatorname{supp}(\chi)$.

This operator is an extension operator because $x \in \Omega \cap \operatorname{supp}(\phi_i)$ implies $x \in U_{j(i)}$ and hence $E_{j(i)}(\phi_i u)(x) = (\phi_i u)(x) = \phi_i(x)u(x)$. Hence,

$$x \in \Omega \quad \Rightarrow \quad Eu(x) = 1\cdot\frac{\sum_{i\in I, x\in\operatorname{supp}(\phi_i)}\phi_i(x)\cdot\phi_i(x)u(x)}{\sum_{i\in I, x\in\operatorname{supp}(\phi_i)}\phi_i(x)^2} = u(x).$$

Moreover, the triangle inequality, the product rule and Hölder's inequality give

$$\|Eu\|_{W^{1,p}(\mathbb{R}^N)} \leq C\left\|\sum_{i\in I}\phi_i(x)E_{j(i)}(\phi_i u)(x)\right\|_{W^{1,p}(\mathbb{R}^N)}$$
$$\leq C\sum_{i\in I}\|\phi_i E_{j(i)}(\phi_i u)\|_{W^{1,p}(\mathbb{R}^N)}$$
$$\leq C\sum_{i\in I}\|\phi_i\|_{W^{1,\infty}(\mathbb{R}^N)}\|E_{j(i)}(\phi_i u)\|_{W^{1,p}(\mathbb{R}^N)}$$
$$\leq C\sum_{i\in I}\|\phi_i\|_{W^{1,\infty}(\mathbb{R}^N)}C_i^*\|\phi_i u\|_{W^{1,p}(\Omega)}$$

$$\le C \sum_{i\in I} \|\phi_i\|^2_{W^{1,\infty}(\mathbb{R}^N)} C_i^* \cdot \|u\|_{W^{1,p}(\Omega)}.$$

□

In the case $k \ge 2$ the proof is even more complicated because the derivatives of order ≥ 2 of the regularized distance function are not bounded any more. For instance, in the case $k = 2$ one needs to bound terms of the form

$$\partial_{ij}(Eu)(x) = \partial_{ij} d(x) \int_1^\infty \partial_N u(x', x_N + 2cd(x)t) \cdot t\phi(t)\, dt$$

in terms of the $\|u\|_{W^{2,p}(\Omega)}$. Here one uses

$$\begin{aligned}
|\partial_{ij}(Eu)(x)| &= |\partial_{ij} d(x)| \left| \int_1^\infty \big(\partial_N u(x', x_N + 2cd(x)t) - \partial_N u(x', x_N + 2cd(x))\big) \cdot t\phi(t)\, dt \right| \\
&= |2cd(x)\partial_{ij} d(x)| \left| \int_1^\infty \left(\int_0^1 \partial_N^2 u(x', x_N + 2cd(x)(1+st))\, ds \right) \cdot t^2 \phi(t)\, dt \right| \\
&\le 2c\|d\,\partial_{ij} d\|_\infty \int_0^1 \left(\int_1^\infty \frac{|\partial_N^2 u(x', x_N + 2cd(x)(1+s)t)|}{t^2}\, dt \right) ds.
\end{aligned}$$

The same techniques as above allow to bound this integral in terms of $\|\partial_N^2 u\|_{L^p(\Omega)}$ and hence in terms of $\|u\|_{W^{2,p}}(\Omega)$.

End Lec 07

6 Sobolev's Embedding Theorem and Applications

In this section we analyze to which $L^q(\Omega)$-spaces a generic function $u \in W^{k,p}(\Omega)$ belongs. By definition we only know $u \in L^p(\Omega)$, but it turns out that this is not the end of the story. Roughly speaking, assuming more and more "weak differentiability" (i.e., large enough k) the functions should become more and more regular, possibly ending up being bounded or even continuous. In Example 2.8 we have seen that the function $u(x) = |x|^\gamma, \gamma < 0$ lies in $W^{1,p}(B)$ if and only if $\gamma > 1 - \frac{N}{p}$. Here, B was the unit ball centered at zero. This prototypical singularity leads to the following observation:

- For $1 \le p < N$ elements of $W^{1,p}(B)$ can be unbounded.
  On the other hand, the larger p is, the milder the singularities have to be.
- For $p > N$ there is the chance that elements of $W^{1,p}(B)$ cannot be unbounded.

We shall prove related statements in this section. To be more precise we seek for the validity of continuous embeddings

$$W^{1,p}(\Omega) \hookrightarrow L^q(\Omega), \quad \text{i.e. } \|u\|_{L^q(\Omega)} \le C\|u\|_{W^{1,p}(\Omega)}$$

under suitable assumptions on p, q, Ω and some constant $C > 0$ that does not depend on u. Later on we will show how this affects the theory of our elliptic model boundary value problem.

We start with a trivial remark: If $\Omega \subset \mathbb{R}^N$ is bounded, more generally: has finite measure, then we actually have $L^p(\Omega) \subset L^q(\Omega)$ for all $q \in [1, p]$. This follows from

$$\|u\|_q = \|u \cdot 1\|_q \le \|u\|_p \|1\|_{\frac{pq}{p-q}} = \|u\|_p \underbrace{|\Omega|^{\frac{1}{q}-\frac{1}{p}}}_{<\infty}.$$

So we see that on bounded domains lower integrability is for free. In typical unbounded domains ($\mathbb{R}^N$, half-spaces, strips, cylinders,...) this is not the case as can be seen from examples of the form $x \mapsto (1+|x|)^{-\alpha}$ for $\alpha > 0$. This has two consequences: First, we will not investigate such embeddings for $q < p$; they are "practically" irrelevant. Second, the final result for bounded domains will be slightly different from the general case.

We turn our attention towards higher integrability where the theory for bounded Lipschitz domains and unbounded domains is essentially[29] the same. As mentioned above, the question is how much integrability / continuity a generic function $u \in W^{k,p}(\Omega)$ admits. We will reduce our analysis to $\mathbb{R}^N$ by means of an extension operator that we constructed in the last section. We start with necessary conditions that are quite easy to obtain. They fit very well to the model singularity $x \mapsto |x|^\gamma$ discussed earlier.

Proposition 6.1. *Let $N, k \in \mathbb{N}$ and $p, q \in [1, \infty]$. If a continuous embedding $W^{k,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ exists, then necessarily*

$$0 \le \frac{1}{p} - \frac{1}{q} \le \frac{k}{N}. \tag{6.1}$$

Beweis:
We assume that the embedding is continuous and take any nontrivial[30] $u \in C_0^\infty(\mathbb{R}^N)$. For $\lambda > 0$ set $u_\lambda(x) := u(\lambda x)$ we have $\partial^\alpha u_\lambda(x) = \lambda^{|\alpha|}(\partial^\alpha u)(\lambda x)$. So we have

$$\|u_\lambda\|_{L^q(\mathbb{R}^N)}^q = \int_{\mathbb{R}^N} |u(\lambda x)|^q \, dx = \lambda^{-N} \int_{\mathbb{R}^N} |u(x)|^q \, dx = |\lambda|^{-N} \|u\|_{L^q(\mathbb{R}^N)}^q$$

as well as

$$\|u_\lambda\|_{W^{k,p}(\mathbb{R}^N)}^p = \sum_{|\alpha| \le k} \|\partial^\alpha u_\lambda\|_{L^p(\mathbb{R}^N)}^p = \sum_{|\alpha| \le k} \lambda^{p|\alpha|} \|(\partial^\alpha u)(\lambda \cdot)\|_{L^p(\mathbb{R}^N)}^p = \sum_{|\alpha| \le k} \lambda^{p|\alpha|-N} \|\partial^\alpha u\|_{L^p(\mathbb{R}^N)}^p.$$

Assuming $W^{k,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ we thus obtain for some $C > 0$

$$\lambda^{-\frac{N}{q}} \|u\|_{L^q(\mathbb{R}^N)} \le C \left(\sum_{|\alpha| \le k} \lambda^{p|\alpha|-N} \right)^{\frac{1}{p}} \|u\|_{W^{k,p}(\mathbb{R}^N)} \le C'(\lambda^{-N} + \lambda^{pk-N})^{\frac{1}{p}} \|u\|_{W^{k,p}(\mathbb{R}^N)}.$$

[29] Wild domains with cusps, fractal structure etc. may cause problems. For Lipschitz domains these exotic phenomena do not occur due to the presence of an extension operator.

[30] Actually one may take any nontrivial function $u \in W^{k,p}(\mathbb{R}^N)$ to carry through the arguments.

This implies (consider $\lambda \to \infty$ resp. $\lambda \to 0$)

$$-\frac{N}{q} \le k - \frac{N}{p} \quad \text{and} \quad -\frac{N}{q} \ge -\frac{N}{p},$$

which is equivalent to (6.1). □

For $1 \le p < \frac{N}{k}$ the conditions (6.1) mean $p \le q \le \frac{Nk}{N-kp}$. For $p \ge \frac{N}{k}$ all $q \ge p$ are allowed. Except for the case $p = \frac{N}{k}$, where a slightly weaker result is true, this is already the answer to the problem as we shall demonstrate in the following. We will concentrate on the proofs for $k = 1$; to prove the statements for higher k one uses inductively

$$u \in W^{k,p}(\mathbb{R}^N) \quad \Leftrightarrow \quad u \in W^{1,p}(\mathbb{R}^N) \text{ and } \partial_1 u, \dots, \partial_N u \in W^{k-1,p}(\mathbb{R}^N).$$

So we come to one of the most important theorems in the Sobolev spaces: Sobolev's Inequality. For $1 < p < N$ it is due to Sobolev himself [22]. Gagliardo [8] extended the result to $p = 1$ and large classes of bounded domains. At almost the same time, Nirenberg [17] proved a more general version on $\mathbb{R}^N$ including the case $p = 1$ as well (knowing that the result holds on "sufficiently nice" domains $\Omega \subset \mathbb{R}^N$). To prove Sobolev's Inequality, we will use the generalized Hölder inequality

$$\left\| |v_1|^{\frac{1}{N-1}} \cdot \ldots \cdot |v_{N-1}|^{\frac{1}{N-1}} \right\|_{L^1(\mathbb{R})} \le \|v_1\|_{L^1(\mathbb{R})}^{\frac{1}{N-1}} \cdot \ldots \cdot \|v_{N-1}\|_{L^1(\mathbb{R})}^{\frac{1}{N-1}}, \tag{6.2}$$

see Exercise 1 on Exercise Sheet 1. Moreover we will need the Inequality of arithmetic and geometric means

$$\prod_{i=1}^{N} a_i^{\frac{1}{N}} \le \frac{1}{N} \sum_{i=1}^{N} a_i \qquad (a_1, \dots, a_N \ge 0). \tag{6.3}$$

Satz 6.2 (Sobolev (1938), Gagliardo (1958), Nirenberg (1959))**.** *Assume $N \in \mathbb{N}, N \ge 2$ and $1 \le p < N$. Then we have the following inequality for all $u \in W^{1,p}(\mathbb{R}^N)$:*

$$\|u\|_{L^{\frac{Np}{N-p}}(\mathbb{R}^N)} \le \frac{p(N-1)}{\sqrt{N}(N-p)} \|\nabla u\|_{L^p(\mathbb{R}^N)}.$$

Beweis:
By Lemma 4.12 and Exercise 4 of Exercise Sheet 1 it suffices to prove this inequality for $u \in C_0^\infty(\mathbb{R}^N)$. The crucial step is to prove the claim for $p = 1$, which we shall do first. For $u \in C_0^\infty(\mathbb{R}^N)$ and $j \in \{1, \dots, N\}$ we have

$$\begin{aligned} |u(x)| &= \left| \int_{-\infty}^{x_j} \partial_j u(x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_N)\, dt \right| \\ &\le \int_{\mathbb{R}} |\partial_j u(x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_N)|\, dt. \end{aligned}$$

This implies

$$|u(x)|^{\frac{N}{N-1}} \leq \prod_{j=1}^{N} \Big(\int_{\mathbb{R}} |\partial_j u(x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_N)| \, dt \Big)^{\frac{1}{N-1}}.$$

Hence,

$$\begin{aligned}
\int_{\mathbb{R}} |u(x)|^{\frac{N}{N-1}} \, dx_1 &\leq \int_{\mathbb{R}} \prod_{i=1}^{N} \Big(\int_{\mathbb{R}} |\partial_i u(x)| \, dx_i \Big)^{\frac{1}{N-1}} \, dx_1 \\
&= \Big(\int_{\mathbb{R}} |\partial_1 u(x)| \, dx_1 \Big)^{\frac{1}{N-1}} \int_{\mathbb{R}} \prod_{i=2}^{N} \Big(\int_{\mathbb{R}} |\partial_i u(x)| \, dx_i \Big)^{\frac{1}{N-1}} \, dx_1 \\
&\overset{(6.2)}{\leq} \Big(\int_{\mathbb{R}} |\partial_1 u(x)| \, dx_1 \Big)^{\frac{1}{N-1}} \prod_{i=2}^{N} \Big(\int_{\mathbb{R}^2} |\partial_i u(x)| \, dx_1 \, dx_i \Big)^{\frac{1}{N-1}}.
\end{aligned}$$

Integrating this inequality now with respect to x_2 yields

$$\begin{aligned}
&\int_{\mathbb{R}^2} |u(x)|^{\frac{N}{N-1}} \, dx_1 \, dx_2 \\
&\leq \int_{\mathbb{R}} \Bigg[\Big(\int_{\mathbb{R}} |\partial_1 u(x)| \, dx_1 \Big)^{\frac{1}{N-1}} \prod_{i=2}^{N} \Big(\int_{\mathbb{R}^2} |\partial_i u(x)| \, dx_1 \, dx_i \Big)^{\frac{1}{N-1}} \Bigg] \, dx_2 \\
&= \Big(\int_{\mathbb{R}^2} |\partial_2 u(x)| \, dx_1 \, dx_2 \Big)^{\frac{1}{N-1}} \cdot \int_{\mathbb{R}} \Bigg[\Big(\int_{\mathbb{R}} |\partial_1 u(x)| \, dx_1 \Big)^{\frac{1}{N-1}} \Big(\prod_{i=3}^{N} \int_{\mathbb{R}^2} |\partial_i u(x)| \, dx_1 \, dx_i \Big)^{\frac{1}{N-1}} \Bigg] \, dx_2 \\
&\overset{(6.2)}{\leq} \prod_{i=1}^{2} \Big(\int_{\mathbb{R}^2} |\partial_i u(x)| \, dx_1 \, dx_2 \Big)^{\frac{1}{N-1}} \prod_{i=3}^{N} \Big(\int_{\mathbb{R}^3} |\partial_i u(x)| \, dx_1 \, dx_2 \, dx_i \Big)^{\frac{1}{N-1}}.
\end{aligned}$$

In this way, one inductively shows for all $k \in \{1, \dots, N\}$

$$\begin{aligned}
&\int_{\mathbb{R}^k} |u(x)|^{\frac{N}{N-1}} \, dx_1 \, dx_2 \, \dots \, dx_k \\
&\leq \prod_{i=1}^{k} \Big(\int_{\mathbb{R}^k} |\partial_i u(x)| \, dx_1 \, dx_2 \, \dots \, dx_k \Big)^{\frac{1}{N-1}} \prod_{i=k+1}^{N} \Big(\int_{\mathbb{R}^{k+1}} |\partial_i u(x)| \, dx_1 \, dx_2 \dots \, dx_k \, dx_i \Big)^{\frac{1}{N-1}}
\end{aligned}$$

where the last product it to be understood as 1 for $k = N$. Finally, we use (6.3) and get

$$\begin{aligned}
\|u\|_{\frac{N}{N-1}} = \Big(\int_{\mathbb{R}^N} |u(x)|^{\frac{N}{N-1}} \, dx \Big)^{\frac{N-1}{N}} &\leq \prod_{i=1}^{N} \Big(\int_{\mathbb{R}^N} |\partial_i u(x)| \, dx \Big)^{\frac{1}{N}} \\
&\overset{(6.3)}{\leq} \frac{1}{N} \sum_{i=1}^{N} \int_{\mathbb{R}^N} |\partial_i u(x)| \, dx \\
&\leq \frac{1}{\sqrt{N}} \int_{\mathbb{R}^N} |\nabla u(x)| \, dx.
\end{aligned}$$

This proves the claim for $p = 1$.

For $1 < p < N$ and nontrivial $u \in C_0^\infty(\mathbb{R}^N)$ consider $v := |u|^{\frac{N(p-1)}{N-p}} u$. Then

$$v \in C_0^1(\mathbb{R}^N) \quad \text{and} \quad \nabla v = \frac{p(N-1)}{N-p} |u|^{\frac{N(p-1)}{N-p}} \nabla u.$$

So the result above implies

$$\begin{aligned}
\|u\|_{\frac{Np}{N-p}}^{\frac{p(N-1)}{N-p}} &= \|v\|_{\frac{N}{N-1}} \\
&\le \frac{1}{\sqrt{N}} \|\nabla v\|_1 \\
&= \frac{p(N-1)}{\sqrt{N}(N-p)} \||u|^{\frac{N(p-1)}{N-p}} |\nabla u|\|_1 \\
&\overset{\text{Hölder}}{\le} \frac{p(N-1)}{\sqrt{N}(N-p)} \||u|^{\frac{N(p-1)}{N-p}}\|_{p'} \||\nabla u|\|_p \\
&= \frac{p(N-1)}{\sqrt{N}(N-p)} \|u\|_{\frac{Np}{N-p}}^{\frac{N(p-1)}{N-p}} \||\nabla u|\|_p.
\end{aligned}$$

This gives the result. □

Bemerkung 6.3.

(a) The best constant in the Sobolev Inequality is known is given by

$$C_S(p) := \pi^{-1/2} N^{-1/p} \left(\frac{p-1}{N-p} \right)^{1-\frac{1}{p}} \left(\frac{\Gamma(N)\Gamma(1+N/2)}{\Gamma(N/p)\Gamma(1+N-N/p)} \right)^{1/N}.$$

In 1976, Talenti [24] proved that for $1 < p < N$ this value is attained precisely[31] for functions $u(x) = (a + b|x - x_0|^{p'})^{1-N/p}$ where $a, b > 0, x_0 \in \mathbb{R}^N$. In the case $p = 1$ we have

$$C_S(1) = \lim_{p \searrow 1} C_S(p) = \frac{\Gamma(1+N/2)^{1/N}}{\sqrt{\pi} N}$$

and a maximizing sequence for the inequality can be chosen to consist of functions converging to the indicator function of a ball in a suitable sense. Federer, Fleming [5] and Rishel [7, Theorem II] proved that the inequality for $p = 1$ is related to the so-called isoperimetric inequality

$$|E|^{\frac{N-1}{N}} \le C_S(1)\, \text{area}(\partial E)$$

for sufficiently regular subsets $E \subset \mathbb{R}^N$. It is maximized by balls.

[31] A related article is [?]. The uniqueness was proved in [?, Corollary 8.2 (b)] using that any maximizer must be a solution of the equation $-\Delta u = u^{\frac{N+2}{N-2}}$ in $\mathbb{R}^N$ that does not change sign. This may be proved with techniques from the Calculus of Variations.

Satz 6.4 (Sobolev's Embedding Theorem)**.** *Assume $N \in \mathbb{N}, N \geq 2$ and $1 \leq p < N$ and $p^* := \frac{Np}{N-p}$. Then there is a continuous embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ precisely for $p \leq q \leq p^*$, i.e., for $0 \leq \frac{1}{p} - \frac{1}{q} \leq \frac{1}{N}$.*

Beweis:
The Sobolev Inequality shows

$$\|u\|_{p^*} \leq C_S(p)\|\nabla u\|_p \leq C_S(p)\|u\|_{1,p}.$$

Moreover, $\|u\|_p \leq \|u\|_{1,p}$. For any given $q \in [p, p^*]$ we find $\theta \in [0,1]$ such that $\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{p^*}$. Then Hölder's inequality[32] gives

$$\|u\|_q \leq \|u\|_p^\theta \|u\|_{p^*}^{1-\theta} \leq C_S(p)^{1-\theta}\|u\|_{1,p}.$$

Since this prefactor does not depend on u, the claim is proved. □

Korollar 6.5. *Assume $N \in \mathbb{N}, N \geq 2$ and $1 \leq p < N$ and $p^* := \frac{Np}{N-p}$. For any bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$ there is a continuous embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $1 \leq q \leq p^*$.*

Beweis:
Theorem 5.6 provides an extension operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$. We thus obtain

$$\|u\|_{L^{p^*}(\Omega)} \leq \|Eu\|_{L^{p^*}(\mathbb{R}^N)} \leq C_S(p)\|Eu\|_{W^{1,p}(\mathbb{R}^N)} \leq C_S(p)\|E\|\|u\|_{W^{1,p}(\Omega)}.$$

Hence, for $1 \leq q \leq p^*$,

$$\|u\|_{L^q(\Omega)} \leq \|u\|_{L^{p^*}(\Omega)}\|1\|_{L^{\frac{1}{q}-\frac{1}{p^*}}(\Omega)} \leq C_S(p)\|E\|\|u\|_{W^{1,p}(\Omega)}|\Omega|^{\frac{1}{p'}}.$$

This proves the claim. □

The Sobolev Inequality is false[33] for $p = N$ and the question is whether an embedding $W^{1,N}(\Omega) \hookrightarrow L^\infty(\Omega)$ holds. The answer is no. In the Exercises we shall prove the following.

Satz 6.6. *Assume $N \in \mathbb{N}, N \geq 2$. Then there is a continuous embedding $W^{1,N}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ precisely for $N \leq q < \infty$.*

[32] Notice

$$\|u\|_q = \||u|^\theta \cdot |u|^{1-\theta}\|_q \leq \||u|^\theta\|_{\frac{p}{\theta}}\||u|^{1-\theta}\|_{\frac{p^*}{1-\theta}} = \|u\|_p^\theta\|u\|_{p^*}^{1-\theta}.$$

This is sometimes called Lyapunov's Inequality.

[33] Heuristically, this follows from $C_S(p) \to \infty$ as $p \nearrow N$. But this is not a proof, just a way of remembering things.

Korollar 6.7. *Assume $N \in \mathbb{N}, N \geq 2$. For any bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$ there is a continuous embedding $W^{1,N}(\Omega) \hookrightarrow L^q(\Omega)$ precisely for $1 \leq q < \infty$.*

There is a sharper statement about the limiting case $p = N$, which is particularly important in two spatial dimensions. The fundamental result in this direction is the Moser-Trudinger Inequality [16, 25] which essentially says that any function $u \in W_0^{1,N}(\Omega)$ satisfies

$$e^{\alpha |u|^{\frac{N}{N-1}}} \in L^1(\Omega)$$

provided that $\Omega \subset \mathbb{R}^N$ is a bounded domain and $\alpha > 0$.

End Lec 08

Example 6.8. We give an indirect proof of the fact that Hölder-domains do not admit extension operators as in Stein's Extension Theorem. To keep the technicalities at a moderate level, we concentrate on the case $N = 2$. We want to show that for $1 \leq p < N = 2$ we have

$$W^{1,p}(\Omega) \not\hookrightarrow L^{p^*}(\Omega), \qquad p^* = \frac{Np}{N-p} = \frac{2p}{2-p}$$

where the non-Lipschitz domain is given by

$$\Omega = \{(x,y) \in \mathbb{R}^2 : |x|^\gamma < y < 1, 0 < |x| < 1\}, \qquad 0 < \gamma < 1.$$

To see this define

$$u(x,y) := y^{-\alpha} \quad \text{where} \quad \alpha := \frac{\gamma + 1}{\gamma p^*}.$$

One checks:

- $(\alpha + 1)p > 1$ because of $p < 2$,
- $\gamma(1 - (\alpha + 1))p > -1$ because of $\gamma < 1$,
- $\gamma(1 - \alpha p^*) = -1$ by definition of α,
- $|\nabla u(x,y)| \leq |\alpha| y^{-p(\alpha+1)}$.

From this we get

$$\begin{aligned}
\int_\Omega |\nabla u(x,y)|^p + |u(x,y)|^p \, d(x,y) &\leq \int_0^1 \int_{|x|^\gamma}^1 (|\alpha| + 1) y^{-(\alpha+1)p} \, dy \, dx \\
&\leq (|\alpha| + 1) \int_0^1 \frac{1}{1 - (\alpha+1)p} \left(1 - x^{\gamma(1-(\alpha+1)p)}\right) dx \\
&\leq \frac{|\alpha| + 1}{p(\alpha+1) - 1} \int_0^1 x^{\gamma(1-(\alpha+1)p)} \, dx \\
&< \infty,
\end{aligned}$$

On the other hand,

$$\int_\Omega |u(x,y)|^{p^*} \, d(x,y) = \int_0^1 \int_{|x|^\gamma}^1 y^{-\alpha p^*} \, dy \, dx$$

$$= \int_0^1 \frac{1}{1-\alpha p^*} \left(1 - x^{\gamma(1-\alpha p^*)}\right) dx$$
$$= \frac{1}{\gamma} \int_0^1 (x^{-1} - 1)\, dx$$
$$= \infty.$$

We infer

$$W^{1,p}(\Omega) \not\hookrightarrow L^{p^*}(\Omega).$$

This implies that Ω does not admit a bounded extension operator $W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$.

6.1 Applications

We return to the boundary value problem (3.3), which was given by

$$-\Delta u(x) + c(x)u(x) = f(x) \quad (x \in \Omega), \qquad u(x) = 0 \quad (x \in \partial\Omega).$$

In Corollary 3.4 we showed that this boundary value problem has a unique solution for $f \in L^2(\Omega), c \in L^\infty(\Omega)$ with $c(x) \geq \mu > 0$. This was proved using the Lax-Milgram Theorem. Using Sobolev's Inequality we may improve this result now. We only state the corresponding result for a bounded Lipschitz domain and only for $N \geq 3$. This is because of $H^1(\Omega) \hookrightarrow L^{\frac{2N}{N-2}}(\Omega)$ only for $N \geq 3$.

Korollar 6.9. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$ be a bounded Lipschitz domain and assume $f \in L^{\frac{2N}{N+2}}(\Omega), c \in L^{\frac{N}{2}}(\Omega)$ with $c(x) \geq \mu > 0$ almost everywhere. Then (3.3) has a unique weak solution $u \in H_0^1(\Omega)$ that satisfies*

$$\|u\|_{1,2} \leq C_S(2) \min\{1, \mu\}^{-1} \|f\|_{\frac{2N}{N+2}}.$$

Beweis:
We recall that we want to solve $a(u,v) = l(v)$ for all $v \in H$ where

$$a(u,v) := \int_\Omega \nabla u(x) \cdot \nabla v(x) + c(x)u(x)v(x)\, dx,$$
$$l(v) := \int_\Omega f(x)v(x)\, dx.$$

Again we need to check the assumptions of the Lax-Milgram-Lemma.

The boundedness of a follows from[34]

$$|a(u,v)| \leq \int_\Omega |\nabla u(x)||\nabla v(x)| + |c(x)||u(x)||v(x)|\, dx$$

[34]We use here $\||\nabla u|\|_2 \leq C_S(2)\|u\|_{1,2}$ for all $u \in H_0^1(\Omega)$, but we had proved it for $u \in H^1(\mathbb{R}^N)$ only. This is no problem because any $u \in H_0^1(\Omega)$ may be extended trivially to $\mathbb{R}^N$ (in striking contrast to $H^1(\Omega)$-functions) so that Sobolev's Inequality applies. The best constant $C > 0$ satisfying $\||\nabla u|\|_2 \leq C\|u\|_{1,2}$ is however smaller, but this is not our issue here.

$$\begin{aligned}
&\le \|\nabla u\|_2\|\nabla v\|_2 + \|c\|_{\frac{N}{2}}\|u\|_{\frac{2N}{N-2}}\|v\|_{\frac{2N}{N-2}}\\
&\le \|\nabla u\|_2\|\nabla v\|_2 + \|c\|_{\frac{N}{2}}C_S(2)^2\|\nabla u\|_2\|\nabla v\|_2\\
&\le (1+\|c\|_{\frac{N}{2}}C_S(2)^2)\|u\|_{1,2}\|v\|_{1,2}
\end{aligned}$$

The proof of coercivity uses $c(x) \ge \mu > 0$ and works as in the proof of Corollary 3.4. Finally, l is a bounded linear functional because

$$|l(v)| \le \|fv\|_1 \le \|f\|_{\frac{2N}{N+2}}\|v\|_{\frac{2N}{N-2}} \le \|f\|_{\frac{2N}{N+2}}C_S(2)\|\nabla v\|_2 \le \|f\|_{\frac{2N}{N+2}}C_S(2)\|v\|_{1,2}.$$

So the Lax-Milgram Lemma proves the claim. □

What do we gain here? Due to $L^2(\Omega) \subset L^{\frac{2N}{N+2}}(\Omega)$ and $L^\infty(\Omega) \subset L^{\frac{N}{2}}(\Omega)$ our assumptions on the coefficients are less restrictive than before. For instance, the function c may be unbounded from above (not below, though!), which was not allowed before.

Question: What would be the corresponding result for $N = 2$? How should an improvement on unbounded domains look like?

7 Morrey's Embedding Theorem and Applications

In the last Section we have shown that $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $1 \le p < N$ and $p \le q \le p^* = \frac{Np}{N-p}$. In the case $p = N$ one obtains the same result for $p \le q < \infty$, but not for $q = \infty$. Now we want to show that in the case $p > N$ actually more is true: $W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega})$ where $\alpha := 1 - \frac{N}{p} \in (0,1)$.

At first sight it seems odd to prove such a result for elements of Sobolev spaces, that are only defined up to a set of measure zero. Actually, elements of Sobolev spaces are, just as elements of Lebesgue spaces, equivalence classes of functions that coincide almost everywhere. Since continuous functions may become discontinuous (and vice versa) after modification on a null set, it does not make sense to claim that any *function* $u \in W^{1,p}(\Omega)$ should be automatically Hölder-continuous. We rather claim that the *equivalence class* $u \in W^{1,p}(\Omega)$ contains a Hölder-continuous function. In other words, we will prove that for any given $u \in W^{1,p}(\Omega)$ there is $\tilde{u} \in C^{0,\alpha}(\overline{\Omega})$ such that $u = \tilde{u}$ almost everywhere.

We define the spaces $C^{m,\alpha}(\overline{\Omega})$ as follows:

$$\begin{aligned}
\|u\|_{C(\overline{\Omega})} &:= \sup_{\overline{\Omega}} |u|,\\
\|u\|_{C^m(\overline{\Omega})} &:= \sum_{\alpha\in\mathbb{N}_0^N, |\alpha|\le m} \|\partial^\alpha u\|_{C(\overline{\Omega})},
\end{aligned}$$

$$[u]_{C^{0,\alpha}(\overline{\Omega})} := \sup_{x,y\in\Omega, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$$
$$\|u\|_{C^{m,\alpha}(\overline{\Omega})} := \|u\|_{C^m(\overline{\Omega})} + \sum_{\alpha\in\mathbb{N}_0^N, |\alpha|=m} [\partial^\alpha u]_{C^{0,\alpha}(\overline{\Omega})}$$

We use the following result without proof (which is not too difficult).

Satz 7.1. *Let $\Omega \subset \mathbb{R}^N$, $m \in \mathbb{N}_0, \alpha \in (0,1]$. Then the spaces*

$$(C^m(\overline{\Omega}), \|\cdot\|_{C^m(\overline{\Omega})}), \qquad (C^{m,\alpha}(\overline{\Omega}), \|\cdot\|_{C^{m,\alpha}(\overline{\Omega})})$$

are Banach spaces.

To prove embeddings into Hölder spaces, we focus on the model situation $\Omega = \mathbb{R}^N$. Using an Extension operator, this turns out to be sufficient. In the following result we denote by $\omega_N := |B_1(0)|$ the volume of the unit ball in $\mathbb{R}^N$.

Satz 7.2 (Morrey)**.** *Let $N < p < \infty$, $u \in W^{1,p}(\mathbb{R}^N)$ and $\alpha := 1 - \frac{N}{p} \in (0,1)$. Then we have for almost all $x, y \in \mathbb{R}^N$*

$$|u(x)| \le \frac{2p-N}{p-N}\omega_N^{-1/p}\|u\|_{W^{1,p}(\mathbb{R}^N)}$$
$$\frac{|u(x)-u(y)|}{|x-y|^\alpha} \le \frac{4p}{p-N}\omega_N^{-1/p}\|\nabla u\|_{L^p(\mathbb{R}^N)}.$$

Beweis:
We first prove these inequalities for $u \in C_0^\infty(\mathbb{R}^N)$. So let $x, y \in \mathbb{R}^N$ be arbitrary and define their midpoint by $m := \frac{x+y}{2}$, set $\rho := \frac{|x-y|}{2} = |x-m| = |y-m|$. Then we have $|B_\rho| = \omega_N \rho^N$ and thus

$$\begin{aligned}
&\omega_N\rho^N|u(x)-u(y)| \\
&= \int_{B_\rho(m)} |u(x)-u(y)|\,dz \\
&\le \int_{B_\rho(m)} |u(x)-u(z)|\,dz + \int_{B_\rho(m)} |u(y)-u(z)|\,dz \\
&\le \int_{B_\rho(m)} \int_0^1 (|\nabla u(x+t(z-x))||x-z| + |\nabla u(y+t(z-y))||y-z|)\,dt\,dz \\
&\le 2\rho \int_{B_\rho(m)} \int_0^1 (|\nabla u(x+t(z-x))| + |\nabla u(y+t(z-y))|)\,dt\,dz \\
&\le 2\rho \int_0^1 \left(\int_{B_{\rho t}(x+t(m-x))} t^{-N}|\nabla u(z)|\,dz + \int_{B_{\rho t}(y+t(m-y))} t^{-N}|\nabla u(z)|\,dz \right) dt \\
&\le 2\rho \int_0^1 t^{-N}\|\nabla u\|_{L^p(B_{\rho t}(x+t(m-x)))} \left(|B_{\rho t}(x+t(m-x))|^{\frac{p-1}{p}} + |B_{\rho t}(y+t(m-y))|^{\frac{p-1}{p}} \right) dt
\end{aligned}$$

$$\leq 2\rho\|\nabla u\|_{L^p(\mathbb{R}^N)} \int_0^1 t^{-N} \cdot 2(\omega_N(\rho t)^N)^{\frac{p-1}{p}}\, dt$$
$$= 4\rho(\omega_N \rho^N)^{\frac{p-1}{p}} \|\nabla u\|_{L^p(\mathbb{R}^N)} \int_0^1 t^{-N/p}\, dt$$
$$\leq \omega_N \rho^N |x-y|^{1-\frac{N}{p}} \cdot \frac{4p}{p-N} \omega_N^{-1/p} \|\nabla u\|_{L^p(\mathbb{R}^N)}.$$

In the last line we used $2\rho = |x-y|$. The second estimate is proved similarly:

$$\begin{aligned}
\omega_N|u(x)| &\leq \int_{B_1(x)} |u(x)-u(y)| + |u(y)|\, dy \\
&\leq \int_{B_1(x)} |u(x)-u(y)|\, dy + \|u\|_{L^p(B_1(x))}|B_1|^{1/p'} \\
&\leq \int_{B_1(x)} \left(\int_0^1 |\nabla u(x+t(y-x))||x-y|\, dt \right) dy + \|u\|_{L^p(\mathbb{R}^N)} \omega_N^{1/p'} \\
&\leq \int_0^1 \left(\int_{B_t(x)} |\nabla u(y)|\, dy \right) t^{-N}\, dt + \|u\|_{L^p(\mathbb{R}^N)} \omega_N^{1/p'} \\
&\leq \int_0^1 \|\nabla u\|_{L^p(B_t(x))} \cdot |B_t(x)|^{1/p'} t^{-N}\, dt + \|u\|_{L^p(\mathbb{R}^N)} \omega_N^{1/p'} \\
&\leq \omega_N^{1/p'} \|\nabla u\|_{L^p(\mathbb{R}^N)} \int_0^1 t^{-N/p}\, dt + \|u\|_{L^p(\mathbb{R}^N)} \omega_N^{1/p'} \\
&\leq \omega_N \cdot \left(\omega_N^{-1/p} \frac{p}{p-N} \|\nabla u\|_{L^p(\mathbb{R}^N)} + \omega_N^{-1/p} \|u\|_{L^p(\mathbb{R}^N)} \right) \\
&\leq \omega_N \cdot \omega_N^{-1/p} \frac{p}{p-N} \|u\|_{W^{1,p}(\mathbb{R}^N)}.
\end{aligned}$$

This prove the inequalities for test functions u. To treat the general case consider a sequence (u_n) of test functions with $u_n \to u$ in $W^{1,p}(\mathbb{R}^N)$ and $u_n \to u$ almost everywhere. Then the above estimate shows that (u_n) is a Cauchy sequence in $C^{0,\alpha}(\mathbb{R}^N)$. Since this is a Banach space, there is $\tilde{u} \in C^{0,\alpha}(\mathbb{R}^N)$ with $u_n \to \tilde{u}$ in $C^{0,\alpha}(\mathbb{R}^N)$. We this conclude $u = \lim_{n\to\infty} u_n = \tilde{u}$ almost everywhere. □

Korollar 7.3. *Let $N \in \mathbb{N}$ and $N < p < \infty$. Then there is a continuous embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow C^{0,\alpha}(\mathbb{R}^N)$ where $\alpha = 1 - \frac{N}{p}$. For bounded Lipschitz domains $\Omega \subset \mathbb{R}^N$ we have $W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega})$.*

Bemerkung 7.4.

(a) We already saw: The result is not true for $p = N$; $W^{1,p}$–functions need not even be bounded.

(b) The embedding $W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega})$ is true for $p = \infty$, so $W^{1,\infty}$-functions coincide with a Lipschitz-continuous function almost everywhere. We did not include this to avoid technicalities. Notice that the reasoning by density in our proof from above does not work, at least not directly, for such functions. The interested reader may have a look at Rademacher's Theorem.

(c) In 1951 Caldéron [3] proved the following. If $u \in W^{1,p}(\Omega)$ with $N < p \le \infty$, then u coincides almost everywhere with some differentiable function. The derivatives of the latter coincide with the corresponding weak derivatives of u almost everywhere. The argument is based on Lebesgue's differentiation theorem.

End Lec 09

8 Continuous Embeddings of Sobolev spaces: A summary

We start with recapitulating the important continuous embeddings of first order Sobolev spaces $W^{1,p}(\mathbb{R}^N)$ with $1 \le p < \infty$. In the past lectures we have proved the following:

(i) (Theorem 6.4) If $1 < p < N$: $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ for $p \le q \le \frac{Np}{N-p}$

(ii) (Theorem 6.6) If $p = N$: $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ for $p \le q < \infty$

(iii) (Corollary 7.3) If $p > N$: $W^{1,p}(\mathbb{R}^N) \hookrightarrow C^{0,\alpha}(\mathbb{R}^N)$ for $\alpha = 1 - \frac{N}{p}$

These result show to which extent functions belonging to $W^{1,p}(\mathbb{R}^N)$ are better than ordinary $L^p(\mathbb{R}^N)$-functions; the existence of a weak gradient in $L^p(\mathbb{R}^N;\mathbb{R}^N)$ regularizes the function. Singularities become milder ($1 < p < N$ or $p = N$) or even become impossible ($p > N$).

What is the consequence for higher order Sobolev spaces? Assume $1 \le p < N$. For any $u \in W^{2,p}(\mathbb{R}^N)$ we have $\partial_1 u, \dots, \partial_N u \in W^{1,p}(\mathbb{R}^N)$ with weak derivatives $\partial_j(\partial_i u) = \partial_{ij} u \in L^p(\mathbb{R}^N)$. Accordingly, we may apply the embeddings for first order Sobolev spaces to get $\partial_1 u, \dots, \partial_N u \in L^q(\mathbb{R}^N)$ for q as in (i). This implies $u \in W^{1,q}(\mathbb{R}^N)$, hence

$$W^{2,p}(\mathbb{R}^N) \hookrightarrow W^{1,q}(\mathbb{R}^N) \qquad \text{for } p \le q \le \frac{Np}{N-p}.$$

Now we can use the embeddings of $W^{1,q}(\mathbb{R}^N)$ to go further, e.g.,

$$W^{2,p}(\mathbb{R}^N) \hookrightarrow W^{1,q}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N) \qquad \text{for } p \le q \le \frac{Np}{N-p},\ q < N,\ q \le r \le \frac{Nq}{N-q}.$$

This gives in the case $1 \le p \le \frac{N}{2}$:

(i) If $1 \le p < \frac{N}{2}$: $W^{2,p}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ for $p \le r \le \frac{Np}{N-2p}$.

(ii) If $p = \frac{N}{2}$: $W^{2,p}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ for $p \le r < \infty$.

For $p > \frac{N}{2}$ we get

(iii) If $\frac{N}{2} < p < N$: $W^{2,p}(\mathbb{R}^N) \hookrightarrow W^{1,\frac{Np}{N-p}}(\mathbb{R}^N) \hookrightarrow C^{0,\alpha}(\mathbb{R}^N)$ for $\alpha = 2 - \frac{N}{p}$.

(iii) If $p = N$: $W^{2,p}(\mathbb{R}^N) \hookrightarrow \bigcap_{p\le q<\infty} W^{1,q}(\mathbb{R}^N) \hookrightarrow \bigcap_{0<\alpha<1} C^{0,\alpha}(\mathbb{R}^N)$.

(iv) If $p > N$: $W^{2,p}(\mathbb{R}^N) \hookrightarrow C^{1,\alpha}(\mathbb{R}^N)$ for $\alpha = 1 - \frac{N}{p}$.
(because $u, \partial_1 u, \dots, \partial_N u \in C^{0,\alpha}(\mathbb{R}^N)$ implies $u \in C^{1,\alpha}(\mathbb{R}^N)$)

In such a way one obtains the following embeddings.

(A) If $1 \le p < \frac{N}{k}$: $W^{k,p}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ for $p \le r \le \frac{Np}{N-kp}$.

(B) If $p = \frac{N}{k}$: $W^{k,p}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ for $p \le r < \infty$.

(C) If $p > \frac{N}{k}$ and $\frac{N}{p} \notin \mathbb{N}$: $W^{k,p}(\mathbb{R}^N) \hookrightarrow C^{l,\alpha}(\mathbb{R}^N)$ for $l = k - \lfloor \frac{N}{p} \rfloor - 1, \alpha := 1 + \lfloor \frac{N}{p} \rfloor - \frac{N}{p}$

The corresponding embeddings on bounded Lipschitz domains are the same up to replacing $p \le r$ by $1 \le r$ in (A) and (B).

An excursion: The Calculus of Variations was invented to prove the existence of minimizers of a given energy functional. In physical applications this can be

$$I : H_0^1(\Omega) \to \mathbb{R}, \quad u \mapsto \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega fu\,dx$$

where, for simplicity, $f \in L^2(\Omega)$. One can show that such functionals indeed have unique minimizers u that are (again unique) weak solutions of the boundary value problem

$$-\Delta u = f \quad \text{on } \Omega, \qquad u = 0 \text{ on } \partial\Omega.$$

In nonlinear models, other functionals are of importance, for instance

$$J : H_0^1(\Omega) \to \mathbb{R}, \quad u \mapsto \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{q}\int_\Omega |u|^q\,dx - \int_\Omega fu\,dx$$

For which $q > 1$ is this well-defined? Sobolev's Embedding Theorem shows that this functional is well-defined (even continuously differentiable) provided that $1 < q \le \frac{2N}{N-2}$. Having proved the existence of a unique minimizer, which one can do with abstract methods[35], one has found a weak solution to the nonlinear boundary value problem

$$-\Delta u + |u|^{q-2}u = f \quad \text{on } \Omega, \qquad u = 0 \text{ on } \partial\Omega.$$

Being interested in other energy functionals, say

$$J : W_0^{1,p}(\Omega) \to \mathbb{R}, \quad u \mapsto \frac{1}{2}\int_\Omega |\nabla u|^p\,dx + \frac{1}{q}\int_\Omega |u|^q\,dx - \int_\Omega fu\,dx$$

one has a good theory available for $p \le q \le \frac{Np}{N-p}$. The conclusion is that Sobolev's Embedding Theorem allows to treat nonlinear problems with the methods of the Calculus of Variations.

End Lec 10

[35] The "Direct Method of the Calculus of Variations" where lower semi-continuity and compact embeddings are heavily exploited. This is the topic on a course on nonlinear boundary value problems.

9 Compact Embeddings: The Rellich-Kondrachov Theorem and beyond

In the previous two sections we discussed the existence of (continuous) embeddings $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ or $W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega})$ under suitable assumptions on p, q, α, Ω. Now we want to show that these embeddings are not only continuous, but even compact. Here the crucial assumption turns out to be that

(i) Ω is bounded and

(ii) $q < \frac{Np}{N-p}$ resp. $\alpha < 1 - \frac{N}{p}$. ("Non-endpoint cases")

We will see in the Exercises that these assumptions are natural in the sense that "typically" the embeddings are not compact for unbounded Ω or Sobolev-critical exponents. Compactness is of utmost importance for the whole theory of analysis, notably elliptic boundary value problems and the calculus of variations. We will later provide some more information about this. We start with the definition of compactness.

Definition 9.1. *A linear operator $K : X \to Y$ between Banach spaces X, Y is called compact if for every bounded sequence $(x_n)_{n\in\mathbb{N}} \subset X$ the sequence $(Kx_n)_{n\in\mathbb{N}}$ has a convergent subsequence in Y.*

We mention a few basic observations about compact operators:

- Every compact operator is bounded, but the converse is not true. (Do not dare to forget this fact!!!)
- If X, Y are finite-dimensional Banach spaces, so $X \simeq \mathbb{R}^k, Y \simeq \mathbb{R}^l$ for some $k, l \in \mathbb{N}_0$, then every bounded linear operator $X \to Y$ is compact.
- The identity map $\iota : X \to X$ is compact if and only if X is finite-dimensional. The proof of this fact relies on Riesz' Lemma.
- The operator $A : l^2(\mathbb{N}) \to l^2(\mathbb{N}),\ (c_n)_{n\in\mathbb{N}} \mapsto (a_n c_n)_{n\in\mathbb{N}}$ is compact if and only if $(a_n)_{n\in\mathbb{N}}$ is a null sequence.
- A linear compact operator $A : L^p(\Omega) \to L^p(\Omega)$ is characterized by the property that there is a sequence $(A_n)_{n\in\mathbb{N}}$ of bounded linear operators with finite-dimensional range (i.e., $\{A_n x : x \in X\}$ is a finite-dimensional subspace of Y) and

$$\|A_n - A\|_{X\to Y} = \sup_{f\in X, \|f\|_X = 1} \|(A_n - A)f\|_Y \to 0 \quad \text{as } n \to \infty.$$

We now want to investigate when the inclusion maps

$$\iota : W^{1,p}(\Omega) \to L^q(\Omega), u \mapsto u, \qquad \iota : W^{1,p}(\Omega) \to C^{0,\alpha}(\overline{\Omega}), u \mapsto u$$

are compact.

9.1 Compact Embeddings into Hölder spaces

The starting point of compactness investigations is the Ascoli-Arzelà Theorem from 1884 [**?**] resp. 1894 Arzelà [**?**]. It relies on the notion of equicontinuity.

Definition 9.2 (Equicontinuity)**.** *Let $K \subset \mathbb{R}^N$ be closed. A sequence $(f_n)_{n\in\mathbb{N}}$ in $C(K)$ is called equicontinuous if for all $\varepsilon > 0$ there is $\delta_\varepsilon > 0$ such that for all $x, y \in K$*

$$|x - y| < \delta_\varepsilon \quad \Rightarrow \quad |f_n(x) - f_n(y)| < \varepsilon \text{ for all } n \in \mathbb{N}.$$

An important class of equicontinuous families of functions are "uniformly Hölder-continuous" functions $(f_n)_{n\in\mathbb{N}}$ satisfying $[f_n]_{C^{0,\alpha}(K)} \le M$ for some $\alpha, M > 0$. Indeed, in that case $|x - y| < \delta_\varepsilon := (M^{-1}\varepsilon)^{1/\alpha}$ implies

$$|f_n(x) - f_n(y)| \le M|x - y|^\alpha < M\delta_\varepsilon^\alpha = \varepsilon.$$

Satz 9.3 (Ascoli(1884), Arzelà (1894))**.** *Let $K \subset \mathbb{R}^N$ be bounded and closed and let $(f_n)_{n\in\mathbb{N}} \subset C(K)$ be a pointwise bounded and equicontinuous sequence. Then $(f_n)_{n\in\mathbb{N}}$ has a uniformly convergent subsequence.*

Beweis:
We choose $x_1, x_2, \ldots$ such that $\mathbb{Q} \cap K = \{x_i : i \in \mathbb{N}\}$. Then $(f_n(x_1))_{n\in\mathbb{N}}$ is (by assumption) a bounded sequence of real numbers. So there is an injective map $\psi_1 : \mathbb{N} \to \mathbb{N}$ such that the subsequence $(f_{\psi_1(n)}(x_1))_{n\in\mathbb{N}}$ converges. Next, $(f_{\psi_1(n)}(x_2))_{n\in\mathbb{N}}$ is a bounded sequence of real numbers and we find another injective map $\psi_2 : \mathbb{N} \to \mathbb{N}$ such that the subsequence $(f_{\psi_1(\psi_2(n))}(x_2))_{n\in\mathbb{N}}$ converges. Since it is a subsequence of the previous subsequence, we get that both $(f_{\psi_1(\psi_2(n))}(x_1))_{n\in\mathbb{N}}, (f_{\psi_1(\psi_2(n))}(x_2))_{n\in\mathbb{N}}$ converge. Inductively, one finds injective maps $\psi_1, \psi_2, \ldots, \psi_k : \mathbb{N} \to \mathbb{N}$ such that the sequences $(f_{\Psi_k(n)})(x_i))_{n\in\mathbb{N}}$ converge for $i = 1, \ldots, k$ where $\Psi_k(n) := \psi_1(\psi_2(\ldots(\psi_k(n))))$. Define the diagonal sequence $g_n(x) := f_{\Psi_n(n)}(x)$. Then $(g_n)_{n\in\mathbb{N}}$ is a subsequence of $(f_n)_{n\in\mathbb{N}}$ and we want to show that it is a Cauchy sequence in $C(K)$.

Indeed, given $\varepsilon > 0$ we choose $x_1, \ldots, x_M \in K$ such that

$$K \subset \bigcup_{i=1}^M B_\delta(x_i) \quad \text{for } \delta = \delta_{\varepsilon/3} \text{ as above.}$$

Then choose $n_0 \in \mathbb{N}$ such that

$$|g_n(x_i) - g_m(x_i)| \le \frac{\varepsilon}{3} \quad \text{for } i = 1, \ldots, M \text{ and } n, m \ge n_0.$$

Then we have for all $x \in K$ and $n, m \ge n_0$

$$|g_n(x) - g_m(x)| \le \min_{i=1,\ldots,M} \left[|g_n(x) - g_n(x_i)| + |g_n(x_i) - g_m(x_i)| + |g_m(x_i) - g_m(x)|\right]$$

$$\leq \frac{\varepsilon}{3} + \min_{i=1,\dots,M} \big[|g_n(x) - g_n(x_i)| + |g_m(x_i) - g_m(x)| \big]$$
$$\leq \varepsilon$$

by choosing x_i (dependent on x) such that $|x - x_i| < \delta_{\varepsilon/3}$, which is possible as we saw above. So $(g_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $(C(K), \|\cdot\|_{C(K)})$ and thus converges uniformly to some continuous function. □

Satz 9.4. *Let $\Omega \subset \mathbb{R}^N$ be bounded and $1 > \alpha > \beta > 0$. Then the embedding $C^{0,\alpha}(\bar{\Omega}) \hookrightarrow C^{0,\beta}(\bar{\Omega})$ is compact*[36].

Beweis:
Let $(f_n)_{n\in\mathbb{N}}$ be bounded in $C^{0,\alpha}(\overline{\Omega})$ with $M := \sup_{n\in\mathbb{N}} \|f_n\|_{0,\alpha} < \infty$. We have seen above that $(f_n)_{n\in\mathbb{N}}$ is then equicontinuous. Moreover, this sequence is pointwise bounded due to $\|f_n\|_{C(\overline{\Omega})} \leq \|f_n\|_{0,\alpha} \leq M$ for all $n \in \mathbb{N}$. So the Ascoli-Arzelà Theorem provides a uniformly convergent subsequence $(f_{n_j})_{j\in\mathbb{N}}$ with limit $f \in C(\bar{\Omega})$. We want to show $f_{n_j} \to f$ in $C^{0,\alpha}(\overline{\Omega})$. To simplify the notation we write f_j instead of f_{n_j},

To see this we estimate as follows for $x \neq y$:

$$\frac{|(f - f_j)(x) - (f - f_j)(y)|}{|x - y|^\beta} \leq \lim_{i\to\infty} \frac{|f_i(x) - f_i(y)| + |f_j(x) - f_j(y)|}{|x - y|^\beta} \leq 2M|x - y|^{\alpha-\beta}, \quad (9.1)$$

$$\frac{|(f - f_j)(x) - (f - f_j)(y)|}{|x - y|^\beta} \leq \frac{2}{|x - y|^\beta} \|f - f_j\|_{C(\bar{\Omega})}. \quad (9.2)$$

For any given $\varepsilon > 0$ choose $j_0 \in \mathbb{N}$ such that

$$\|f - f_j\|_{C(\bar{\Omega})} < \min\Big\{\frac{\varepsilon}{2}, \frac{\varepsilon}{4}\Big(\frac{\varepsilon}{4M}\Big)^{\frac{\beta}{\alpha-\beta}}\Big\} \qquad \text{for all } j \geq j_0.$$

Then we get for $x \neq y \in \overline{\Omega}$

$$|x - y| < \Big(\frac{\varepsilon}{4M}\Big)^{\frac{1}{\alpha-\beta}} \quad \Rightarrow \quad \frac{|(f - f_j)(x) - (f - f_j)(y)|}{|x - y|^\beta} \overset{(9.1)}{\leq} 2M|x - y|^{\alpha-\beta} < \frac{\varepsilon}{2},$$

$$|x - y| \geq \Big(\frac{\varepsilon}{4M}\Big)^{\frac{1}{\alpha-\beta}} \quad \Rightarrow \quad \frac{|(f - f_j)(x) - (f - f_j)(y)|}{|x - y|^\beta} \overset{(9.2)}{\leq} \frac{2\|f - f_j\|_{C(\bar{\Omega})}}{|x - y|^\beta} < \frac{\varepsilon}{2}.$$

This implies

$$\|f - f_j\|_{0,\beta} = \|f - f_j\|_{C(\bar{\Omega})} + [f - f_j]_{0,\beta} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which is all we had to show. □

With a little bit of technical work, one may prove the more general statement that $C^{k_1,\alpha_1}(\bar{\Omega}) \hookrightarrow C^{k_2,\alpha_2}(\bar{\Omega})$ is compact provided that $k_1, k_2 \in \mathbb{N}_0$ und $\alpha_1, \alpha_2 \in (0, 1)$ satisfy $k_1 + \alpha_1 > k_2 + \alpha_2$.

[36] We haven't even proved the weaker statement $C^{0,\alpha}(\bar{\Omega}) \subset C^{0,\beta}(\bar{\Omega})$ yet. It is a consequence of this result.

Korollar 9.5. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain and assume $N < p < \infty$. Then the embedding $W^{1,p}(\Omega) \hookrightarrow C^{0,\beta}(\overline{\Omega})$ is compact provided that $0 < \beta < 1 - \frac{N}{p}$. In particular, $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for all $q \in [1, \infty]$.*

Beweis:
Let $(u_n)_{n\in\mathbb{N}}$ be bounded in $W^{1,p}(\Omega)$. Then Morrey's Embedding Theorem shows that there is a $C > 0$ such that

$$\sup_{n\in\mathbb{N}} \|u_n\|_{C^{0,\alpha}(\overline{\Omega})} \le C \sup_{n\in\mathbb{N}} \|u_n\|_{W^{1,p}(\Omega)} < \infty$$

for $\alpha := 1 - \frac{N}{p} > 0$. So Theorem 9.4 implies that $(u_n)_{n\in\mathbb{N}}$ has a subsequence that converges in $C^{0,\beta}(\overline{\Omega})$. This proves the first claim. In view of

$$\|u_n - u\|_{L^q(\Omega)} \le \|u_n - u\|_{L^\infty(\Omega)} |\Omega|^{\frac{1}{q}} \le \|u_n - u\|_{C^{0,\beta}(\overline{\Omega})} |\Omega|^{\frac{1}{q}}$$

for all $q \in [1, \infty]$ the second claim holds as well. □

Notice that this has an abstract generalization: The concatenation of bounded linear operator and a compact linear operator is a compact linear operator.

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9.2 Compact Embeddings into Lebesgue spaces

We now prove the (more important) statement that Sobolev's Embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for $1 \le q < p^*$. This is the Rellich-Kondrachov Theorem – Rellich [19] proved it in the special case $p = q = 2$ and Kondrachov [11] extended this result to general p, q. A modern proof may be based on a characterization of precompact subsets of $L^p(\Omega)$ respectively $L^p(\mathbb{R}^N)$. We use without proof the following characterization of such sets.

Proposition 9.6. *Let $(X, \|\cdot\|_X)$ be a Banach space and $V \subset X$ a subset. Then the following statements are equivalent:*

- *(Precompactness) Every sequence in V has a convergent subsequence in X.*
- *(Total boundedness) For all $\varepsilon > 0$ the set $\overline{V}$ can be covered by finitely many balls in X with radius ε.*

We are going to use the Fréchet-Kolmogorov-Riesz criterion to check the compactness of the embedding. This criterion requires for two preliminary results about estimates for $f(\cdot + h) - f$ when $h \in \mathbb{R}^N$ is given (and small).

Proposition 9.7. *Let $f \in L^p(\mathbb{R}^N)$. Then $f(\cdot + h) \to f$ in $L^p(\mathbb{R}^N)$ as $|h| \to 0$.*

Beweis:
Choose $\varepsilon > 0$ and $g \in C_0^\infty(\mathbb{R}^N)$ such that $\|f - g\|_p < \frac{\varepsilon}{3}$, see Theorem 4.8. Since g is uniformly continuous with compact support, the Dominated Convergence Theorem yields a $\delta > 0$ such that $|h| \leq \delta$ implies $\|g(\cdot + h) - g\|_p < \frac{\varepsilon}{3}$, whence

$$\begin{aligned}\|f(\cdot + h) - f\|_p &\leq \|f(\cdot + h) - g(\cdot + h)\|_p + \|g(\cdot + h) - g\|_p + \|g - f\|_p \\ &\leq 2\|f - g\|_p + \|g(\cdot + h) - g\|_p \\ &< \varepsilon.\end{aligned}$$

□

Proposition 9.8. *Let $f \in W^{1,p}(\mathbb{R}^N)$. Then we have for all $h \in \mathbb{R}^N$*

$$\|f(\cdot + h) - f\|_p \leq |h| \|\nabla f\|_p.$$

Beweis:
By density (Lemma 4.12), it suffices to prove this inequality for $C_0^\infty(\mathbb{R}^N)$. We use

$$|f(x+h) - f(x)| = \left| \int_0^1 \nabla f(x + th) \cdot h \, dt \right| \leq |h| \int_0^1 |\nabla f(x + th)| \, dt \leq |h| \left(\int_0^1 |\nabla f(\cdot + th)|^p \, dt \right)^{\frac{1}{p}}.$$

Integrating this over all $x \in \mathbb{R}^N$ we get

$$\|f(\cdot + h) - f\|_p \leq |h| \left(\int_{\mathbb{R}^N} \int_0^1 |\nabla f(x + th)|^p \, dt \, dx \right)^{\frac{1}{p}} = |h| \left(\int_0^1 \|\nabla f\|_p^p \, dt \right)^{\frac{1}{p}} = |h| \|\nabla f\|_p.$$

□

The historical background of the following result is beautifully described in the survey paper [10].

Satz 9.9 (Fréchet (1908), Kolmogorov (1931), M. Riesz (1933))**.** *Assume $1 \leq p < \infty$ and $N \in \mathbb{N}$. Then a family $\mathfrak{F}$ is precompact in $L^p(\mathbb{R}^N)$ if and only if the following conditions hold:*

(i) $\mathfrak{F}$ is bounded in $L^p(\mathbb{R}^N)$.

(ii) For all $\varepsilon > 0$ there is a $\delta_\varepsilon > 0$ such that $\|f(\cdot + h) - f\|_p < \varepsilon$ for all $f \in \mathfrak{F}, h \in \mathbb{R}^N, |h| \leq \delta$.

(iii) For all $\varepsilon > 0$ there is a compact subset $K_\varepsilon \subset \mathbb{R}^N$ such that $\|f\|_{L^p(\mathbb{R}^N \setminus K_\varepsilon)} < \varepsilon$ for all $f \in \mathfrak{F}$.

Beweis:
We first show that a precompact family $\mathfrak{F}$ satisfies (i),(ii),(iii). So let $\varepsilon > 0$. Then there are $g_1, \dots, g_m \in L^p(\mathbb{R}^N)$ such that

$$\mathfrak{F} \subset \bigcup_{i=1}^{m} B_{\varepsilon/3}(g_i). \tag{9.3}$$

So (i) follows from

$$\|f\|_p \le \max_{i=1,\dots,m} \|g_i\|_p + \frac{\varepsilon}{3} \quad \text{for all } f \in \mathfrak{F}.$$

Moreover, Proposition 9.7 provides a $\delta_\varepsilon > 0$ such that

$$\max_{i=1,\dots,m} \|g_i(\cdot + h) - g_i\|_p < \frac{\varepsilon}{3} \qquad \text{for } |h| \le \delta_\varepsilon.$$

For any given $f \in \mathfrak{F}$ we may then choose (according to (9.3)) $i \in \{1, \dots, m\}$ such that $\|f - g_i\|_p < \frac{\varepsilon}{3}$. Hence, (ii) results from

$$\begin{aligned} \|f(\cdot + h) - f\|_p &\le \|f(\cdot + h) - g_i(\cdot + h)\|_p + \|g_i(\cdot + h) - g_i\|_p + \|g_i - f\|_p \\ &\le \varepsilon \qquad \text{for all } f \in \mathfrak{F}. \end{aligned}$$

To prove (iii) let us choose $\varphi_1, \dots, \varphi_m \in C_0^\infty(\mathbb{R}^N)$ such that $\|g_i - \varphi_i\|_p < \frac{2\varepsilon}{3}$, see Theorem 4.8, set $K_\varepsilon := \bigcup_{i=1}^m \operatorname{supp}(\varphi_i)$. Choosing g_i as above for any given $f \in \mathfrak{F}$ we get

$$\begin{aligned} \|f\|_{L^p(\mathbb{R}^N \setminus K_\varepsilon)} &= \|f - \varphi_i\|_{L^p(\mathbb{R}^N \setminus K_\varepsilon)} \\ &\le \|f - g_i\|_{L^p(\mathbb{R}^N \setminus K_\varepsilon)} + \|g_i - \varphi_i\|_{L^p(\mathbb{R}^N \setminus K_\varepsilon)} \\ &\le \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} \\ &= \varepsilon. \end{aligned}$$

This finishes the first part of the proof.

Now assume that $\mathfrak{F} \subset L^p(\mathbb{R}^N)$ satisfies (i),(ii),(iii), let $\varepsilon > 0$ be arbitrary. The strategy is to approximate the family by a family of continuous functions to which we can apply the Ascoli-Arzelà Theorem. Proposition 4.7 and (ii) show that a nonnegative $\rho \in C_0^\infty(\mathbb{R}^N)$ with $\|\rho\|_1 = 1$ and sufficiently small support may be chosen in such a way that the following holds for all(!)[37] $f \in \mathcal{F}$:

$$\begin{aligned}
\|\rho * f - f\|_p &= \left(\int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} \rho(x-y) f(y)\, dy - f(x) \right|^p dx \right)^{\frac{1}{p}} \\
&\le \left(\int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} \rho(h)^{\frac{1}{p'}} \cdot \rho(h)^{\frac{1}{p}} (f(x-h) - f(x))\, dh \right|^p dx \right)^{\frac{1}{p}} \\
&\le \left(\int_{\mathbb{R}^N} \|\rho^{\frac{1}{p'}}\|_{p'}^p \|\rho^{\frac{1}{p}} (f(x-\cdot) - f(x))\|_p^p\, dx \right)^{\frac{1}{p}} \\
&\le \|\rho\|_1^{\frac{1}{p'}} \left(\int_{\mathbb{R}^N} \rho(h) \left(\int_{\mathbb{R}^N} |f(x-h) - f(x)|^p\, dx \right) dh \right)^{\frac{1}{p}} \qquad \text{(Fubini)} \\
&\le 1 \cdot \sup_{h \in \operatorname{supp}(\rho)} \|f(\cdot + h) - f\|_p \cdot \left(\int_{\mathbb{R}^N} \rho(h)\, dh \right)^{\frac{1}{p}} \\
&\le \sup_{h \in \operatorname{supp}(\rho)} \|f(\cdot + h) - f\|_p < \frac{\varepsilon}{3} \qquad \text{for all } f \in \mathfrak{F}.
\end{aligned} \tag{9.4}$$

Having chosen ρ in such a way we consider the family of continuous(!) functions

$$\mathfrak{G} := \{(\rho * f)|_K : f \in \mathfrak{F}\} \subset C(K)$$

where $K := K_{\varepsilon/3}$ according to (iii). Let us show that this family is pointwise bounded and equicontinuous.

Then $\mathfrak{G}$ is pointwise bounded due to

$$\|\rho * f\|_{C(K)} \le \|\rho * f\|_\infty \le \|\rho\|_{p'} \|f\|_p \le \|\rho\|_{p'} M \quad \text{for all } f \in \mathfrak{F}.$$

Moreover, it is equicontinuous due to

$$\begin{aligned}
\sup_{x,y \in K, |x-y| < \delta} |(\rho * f)(x) - (\rho * f)(y)| &= \sup_{x,y \in K, |x-y| < \delta} \left| \int_{\mathbb{R}^N} \rho(x-z) f(z)\, dz - \int_{\mathbb{R}^n} \rho(y-z) f(z)\, dz \right| \\
&\le \sup_{x,y \in K, |x-y| < \delta} \int_{\mathbb{R}^N} |\rho(x-z)| |f(z) - f(y-x+z)|\, dz \\
&\le \sup_{x,y \in K, |x-y| < \delta} \|\rho(x-\cdot)\|_{p'} \|f - f(y-x+\cdot)\|_p \\
&= \|\rho\|_{p'} \sup_{x,y \in K, |x-y| < \delta} \|f - f(y-x+\cdot)\|_p \\
&= o(1) \qquad \text{as } \delta \to 0 \text{ uniformly w.r.t. } f \in \mathfrak{F}.
\end{aligned}$$

So the Ascoli-Arzelà Theorem shows that $\mathfrak{G}$ is precompact in $C(K)$. This implies that for $\tilde{\varepsilon} := \frac{\varepsilon}{3|K|^{1/p}}$ there are functions $g_1, \dots, g_m \in C(K)$ such that

$$\mathfrak{G} \subset \bigcup_{i=1}^m \{h \in C(K) : \|h - g_i\|_{C(K)} < \tilde{\varepsilon}\}.$$

[37] Notice the subtle difference: In Proposition 4.7 we showed $\|\rho * f - f\|_p$ can be arbitrarily small for any given $f \in L^p(\mathbb{R}^N)$. This is, however, not sufficient to conclude because we need a uniform approximation property for all $f \in \mathfrak{F}$. So we need "$\exists \rho \forall f$ instead of "$\forall f \exists \rho$".

Extending the functions g_i trivially to $\mathbb{R}^N$, we obtain for all $f \in \mathfrak{F}$

$$\begin{aligned}
\min_{i=1,\dots,m} \|f - g_i\|_{L^p(\mathbb{R}^N)} &= \min_{i=1,\dots,m} \left(\|f\|_{L^p(\mathbb{R}^N \setminus K)} + \|f - g_i\|_{L^p(K)} \right) \\
&\le \frac{\varepsilon}{3} + \min_{i=1,\dots,m} \left(\|f - \rho * f\|_{L^p(K)} + \| \underbrace{(\rho * f)|_K}_{\in \mathfrak{G}} - g_i\|_{L^p(K)} \right) \\
&\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \tilde{\varepsilon} \cdot \|1\|_{L^p(K)} \\
&= \varepsilon.
\end{aligned}$$

Hence

$$\mathfrak{F} \subset \bigcup_{i=1}^{m} \{h \in L^p(\mathbb{R}^N) : \|h - g_i\|_{L^p(\mathbb{R}^N)} < \varepsilon\},$$

which is all we had to show. □

Satz 9.10 (Rellich-Kondrachov Theorem). *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain and $1 \le p < N$. Then the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for $1 \le q < \frac{Np}{N-p}$.*

Beweis:
The first and main step is to prove the compactness of $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$. We have to show that every bounded sequence $(u_n)_{n\in\mathbb{N}}$ in $W^{1,p}(\Omega)$ has a subsequence that converges in $L^p(\Omega)$. To this end we show that

$$\mathfrak{F} := \{f_n : n \in \mathbb{N}\} \quad \text{where } f_n := E(u_n)\chi$$

is precompact provided that $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$ denotes Stein's Extension operator and $\chi \in C_0^\infty(\mathbb{R}^N)$ satisfies $\chi(x) = 1$ for $x \in \overline{\Omega}$, set $K := \operatorname{supp}(\chi)$.

We show that (i),(ii),(iii) from Theorem 9.9 are satisfied:

(i) We have

$$\|f_n\|_p = \|(Eu_n)\chi\|_p \le \|\chi\|_\infty \|Eu_n\|_p \le \|\chi\|_\infty \|E\| \|u_n\|_{1,p} \le C$$

(ii) Proposition 9.8 gives

$$\begin{aligned}
\|f_n(\cdot + h) - f_n\|_p &= \|((Eu_n)\chi)(\cdot + h) - (Eu_n)\chi\|_p \\
&\le |h| \|\nabla((Eu_n)\chi)\|_p \\
&\le |h| \|Eu_n\chi\|_{1,p} \\
&\le |h| \|Eu_n\|_{1,p} \|\chi\|_{1,\infty} && \text{(Product rule)} \\
&\le C|h|.
\end{aligned}$$

(iii) This follows from $\|f_n\|_{L^p(\mathbb{R}^N \setminus K)} = 0$.

So Theorem 9.9 shows that $\mathfrak{F}$ is precompact in $L^p(\mathbb{R}^N)$ and hence $f_n \to f$ in $L^p(\mathbb{R}^N)$ after passing to a subsequence. In particular, for $u := f\mathbb{1}_\Omega$,

$$\|u_n - u\|_{L^p(\Omega)} = \|f_n - f\|_{L^p(\Omega)} \le \|f_n - f\|_{L^p(\mathbb{R}^N)} \to 0 \quad \text{as } n \to \infty.$$

This proves that $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact.

To treat general exponents $1 \le q < p^*$ we use Lyapunov's Inequality and choose $\theta \in (0,1)$ according to $\frac{1}{q} = \frac{\theta}{1} + \frac{1-\theta}{p^*}$. Then

$$\begin{aligned}\|u_n - u\|_{L^q(\Omega)} &\le \|u_n - u\|_{L^1(\Omega)}^{\theta} \|u_n - u\|_{L^{p^*}(\Omega)}^{1-\theta} \\ &\le |\Omega|^{\frac{\theta(p-1)}{p}} \|u_n - u\|_{L^p(\Omega)}^{\theta} \cdot C\|u_n - u\|_{W^{1,p}(\Omega)}^{1-\theta} \\ &\le C'\|u_n - u\|_{L^p(\Omega)}^{\theta}.\end{aligned}$$

Since θ is bigger than zero (here we use $q < p^*$) we find $u_n \to u$ is $L^q(\Omega)$. □

We now comment on why the compactness is important. The short answer is that it shows that the solution theory of elliptic boundary value problems can be reduced to the solution theory for linear problems of the form

$$(I - K)u = f, \qquad u \in H_0^1(\Omega)$$

where $f \in L^2(\Omega)$ and $K : L^2(\Omega) \to L^2(\Omega)$ is compact. The solution theory for such equation is well-known; it is governed by Fredholm's Alternative. Without the compactness assumption this theory breaks down.

To see why these equations appear consider

$$(-\Delta + 1)u + c(x)u = f(x), \qquad u \in H_0^1(\Omega).$$

Using the "solution operator"[38] $(-\Delta+1)^{-1} : L^2(\Omega) \to H_0^1(\Omega)$ from Corollary 3.2 we obtain the equivalent problem

$$u + (-\Delta + 1)^{-1}(cu) = (-\Delta + 1)^{-1}f, \qquad u \in H_0^1(\Omega).$$

Assuming $c \in L^\infty(\Omega)$ this equation of the form above with

$$K : H_0^1(\Omega) \to H_0^1(\Omega), \quad \phi \mapsto (-\Delta + 1)^{-1}(c \cdot \iota\phi).$$

This is a compact (and self-adjoint) operator as a concatenation of the bounded operators $(-\Delta + 1)^{-1} : L^2(\Omega) \to H_0^1(\Omega), L^2(\Omega) \ni \phi \mapsto c\phi \in L^2(\Omega)$ and the compact Embedding operator $\iota : H_0^1(\Omega) \to L^2(\Omega)$.

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[38] It maps $f \in L^2(\Omega)$ to the unique $H_0^1(\Omega)$-solution of $-\Delta u + u = f$. It is therefore a kind of inverse (a right inverse) of the differential operator $-\Delta + 1$ in a weak sense.

10 Poincaré's Inequality and Applications

In this section we want to remove some insufficiency in our analysis of the boundary value problem

$$-\Delta u + c(x)u = f(x) \quad \text{in } \Omega, \qquad u \in H_0^1(\Omega).$$

The very important case $c \equiv 0$ was not covered by this discussion because our approach required $c(x)$ to be positive so that the bilinear form

$$(u,v) \mapsto \int_\Omega \nabla u \cdot \nabla v + c(x)uv\,dx \tag{10.1}$$

is bounded and coercive on $H_0^1(\Omega)$. Being given these properties we deduced the existence of solutions to this boundary value problem from the Lax-Milgram Theorem resp. Riesz' Representation Theorem. On the other hand, it is known from Classical PDE Theory, notably Perron's method, that problems of the form $-\Delta u = f$ are equally well-behaved, at least for continuous right hand sides and bounded domains $\Omega \subset \mathbb{R}^N$. So the question is immediate whether the weak solution approach using Sobolev spaces may be refined to cover this case is well. This is the topic we want to discuss here.

According to the above, we may concentrate on weakest possible conditions on c that make the bilinear form (10.1) coercive. Working in the space $H^1(\Omega)$ we cannot do much about the case $c \equiv 0$ because the inequality

$$\left(\int_\Omega |\nabla u|^2\,dx\right)^{1/2} \geq \alpha \|u\|_{H^1(\Omega)} \qquad (u \in H^1(\Omega))$$

cannot hold for any positive α. In fact, nontrivial constant functions belong to $H^1(\Omega)$ and give zero on the left and something positive on the right, contradiction! Here we used that Ω is a bounded domain. But it turns out that the estimate

$$\left(\int_\Omega |\nabla u|^2\,dx\right)^{1/2} \geq \alpha \|u\|_{H^1(\Omega)} \qquad (u \in H_0^1(\Omega))$$

is true for some positive α. This will be a consequence of Poincaré's inequality to be proved below. As a consequence, the counterexample of constant functions does not work any more and so the only constant function belonging to $H_0^1(\Omega)$ is the trivial one. In particular this tells us that $H_0^1(\Omega)$ is a strict subspace of $H^1(\Omega)$.

Such an inequality was used in the proof of Sobolev's Embedding Theorem on $\mathbb{R}^N$ where $H^1(\mathbb{R}^N) = H_0^1(\mathbb{R}^N)$. Using the Mean Value Theorem we expressed $u(x)$ in terms of its derivatives only and estimated the integrals using Hölder's Inequality. This worked because all elements of the dense subspace $C_0^\infty(\mathbb{R}^N)$ vanish at infinity. We will see below that the same sort of idea works in a general bounded domain Ω as long as "u vanishes somewhere in Ω".

Satz 10.1 (Poincaré (1890), Friedrichs (1928)). *Assume* $\Omega \subset \{x \in \mathbb{R}^N : a < x \cdot v < b\}$ *for some* $v \in \mathbb{R}^N, |v| = 1$. *Then, we have for* $1 \le p < \infty$ *and* $u \in W_0^{1,p}(\Omega)$

$$\|u\|_p \le \frac{p}{2}(b-a)\|\partial_v u\|_p.$$

Beweis. Since $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega)$ (by definition), it suffices to prove this estimate for $u \in C_0^\infty(\Omega)$. Choose $x_0 \in \mathbb{R}^N$ such that $x_0 \cdot v = \frac{a+b}{2}$, so

$$|(x - x_0) \cdot v| \le \frac{b-a}{2} \qquad \text{for all } x \in \Omega.$$

In the case $p > 1$ we use the following identity[39]:

$$\partial_v \left((\cdot - x_0) \cdot v |u|^p \right) = |u|^p + p(\cdot - x_0) \cdot v |u|^{p-2} u \partial_v u \quad \text{in } \Omega.$$

Integrating this over Ω gives

$$0 = \int_\Omega \partial_v \left((\cdot - x_0) \cdot v |u|^p \right) dx = \int_\Omega |u|^p + p(\cdot - x_0) \cdot v |u|^{p-2} \partial_v u \, dx.$$

Hence,

$$\begin{aligned} \|u\|_p^p &\le p \int_\Omega |(x - x_0) \cdot v| |u|^{p-1} |\partial_v u| \, dx \\ &\le \frac{p}{2}(b-a) \int_\Omega |u|^{p-1} |\partial_v u| \, dx \\ &\le \frac{p}{2}(b-a) \|u\|_p^{p-1} \|\partial_v u\|_p. \end{aligned}$$

This gives the claim in the case $p > 1$. The claim for $p = 1$ follows from the Dominated Convergence Theorem as $p \searrow 1$. □

Given this result we define

$$\begin{aligned} \langle u, v \rangle_{H_0^1(\Omega)} &:= \int_\Omega \nabla u \cdot \nabla v \, dx, \\ \|u\|_{H_0^1(\Omega)} &:= \sqrt{\langle u, u \rangle_{H_0^1(\Omega)}}, \\ \|u\|_{W_0^{1,p}(\Omega)} &:= \left(\int_\Omega |\nabla u|^p \, dx \right)^{1/p} \end{aligned}$$

Korollar 10.2. *Assume that* $\Omega \subset \mathbb{R}^N$ *is a bounded domain and* $1 \le p < \infty$. *Then* $\|\cdot\|_{W_0^{1,p}(\Omega)}$ *is a norm on* $W_0^{1,p}(\Omega)$ *that is equivalent to* $\|\cdot\|_{W^{1,p}(\Omega)}$.

[39] This identity holds in the weak sense. It is not immediate, but rather follows by approximation of $t \mapsto |t|^p$ by smooth versions such as $t \mapsto (t^2 + \varepsilon^2)^{p/2} - \varepsilon^p$.

Beweis:
We only show

$$\beta\|u\|_{W^{1,p}(\Omega)} \geq \|u\|_{W_0^{1,p}(\Omega)} \geq \alpha\|u\|_{W^{1,p}(\Omega)} \qquad (u \in W_0^{1,p}(\Omega))$$

for some $\alpha, \beta > 0$. In fact we can choose $\beta = 1$ because of

$$\|u\|^p_{W_0^{1,p}(\Omega)} = \int_\Omega |\nabla u|^p \, dx \leq \int_\Omega |\nabla u|^p \, dx + \int_\Omega |u|^p \, dx = \|u\|^p_{W^{1,p}(\Omega)}$$

The nontrivial opposite bound is a consequence of Poincaré's Inequality. To see this choose $a, b \in \mathbb{R}$ and v as in Theorem 10.1. Then

$$\|u\|^p_{W^{1,p}(\Omega)} = \|u\|^p_{L^p(\Omega)} + \||\nabla u|\|^p_{L^p(\Omega)} \leq \left|\frac{p}{2}(b-a)\right|^p \|\partial_v u\|^p_{L^p(\Omega)} \leq \left|\frac{p}{2}(b-a)\right|^p \|\nabla u\|^p_{L^p(\Omega)}$$

We conclude that one possible choice is

$$\alpha := \left(\left(\frac{p}{2}(b-a)\right)^p + 1\right)^{-\frac{1}{p}}.$$

□

Attention: $\|\cdot\|_{W_0^{1,p}(\Omega)}$ is not a norm on $W^{1,p}(\Omega)$, only on $W^{1,p}(\Omega)$. Given that the norms are equivalent on $W_0^{1,p}(\Omega)$, we know that this space is a Banach space when equipped with this new norm. The quantity

$$C_P(\Omega, p) := \sup_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\|u\|_{L^p(\Omega)}}{\||\nabla u|\|_{L^p(\Omega)}}$$

is called the Poincaré constant. It is particularly important in the case $p = 2$.

10.1 Applications to boundary value problems

Let's draw the consequences for our boundary value problem (3.3), which was given by

$$-\Delta u(x) + c(x)u(x) = f(x) \quad (x \in \Omega), \qquad u(x) = 0 \quad (x \in \partial\Omega).$$

Korollar 10.3. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$ be a bounded Lipschitz domain and assume $f \in L^{\frac{2N}{N+2}}(\Omega), c \in L^{\frac{N}{2}}(\Omega)$ with $c \geq 0$ almost everywhere in Ω. Then (3.3) has a unique weak solution $u \in H_0^1(\Omega)$ that satisfies*

$$\|u\|_{H_0^1(\Omega)} \leq C_S(2)\|f\|_{\frac{2N}{N+2}}.$$

Beweis:
We recall that we want to solve $a(u,v) = l(v)$ for all $v \in H_0^1(\Omega)$ where

$$\begin{aligned} a(u,v) &:= \int_\Omega \nabla u(x) \cdot \nabla v(x) + c(x)u(x)v(x)\,dx, \\ l(v) &:= \int_\Omega f(x)v(x)\,dx. \end{aligned}$$

Again we need to check the assumptions of the Lax-Milgram-Lemma. We mainly repeat our earlier analysis, but now we work on the Hilbert space $(H_0^1(\Omega), \langle\cdot,\cdot\rangle_{H_0^1(\Omega)})$ instead of $(H_0^1(\Omega), \langle\cdot,\cdot\rangle_{H^1(\Omega)})$.

The boundedness of a follows from[40]

$$\begin{aligned} |a(u,v)| &\le \int_\Omega |\nabla u(x)||\nabla v(x)| + |c(x)||u(x)||v(x)|\,dx \\ &\le \||\nabla u|\|_2 \||\nabla v|\|_2 + \|c\|_{\frac{N}{2}} \|u\|_{\frac{2N}{N-2}} \|v\|_{\frac{2N}{N-2}} \\ &\le \||\nabla u|\|_2 \||\nabla v|\|_2 + \|c\|_{\frac{N}{2}} C_S(2)^2 \||\nabla u|\|_2 \||\nabla v|\|_2 \\ &= (1 + \|c\|_{\frac{N}{2}} C_S(2)^2) \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)} \end{aligned}$$

The linear functional l is bounded because of

$$|l(v)| \le \|fv\|_1 \le \|f\|_{\frac{2N}{N+2}} \|v\|_{\frac{2N}{N-2}} \le \|f\|_{\frac{2N}{N+2}} C_S(2) \||\nabla v|\|_2 = C_S(2) \|f\|_{\frac{2N}{N+2}} \|v\|_{H_0^1(\Omega)}.$$

We are left with checking the coercivity of a. We have

$$a(u,u) = \int_\Omega |\nabla u(x)|^2 + c(x)|u(x)|^2\,dx \ge \||\nabla u|\|_2^2 \ge \|u\|_{H_0^1(\Omega)}^2.$$

So the Lax-Milgram Lemma gives the claim. □

The improvement is that our earlier version required $c(x) \ge \mu > 0$ for some $\mu > 0$. Even this can be further improved to $c(x) > -\lambda_1(\Omega)$ a.e. where $\lambda_1(\Omega)$ denotes the smallest positive eigenvalue of the Dirichlet-Laplacian. Notice that Corollary 10.3 estimates the solution in the norm $\|\cdot\|_{H_0^1(\Omega)}$, which is different from the $H^1(\Omega)$-norm that we used earlier.

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[40] If helpful (and $c \in L^\infty(\Omega)$), one may as well use

$$\int_\Omega |c(x)||u(x)||v(x)|\,dx \le \|c\|_\infty \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \le C_P(\Omega,2)^2 \|c\|_\infty \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}.$$

Similarly for the estimate of l.

10.2 Some limitations and generalizations

Our first remark is concerned with the failure of Poincarés Inequality in sufficiently "thick domains". We saw that it holds if a given domain has finite length in one direction, so the following result may be seen as a kind of converse to that statement.

Satz 10.4. *Let $\Omega \subset \mathbb{R}^N$ be open such that there is a sequece $(x_n) \subset \Omega$ and $(r_n) \subset \mathbb{R}_+$ such that $B_{r_n}(x_n) \subset \Omega$ and $r_n \to \infty$. Then Poincaré's Inequality cannot hold on $W_0^{1,p}(\Omega)$ for any given $p \in [1, \infty)$.*

Beweis:
Choose a nontrivial test function $\chi \in C_0^\infty(B_1(0))$ and define $u_n(x) := \chi(\frac{x-x_n}{r_n})$. Then $u_n \in C_0^\infty(\Omega) \subset W_0^{1,p}(\Omega)$ for all $1 \le p < \infty$ and

$$\frac{\|u_n\|_{L^p(\Omega)}}{\|\nabla u_n\|_{L^p(\Omega)}} = \frac{\|u_n\|_{L^p(\mathbb{R}^N)}}{\|\nabla u_n\|_{L^p(\mathbb{R}^N)}} = \frac{r_n^{\frac{N}{p}} \|\chi\|_{L^p(\mathbb{R}^N)}}{r_n^{\frac{N-p}{p}} \|\nabla\chi\|_{L^p(\mathbb{R}^N)}} = r_n \frac{\|\chi\|_{L^p(\Omega)}}{\|\nabla\chi\|_{L^p(\Omega)}} \to +\infty$$

□

Next we turn towards more abstract versions of Poincaré's Inequality that will imply Wirtinger's Inequality. We start with the following version.

Satz 10.5. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain and let $V \subset W^{1,p}(\Omega)$ be a closed subspace such that the only constant function in V is the trivial one. Then, for each $p \in (1, \infty)$, there is $C > 0$ such that*

$$\|u\|_{L^p(\Omega)} \le C\|\nabla u\|_{L^p(\Omega)} \quad \textit{for all } u \in V.$$

Beweis:
We argue by contradiction and assume that there is a bounded sequence $(u_n) \subset V$ such that

$$\|u_n\|_{L^p(\Omega)} \to 1 \qquad \text{and} \qquad \|\nabla u_n\|_{L^p(\Omega)} \to 0.$$

Then (u_n) is bounded in $W^{1,p}(\Omega)$ and the Rellich-Kondrachov Theorem provides a subsequence (u_{n_k}) and $u \in L^p(\Omega)$ such that $u_{n_k} \to u$ in $L^p(\Omega)$ as $k \to \infty$. In particular, $\|u\|_{L^p(\Omega)} = \lim_{k\to\infty} \|u_{n_k}\|_{L^p(\Omega)} = 1$ and one even can ensure[41] $u \in V$. Furthermore,

[41] This fact is not obvious, and I don't see how to prove it with the methods developed so far. Here is the reasoning that works but requires extra knowledge:
As a closed subspace of $W^{1,p}(\Omega)$ the space V, equipped with the same norm, is a separable reflexive Banach space. This is true due to $1 < p < \infty$ and proofs may be found in [Adams]. In such spaces, bounded sequences have *weakly* convergent subsequence by the Banach-Alaoglu-Theorem. (This result is very important for the Calculus of Variations!) So $u_{n_k} \rightharpoonup u$ for some $u \in V$ and the compactness of the embedding $\iota : V \to L^p(\Omega)$ implies $\|\iota(u_{n_k}) - \iota(u)\|_{L^p(\Omega)}$ in $L^p(\Omega)$.

$\|\nabla u_{n_k}\|_{L^p(\Omega)} \to 0$ implies

$$\begin{aligned} 0 &= -\lim_{k\to\infty} \int_\Omega \partial_j u_{n_k} \phi \, dx \\ &= -\lim_{k\to\infty} \int_\Omega u_{n_k} \partial_j \phi \, dx \\ &= -\int_\Omega u \partial_j \phi \, dx \qquad \text{for all } \phi \in C_0^\infty(\Omega), \, j \in \{1, \dots, N\}, \end{aligned}$$

so the weak gradient of u is identically zero on Ω. From the Exercises we conclude that $u \in V$ must be constant. Our assumption on V then implies $u = 0$, which contradicts $\|u\|_{L^p(\Omega)} = 1$. We thus conclude that our assumption was false, i.e., there is some Poincaré Inequality in V, which is all we had to show. □

We point out that the same argument yields Poincaré Inequalities of the form $\|u\|_{L^q(\Omega)} \le C\|\nabla u\|_{L^p(\Omega)}$ for general exponents $q \in [1, p^*)$. The classical Poincaré Inequality corresponds to the choice $V = W_0^{1,p}(\Omega)$. Another important inequality, called Wirtinger's Inequality, arises from the choice $V = \{u \in W^{1,p}(\Omega) : \int_\Omega u \, dx = 0\}$. This subset is indeed closed because $u_n \to u \in W^{1,p}(\Omega)$ with $u_n \in V$ implies $\|u_n - u\|_{L^1(\Omega)} \to 0$ and in particular

$$\int_\Omega u \, dx = \lim_{n\to\infty} \int_\Omega u_n \, dx = 0.$$

Korollar 10.6 (Wirtinger's Inequality). *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain and $p \in (1, \infty)$. Then there is $C > 0$ such that*

$$\left\| u - \fint_\Omega u \, dx \right\|_{L^p(\Omega)} \le C\|\nabla u\|_{L^p(\Omega)} \qquad \text{for all } u \in W^{1,p}(\Omega).$$

Here: $\fint_\Omega u \, dx = \frac{1}{|\Omega|} \int_\Omega u \, dx$.

Beweis:
For all $u \in W^{1,p}(\Omega)$ the function $v := u - \fint_\Omega u \, dx$ satisfies $v \in V := \{u \in W^{1,p}(\Omega) : \int_\Omega u \, dx = 0\}$ and $\nabla v = \nabla u$. Hence, denoting by C the Poincaré constant of the subspace V given by Theorem 10.5, we get

$$\left\| u - \fint_\Omega u \, dx \right\|_{L^p(\Omega)} = \|v\|_{L^p(\Omega)} \le C\|\nabla v\|_{L^p(\Omega)} = C\|\nabla u\|_{L^p(\Omega)} \qquad \text{for all } u \in W^{1,p}(\Omega).$$

□

Note that we may replace $\fint_\Omega u \, dx$ by $\fint_A u \, dx$ for any $A \subset \Omega$ of positive measure. We now present some nice application of the optimal Wirtinger Inequality in one spatial dimension. The latter reads

$$\int_0^{2\pi} \left(f(s) - \fint_0^{2\pi} f(t) \, dt \right)^2 ds \le \int_0^{2\pi} f'(s)^2 \, ds \qquad (f \in H^1(\Omega)) \tag{10.2}$$

We follow [12]. The setting is as follows. Let $\gamma(t) := (X(t), Y(t))$ for $t \in [0, 2\pi]$ where $X, Y : \mathbb{R} \to \mathbb{R}$ are smooth 2π-periodic functions that are parametrized by arclength. In particular, its length is 2π. From the Divergence Theorem we know[42] that the area of the encircled 2D-domain Ω is given by

$$
\begin{aligned}
|\Omega| &= \frac{1}{2}\int_\Omega \operatorname{div}(x,y)\, d(x,y) \\
&= \frac{1}{2}\int_{\partial\Omega} (x,y)\cdot \nu(x,y)\, d\sigma(x,y) \\
&= \frac{1}{2}\int_0^{2\pi} (X(s),Y(s)) \cdot \underbrace{\frac{(Y'(s),-X'(s))}{|(Y'(s),-X'(s))|}}_{=\nu(X(s),Y(s))} \underbrace{|(X'(s),Y'(s)|}_{=1}\, ds \\
&= \frac{1}{2}\int_0^{2\pi} (X(s),Y(s)) \cdot (Y'(s),-X'(s))\, ds \\
&= \frac{1}{2}\int_0^{2\pi} X(s)Y'(s) - X'(s)Y(s)\, ds \\
&= \int_0^{2\pi} X(s)Y'(s)\, ds.
\end{aligned}
$$

Here the last equality follows from integration by parts and that (X, Y) are 2π-periodic so that the boundary terms vanish. We want to show that the largest area comes a circular circumference, proving thus the isoperimetric inequality in some special case.

We set $\bar{X} := ⨍_0^{2\pi} X(s)\, ds$ we get

$$
\begin{aligned}
|\Omega| &= \int_0^{2\pi} X(s)Y'(s)\, ds \\
&= \int_0^{2\pi} (X(s)-\bar{X})Y'(s)\, ds \\
&\le \frac{1}{2}\int_0^{2\pi} ((X(s)-\bar{X})^2 + Y'(s)^2)\, ds \\
&\overset{(10.2)}{\le} \frac{1}{2}\int_0^{2\pi} (X'(s)^2 + Y'(s)^2)\, ds \\
&= \frac{1}{2}\int_0^{2\pi} 1\, ds \\
&= \pi.
\end{aligned}
$$

Let us discuss the equality case. Since $ab = \frac{1}{2}(a^2+b^2)$ if and only if $a + b = 0$, we obtain

$$X(s) - \bar{X} + Y'(s) = 0 \qquad \text{for almost all } s \in [0, 2\pi].$$

[42] We rather assume that Ω is such that the Divergence Theorem applies. This can be ensured if γ does not intersect itself, i.e. $\gamma(t) = \gamma(s)$ for $s, t \in \mathbb{R}$ if and only if $s - t \in 2\pi\mathbb{Z}$.

Similarly, starting from the formula $A = -\int_0^{2\pi} X'(s)Y(s)\,dx$ instead of $A = \int_0^{2\pi} X'(s)Y(s)\,ds$, we infer

$$Y(s) - \bar{Y} - X'(s) = 0 \qquad \text{for almost all } s \in [0, 2\pi].$$

We thus obtain that $\tilde{X} := X - \bar{X}, \tilde{Y} := Y - \bar{Y}$ satisfy

$$\tilde{X}''(s) = \tilde{Y}'(s) = -\tilde{X}(s), \qquad \tilde{Y}''(s) = -\tilde{X}'(s) = -\tilde{Y}(s) \qquad \text{for almost all } s \in [0, 2\pi].$$

So there are $\alpha, \beta \in \mathbb{R}$ such that

$$\begin{aligned}
\begin{pmatrix} X(s) \\ Y(s) \end{pmatrix} &= \begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} + \begin{pmatrix} \tilde{X}(s) \\ \tilde{Y}(s) \end{pmatrix} \\
&= \begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} + \begin{pmatrix} \alpha \sin(s) + \beta \cos(s) \\ \alpha \cos(s) - \beta \sin(s) \end{pmatrix} \\
&= \begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} + \sqrt{\alpha^2 + \beta^2} \begin{pmatrix} \sin(s + s_0) \\ \cos(s + s_0) \end{pmatrix} \\
\text{where} \quad \cos(s_0) &= \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}}, \sin(s_0) = \frac{\beta}{\sqrt{\alpha^2 + \beta^2}}.
\end{aligned}$$

Since X, Y are parametrized by arclength we even know $\alpha^2 + \beta^2 = 1$. We thus conclude

$$\begin{pmatrix} X(s) \\ Y(s) \end{pmatrix} = \begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} + \begin{pmatrix} \sin(s + s_0) \\ \cos(s + s_0) \end{pmatrix} \qquad (s_0 \in \mathbb{R}).$$

So equality can only hold if (X, Y) describes the unit circle, and it does! The conclusion is that among "all curves" with length 2π the circle has the largest area, which is π.

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11 Trace Theorem

In this section we are going to talk about traces of functions belonging to Sobolev spaces $W^{1,p}(\Omega)$ where $1 \le p < \infty$ and $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain. As before, the discussion extends to higher order Sobolev spaces $W^{k,p}(\Omega)$ (as in Section **??**) using $\partial_1 u, \ldots, \partial_N u \in W^{k-1,p}(\Omega)$ for all $u \in W^{k,p}(\Omega)$. A "trace" is nothing but a reasonable generalization of the "restriction" of a given function to a given subset of $\Gamma \subset \overline{\Omega}$ having lower dimension. For the discussion of elliptic boundary value problems the particular $\Gamma = \partial\Omega$ is the most important one. For continuous functions $u \in C(\overline{\Omega})$ the definition of $u|_{\partial\Omega}$ is evident. Accordingly, for $p > N$ and $u \in W^{1,p}(\Omega) \subset C^{0,\alpha}(\overline{\Omega}), \alpha = 1 - \frac{N}{p}$ there is no problem either because the trace of u is simply defined as the restriction of the $C^{0,\alpha}(\overline{\Omega})$-representative of u to the boundary[43]. But this is not enough, because in the

[43]In other words, denoting the trace operator by γ, $\gamma u = \tilde{u}|_{\partial\Omega}$ where $\tilde{u} \in C^{0,\alpha}(\overline{\Omega})$ is uniquely determined by $\tilde{u} = u$ almost everywhere

most important case $p = 2$ this restricts all considerations to the case $N = 1$. In particular, no PDE theory can be built on that. So the task is to find a meaning of a "trace" for functions $u \in W^{1,p}(\Omega)$ where $1 \le p \le N$.

Definition 11.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $1 \le p \le N$. Then a bounded linear operator $\gamma : W^{1,p}(\Omega) \to L^q(\partial\Omega)$ is called trace operator if $\gamma u = u|_{\partial\Omega}$ for all $u \in C(\overline{\Omega}) \cap W^{1,p}(\Omega)$. The function γu is called the trace of u onto $\partial\Omega$.*

Our aim is to show that bounded Lipschitz domains admit such trace operators. In the proof we will need the following technical fact taken from [9, Lemma 1.5.1.9].

Proposition 11.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain with unit outer normal vector field $\nu : \partial\Omega \to \mathbb{R}^N$ and surface measure σ. Then there is a smooth vector field $F \in C^\infty(\overline{\Omega}; R^N)$ such that $F(x) \cdot \nu(x) \ge 1$ for σ-almost all $x \in \partial\Omega$ and there is $t^* > 0$ such that $x + tF(x) \in \overline{\Omega}^c$ for $0 < t < t^*$, $x + tF(x) \in \Omega$ for $-t^* < t < 0$ for all $x \in \partial\Omega$.*

The idea of the proof is to define F locally, i.e., within sufficiently small but finitely many neighbourhoods of boundary pieces, as a mollified (smoothened) version of the unit normal vector field $\nu : \partial\Omega \to \mathbb{R}^N$ itself. If I have time: see the Appendix for a proof. It is particularly important that F is defined and smooth on $\overline{\Omega}$ and not only on $\partial\Omega$.

Satz 11.3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain and $1 \le p < N$. Then there is a trace operator $\gamma : W^{1,p}(\Omega) \to L^q(\partial\Omega)$ for $1 \le q \le \frac{(N-1)p}{N-p}$. It is compact provided that $1 \le q < \frac{(N-1)p}{N-p}$.*

Beweis:
In view of Theorem 4.11 we first consider prove that

$$\|\gamma u\|_{L^q(\partial\Omega)} = \|u\|_{L^q(\partial\Omega)} \le C\|u\|_{W^{1,p}(\Omega)} \quad \text{for all } u \in C_0^\infty(\mathbb{R}^N).$$

This turns out to be the main step of the proof. We focus on $q > 1$; the general case then follows from the above inequality for all $q > 1$ and the Dominated Convergence Theorem as $q \searrow 1$. The Divergence Theorem and finally Sobolev's Embedding Theorem give

$$\begin{aligned}
\|\gamma u\|^q_{L^q(\partial\Omega)} &= \int_{\partial\Omega} |u|^q d\sigma \\
&\le \int_{\partial\Omega} |u|^q F \cdot \nu d\sigma \\
&= \int_{\Omega} \operatorname{div}(|u|^q F)\, dx \\
&= \int_{\Omega} \left(q|u|^{q-2}u\langle\nabla u, F\rangle + |u|^q \operatorname{div}(F)\right) dx
\end{aligned}$$

$$\leq C\|F\|_{C^1(\overline{\Omega};\mathbb{R}^N)} \int_\Omega (q|u|^{q-1}|\nabla u| + |u|^q)\,dx$$
$$\leq C\|F\|_{C^1(\overline{\Omega};\mathbb{R}^N)} \left(q\|u\|^{q-1}_{\frac{(q-1)p}{p-1}} \|\nabla u\|_p + \|u\|_q^q \right)$$
$$\leq C'\|F\|_{C^1(\overline{\Omega};\mathbb{R}^N)} \|u\|^q_{W^{1,p}(\Omega)}.$$

This allows to define the trace operator $\gamma : W^{1,p}(\Omega) \to L^q(\Omega)$ by density:

$$\gamma u := \lim_{\|\tilde{u}-u\|_{W^{1,p}(\Omega)}\to 0, \tilde{u}\in C_0^\infty(\mathbb{R}^N)} \tilde{u}|_{\partial\Omega},$$

see Exercise 4 on Exercise sheet 1.

It remains to prove the compactness statement. This is a consequence of the estimate

$$\|\gamma u\|^q_{L^q(\partial\Omega)} \leq \|F\|_{C^1(\overline{\Omega};\mathbb{R}^N)} \left(q\|u\|^{q-1}_{\frac{(q-1)p}{p-1}} \|\nabla u\|_p + \|u\|_q^q \right)$$

that we have proved above. Indeed, if $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in $W^{1,p}(\Omega)$ then the Rellich-Kondratchov Theorem provides a subsequence, again denoted by $(u_n)_{n\in\mathbb{N}}$ for simplicity that converges in $L^{\frac{(q-1)p}{p-1}}(\Omega)$ and in $L^q(\Omega)$. Here we used $\frac{(q-1)p}{p-1} < \frac{Np}{N-p}$, which follows from $q < \frac{(N-1)p}{N-p}$. Hence,

$$\begin{aligned}
&\|\gamma(u_n) - \gamma(u_m)\|_{L^q(\partial\Omega)} \\
&= \|\gamma(u_n - u_m)\|_{L^q(\partial\Omega)} \\
&\leq \|F\|_{C^1(\overline{\Omega};\mathbb{R}^N)} \left(q\, \underbrace{\|u_n - u_m\|^{q-1}_{\frac{(q-1)p}{p-1}}}_{\to 0} \underbrace{\|\nabla(u_n - u_m)\|_p}_{\text{bounded}} + \underbrace{\|u_n - u_m\|_q^q}_{\to 0} \right) \\
&\to 0 \qquad \text{as } n, m \to \infty.
\end{aligned}$$

So $(\gamma(u_n))$ is a Cauchy sequence and thus converges. This proves the compactness. □

Formally, this result does not include the case $p = N$. As in the case of the Sobolev Embedding Theorem, one may instead use the results for $p < N$ to deal with this case because of $W^{1,N}(\Omega) \subset W^{1,p}(\mathbb{R}^N)$ for all $p \in [1, N)$, which is a consequence of the usual embeddings of Lebesgue spaces on bounded domains. So we see that the trace operator γ can be defined for all Sobolev spaces $W^{1,p}(\Omega)$ and hence the "boundary values" of any such function makes sense in the sense of an $L^q(\partial\Omega)$-function. We are going to show that $u \in W^{1,p}(\Omega), \gamma u = 0$ is equivalent to $u \in W_0^{1,p}(\Omega)$, so prescribing zero boundary data requiring the trace to be zero resp. requiring $W_0^{1,p}(\Omega)$ is equivalent. One may rephrase this as $\ker(\gamma) = W_0^{1,p}(\Omega)$. To this end we first generalize the integration-by-parts rule.

Proposition 11.4 (Integration by parts). *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain and $u \in W^{1,p}(\Omega), v \in W^{1,q}(\Omega)$ for $1 \leq p, q < N$ such that $\frac{1}{p} + \frac{1}{q} \leq \frac{N+1}{N}$. Then, for all $j = 1, \dots, N$,*

$$\int_\Omega \partial_j uv \, dx = \int_{\partial\Omega} \gamma(u)\gamma(v)\nu_i \, d\sigma - \int_\Omega u\partial_j v \, dx$$

Beweis:
By Theorem 4.11 we choose $u_n, v_n \in C_0^\infty(\mathbb{R}^N)$ such that $u_n \to u$ in $W^{1,p}(\Omega)$ und $v_n \to v$ in $W^{1,q}(\Omega)$. The classical integration-by-parts rule gives

$$\begin{aligned}\int_\Omega \partial_j u_n v_n \, dx &= \int_{\partial\Omega} u_n v_n \nu_j \, d\sigma - \int_\Omega u_n \partial_j v_n \, dx \\ &= \int_{\partial\Omega} \gamma(u_n)\gamma(v_n)\nu_j \, d\sigma - \int_\Omega u_n \partial_j v_n \, dx\end{aligned}$$

Since $u_n \to u$ in $W^{1,p}(\Omega)$ and $v_n \to v$ in $W^{1,q}(\Omega)$ we have

$$\gamma(u_n) \to \gamma(u) \text{ in } L^{\frac{(N-1)p}{N-p}}(\partial\Omega), \qquad \gamma(v_n) \to \gamma(v) \text{ in } L^{\frac{(N-1)q}{N-q}}(\partial\Omega).$$

Moreover,

$$\begin{aligned}\partial_j u_n &\to \partial_j u \text{ in } L^p(\Omega), & u_n &\to u \text{ in } L^{\frac{Np}{N-p}}(\Omega), \\ \partial_j v_n &\to \partial_j v \text{ in } L^q(\Omega), & v_n &\to v \text{ in } L^{\frac{Nq}{N-q}}(\Omega)\end{aligned}$$

Next define $r \in [1, \infty]$ via $\frac{1}{r} := \frac{N+1}{N} - \frac{1}{p} - \frac{1}{q}$, which is possible in view of $\frac{1}{p} + \frac{1}{q} \leq \frac{N+1}{N}$ and $\frac{1}{p} + \frac{1}{q} > \frac{2}{N} > \frac{1}{N}$. Then

$$\begin{aligned}\left|\int_\Omega \partial_j u_n v_n \, dx - \int_\Omega \partial_j uv \, dx\right| &\leq \int_\Omega |\partial_j u_n - \partial_j u||v_n| + |\partial_j u||v_n - v| \, dx \\ &\leq \|\partial_j u_n - \partial_j u\|_p \|v_n\|_{\frac{Nq}{N-q}} \|1\|_r + \|\partial_j u\|_p \|v_n - v\|_{\frac{Nq}{N-q}} \|1\|_r \\ &\to 0 \qquad (n \to \infty), \\ \left|\int_\Omega u_n \partial_j v_n \, dx - \int_\Omega u\partial_j v \, dx\right| &\leq \int_\Omega |\partial_j v_n - \partial_j v||u_n| + |\partial_j v||u_n - u| \, dx \\ &\leq \|\partial_j v_n - \partial_j v\|_q \|u_n\|_{\frac{Np}{N-p}} \|1\|_r + \|\partial_j v\|_p \|u_n - u\|_{\frac{Np}{N-p}} \|1\|_r \\ &\to 0 \qquad (n \to \infty).\end{aligned}$$

Moreover,

$$\begin{aligned}&\left|\int_{\partial\Omega} \gamma(u_n)\gamma(v_n)\nu_j \, d\sigma - \int_{\partial\Omega} \gamma(u)\gamma(v)\nu_j \, d\sigma\right| \\ &\leq \int_{\partial\Omega} |\gamma(u_n - u)||\gamma(v_n)| + |\gamma(u)||\gamma(v_n - v)| \, d\sigma \\ &\leq \|\gamma(u_n - u)\|_{L^{\frac{(N-1)p}{N-p}}(\partial\Omega)} \|\gamma(v_n)\|_{L^{\frac{(N-1)q}{N-q}}(\partial\Omega)} + \|\gamma(u)\|_{L^{\frac{(N-1)p}{N-p}}(\partial\Omega)} \|\gamma(v_n - v)\|_{L^{\frac{(N-1)p}{N-p}}(\partial\Omega)} \\ &\leq C(\|u_n - u\|_{W^{1,p}(\Omega)} \|v_n\|_{W^{1,q}(\Omega)} + \|v_n - v\|_{W^{1,q}(\Omega)} \|u\|_{W^{1,p}(\Omega)})\end{aligned}$$

$\to 0 \qquad (n \to \infty).$

We thus conclude

$$\begin{aligned}\int_\Omega \partial_j u v \, dx &= \lim_{n\to\infty} \int_\Omega \partial_j u_n v_n \, dx \\ &= \lim_{n\to\infty} \int_{\partial\Omega} \gamma(u_n)\gamma(v_n)\nu_j \, d\sigma - \lim_{n\to\infty} \int_\Omega u_n \partial_j v_n \, dx \\ &= \int_{\partial\Omega} \gamma(u)\gamma(v)\nu_j \, d\sigma - \int_\Omega u \partial_j v \, dx\end{aligned}$$

□

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Proposition 11.5. *Set $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain, $1 \le p < \infty$. Then $\gamma(C^\infty(\overline{\Omega}))$ is dense in $L^p(\partial\Omega)$.*

Beweis:
Let $v \in L^p(\partial\Omega)$ and $\varepsilon > 0$. Then choose $w \in C(\partial\Omega)$ such that $\|v-w\|_{L^p(\partial\Omega)} < \varepsilon$. This can be justified as in Proposition 4.6, replacing the Lebesgue measure by the surface measure σ. (It satisfies a regularity property analogous to the one from Lemma 4.5.) To approximate w we use Tietze's Extension Theorem, see Theorem 13.6 in the appendix. It provides a continuous function $W \in C(\mathbb{R}^N)$ such that $W|_{\partial\Omega} = w$. Multiplying this function with a cutoff-function function having compact support which is identically 1 on $\partial\Omega$, we may even w.l.o.g. assume that W has compact support. Finally, define $V_\varepsilon := \rho_\varepsilon * W$. Then $V_\varepsilon \in C^\infty(\mathbb{R}^N) \subset C^\infty(\overline{\Omega})$ and $\gamma(V_\varepsilon) = V_\varepsilon, \gamma(W) = W|_{\partial\Omega} = w$ imply

$$\begin{aligned}\|\gamma(V_\varepsilon) - v\|_{L^p(\partial\Omega)} &\le \|V_\varepsilon - W\|_{L^p(\partial\Omega)} + \|w - v\|_{L^p(\partial\Omega)} \\ &\le \sup_{x\in\partial\Omega} |(\rho_\varepsilon * W)(x) - W(x)| \cdot |\partial\Omega|^{\frac{1}{p}} + \varepsilon \\ &\le \sup_{x\in\partial\Omega} \int_{\mathbb{R}^N} \rho_\varepsilon(x-y)|W(y) - W(x)| \, dy \cdot |\partial\Omega|^{\frac{1}{p}} + \varepsilon \\ &\le \sup_{x\in\partial\Omega, |y-x|\le\varepsilon} |W(y) - W(x)| \cdot |\partial\Omega|^{\frac{1}{p}} + \varepsilon \\ &\to 0 \qquad \text{as } \varepsilon \to 0.\end{aligned}$$

Here we used that W is uniformly continuous (why?). □

Lemma 11.6. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain, $1 \le p < \infty$ and $u \in W^{1,p}(\Omega)$. Then the following statements are equivalent:*

(i) $u \in W_0^{1,p}(\Omega)$.

(ii) $u \in \ker(\gamma)$, *i.e.*, $\gamma u = 0$.

(iii) The trivial extension U of u belongs to $W^{1,p}(\mathbb{R}^N)$ with

$$\partial_j U = \partial_j u \cdot \mathbb{1}_\Omega \qquad \textit{for } j = 1, \dots, N.$$

(iv) There is $C > 0$ such that $|\int_\Omega u \partial_i \phi \, dx| \le C \|\phi\|_{L^{p'}(\Omega)}$ for alle $\phi \in C_0^\infty(\mathbb{R}^N)$.

Beweis:
(i) → (ii) That's trivial: If $(u_n) \subset C_0^\infty(\Omega)$ satisfies $u_n \to u$ in $W^{1,p}(\Omega)$, then $\gamma(u_n) = u_n|_{\partial\Omega} = 0$ and the Trace Theorem (for $q := p$) imply

$$\|\gamma u\|_{L^p(\partial\Omega)} = \|\gamma(u - u_n)\|_{L^p(\partial\Omega)} \le C\|u - u_n\|_{W^{1,p}(\Omega)} \to 0,$$

hence $\gamma u = 0$.

(ii)→ (iii) For all $\phi \in C_0^\infty(\mathbb{R}^N)$ we have by Proposition 11.4

$$\int_{\mathbb{R}^N} U \partial_j \phi \, dx = \int_\Omega u \partial_j \phi \, dx = \int_{\partial\Omega} \underbrace{\gamma(u)}_{=0} \phi \nu_j \, d\sigma - \int_\Omega (\partial_j u) \phi \, dx = -\int_{\mathbb{R}^N} (\partial_j u \cdot \mathbb{1}_\Omega) \phi \, dx$$

This shows that U has a j-th weak derivative on $\mathbb{R}^N$ given by $\partial_j U = \partial_j u \cdot \mathbb{1}_\Omega$. In particular, $U \in W^{1,p}(\mathbb{R}^N)$.

(iii)→(i) We choose the vector field $F \in C^\infty(\overline{\Omega}; \mathbb{R}^N)$ as in Proposition 11.2. Let $U \in W^{1,p}(\mathbb{R}^N)$ be the trivial extension of u and set $U_\varepsilon(x) := U(x + \varepsilon F(x))$. As in the proof of Proposition 9.7 one finds

$$\|U_\varepsilon - U\|_{W^{1,p}(\Omega)} \to 0 \qquad \text{as } \varepsilon \to 0.$$

The define $u_\varepsilon \in C^\infty(\mathbb{R}^N)$ via

$$u_\varepsilon(x) := \big(\rho_{\delta_\varepsilon} * (U_\varepsilon \cdot \mathbb{1}_\Omega)\big)(x) = \int_\Omega \rho_{\delta_\varepsilon}(x - y) U_\varepsilon(y) \, dy.$$

We can choose $\delta_\varepsilon > 0$ so small that u_ε vanishes in a neighbourhood of $\partial\Omega$. Indeed, otherwise there would be sequences $(x_n), (y_n)$ such that $x_n \to x \in \partial\Omega$ with $|x_n - y_n| \le \frac{1}{n}, y_n \in \Omega, y_n + \varepsilon F(y_n) \in \Omega$ (so that $(U_\varepsilon \cdot \mathbb{1}_\Omega)(y_n) \ne 0$). But this implies $x \in \partial\Omega, x + \varepsilon F(x) \in \overline{\Omega}$, which is impossible by Proposition 11.2. Accordingly, $\operatorname{supp}(u_\varepsilon) \cap \partial\Omega = \varnothing$, hence

$$u_\varepsilon \cdot \mathbb{1}_\Omega \in C_0^\infty(\Omega).$$

Additionally, shrinking δ_ε further if necessary, we may assume

$$\|u_\varepsilon - U_\varepsilon\|_{W^{1,p}(\mathbb{R}^N)} \le \varepsilon$$

see Corollary 4.13. We thus conclude

$$\|u_\varepsilon \cdot \mathbb{1}_\Omega - u\|_{W^{1,p}(\Omega)} \le \|u_\varepsilon - U\|_{W^{1,p}(\Omega)} \le \|u_\varepsilon - U_\varepsilon\|_{W^{1,p}(\mathbb{R}^N)} + \|U_\varepsilon - U\|_{W^{1,p}(\Omega)} \to 0 \quad \text{as } \varepsilon \to 0.$$

This proves $u \in W_0^{1,p}(\Omega)$.

(iii) → (iv) We have for all $\phi \in C_0^\infty(\mathbb{R}^N)$

$$\begin{aligned}\left|\int_\Omega u\partial_j\phi\,dx\right| &= \left|\int_{\mathbb{R}^N} U\partial_j\phi\,dx\right| \\ &= \left|\int_{\mathbb{R}^N} (\partial_j U)\phi\,dx\right| \\ &= \left|\int_{\mathbb{R}^N} (\partial_j u \mathbb{1}_\Omega)\phi\,dx\right| \\ &\le \int_\Omega |\partial_j u||\phi|\,dx \\ &\le \|\partial_j u\|_{L^p(\Omega)}\|\phi\|_{L^{p'}(\Omega)}.\end{aligned}$$

(iv) → (ii) Integration by parts gives for all $\phi \in C_0^\infty(\mathbb{R}^N) \subset C^\infty(\overline{\Omega})$

$$C\|\phi\|_{L^{p'}(\Omega)} \ge \left|\int_\Omega u\partial_j\phi\,dx\right| = \left|\int_{\partial\Omega}\gamma(u)\phi\nu_j\,d\sigma - \int_\Omega \partial_j u\phi\,dx\right|$$

For $j \in \mathbb{N}$ let $K_j := \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) \ge \frac{1}{j}\}$ and denote by $\chi_j : \mathbb{R}^N \to [0,1]$ a smooth function satisfying $\chi_j(x) = 0$ for $x \in K_j$ and $\chi_j(x) = 1$ for $x \in \partial\Omega$. Such a function exists, see Theorem 4.3. Replacing ϕ by $\phi\cdot\chi_j$, we thus obtain for all $j \in \mathbb{N}$

$$\|\partial_j u\|_{L^p(\Omega)}\|\phi\chi_j\|_{L^{p'}(\Omega)} \ge \left|\int_{\partial\Omega}\gamma(u)\phi\underbrace{\chi_j}_{=1}\nu_j\,d\sigma - \int_\Omega \partial_j u\phi\chi_j\,dx\right|.$$

Sending j to infinity we obtain from the Dominated Convergence Theorem

$$0 \ge \left|\int_{\partial\Omega}\gamma(u)\phi\nu_j\,d\sigma\right|.$$

This holds for all $\phi \in C^\infty(\overline{\Omega})$ and Proposition 11.5 thus implies $\gamma(u)\nu_j = 0$ for all $j = 1,\dots,N$. This gives $\gamma(u) = 0$, which is (ii). □

So we conclude that the functions from $W_0^{1,p}(\Omega)$ are the ones for which the corresponding trace is zero. One may use the trace operator to refine Poincaré's Inequality. We show that it is not necessary that the functions vanish on the whole of $\partial\Omega$, but a reasonably large piece of it is already sufficient. The following corollary thus contains the classical Poincaré Inequality from Theorem **??** as the sepcial case $\Gamma = \partial\Omega$.

Korollar 11.7. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain and assume that $\Gamma \subset \partial\Omega$ has positive surface measure ($\sigma(\Gamma) > 0$). Then, for all $p \in (1,\infty)$, there is a $C > 0$ such that*

$$\left\|u - \frac{1}{|\Gamma|}\int_\Gamma \gamma(u)\,d\sigma\right\|_{L^p(\Omega)} \le C\|\nabla u\|_{L^p(\Omega)} \qquad \text{for all } u \in W^{1,p}(\Omega)$$

Beweis:
This is a consequence of Theorem 10.5 applied to $v := u - \frac{1}{|\Gamma|}\int_\Gamma \gamma(u)\,d\sigma$ belonging to

$$V = \left\{ v \in W^{1,p}(\Omega) : \int_\Gamma \gamma(v)\,d\sigma = 0 \right\}.$$

In fact, one may check that this subspace is closed and the only constant function in V is the trivial one because $u \equiv c$ implies

$$0 = \int_\Gamma \gamma(u)\,d\sigma = c\,\sigma(\Gamma), \quad \text{whence } c = 0.$$

(Here, one sees why $\sigma(\Gamma) > 0$ is required.) □

We sketch some application to our favourite elliptic boundary value problem where now nontrivial boundary conditions will be allowed. Generalizing the previous approach via the Lax-Milgram-Lemma we may now try to solve $a(u,v) = l(v)$ where

$$\begin{aligned} a(u,v) &:= \int_\Omega \nabla u(x) \cdot \nabla v(x) + c(x)u(x)v(x)\,dx + \int_{\partial\Omega} (\gamma u)(x)(\gamma v)(x)\,d\sigma(x), \\ l(v) &:= \int_\Omega f(x)v(x)\,dx + \int_{\partial\Omega} \kappa(x)(\gamma v)(x)\,d\sigma(x). \end{aligned}$$

The integrals over Ω may be analyzed as before whereas the boundary integral makes sense thanks to the trace theorem if we assume κ to be bounded[44]. The coercivity of the bilinear form on $H^1(\Omega)$ holds for instance if c is positive, but even $c \geq 0$ works (→ Exercises). The more interesting question is which boundary value problem the solution solves. One finds as before

$$-\Delta u + c(x)u = f(x) \quad \text{in } \Omega$$

in the weak sense and the "contribution over Ω" in the bilinear form is zero. (Reason: the equation holds for all $v \in H^1(\Omega)$, so also for all $v \in H_0^1(\Omega)$. For those functions the integral terms are zero.) It remains to study the integrals coming from the boundary $\partial\Omega$. We find

$$\int_{\partial\Omega} ((\gamma u)(x) - \kappa(x))(\gamma v)(x)\,d\sigma(x) = 0 \quad \text{for all } v \in H^1(\Omega).$$

Proposition 11.5 implies $\gamma u = \kappa$. In other words, the solution coming from the Lax-Milgram-Lemma is the unique weak solution to

$$-\Delta u + c(x)u = f(x) \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = \kappa \quad \text{on } \partial\Omega.$$

Quite remarkable: For $\kappa \equiv 0$ we thus obtain the unique solution in $H_0^1(\Omega)$ with a different functional and working on $H^1(\Omega)$. In other words, the following two bilinear forms give the same solution:

$$a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R},\ (u,v) \mapsto \int_\Omega \nabla u \cdot \nabla v + c(x)u(x)v(x)\,dx + \int_{\partial\Omega} (\gamma u)(x)(\gamma v)(x)\,d\sigma(x),$$

[44] What is the optimal integrability condition in view of Theorem 11.3?

$$\tilde{a} : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R},\ (u,v) \mapsto \int_\Omega \nabla u \cdot \nabla v + c(x)u(x)v(x)\,dx$$

How is that possible? One can show that the Lax-Milgram Lemma provides the uniquely determined **minimizers** of the corresponding functionals

$$I : H^1(\Omega) \to \mathbb{R}, \quad u \mapsto \frac{1}{2}\int_\Omega |\nabla u|^2 + c(x)u(x)^2\,dx + \frac{1}{2}\int_{\partial\Omega} (\gamma u)(x)^2\,d\sigma(x) - \int_\Omega f(x)u(x)\,dx$$
$$\tilde{I} : H_0^1(\Omega) \to \mathbb{R}, \quad u \mapsto \frac{1}{2}\int_\Omega |\nabla u|^2 + c(x)u(x)^2\,dx - \int_\Omega f(x)u(x)\,dx$$

One can see that the minimizer of I "wants to make $|\gamma u|$ as small as possible" keeping the other terms fixed. So it is not surprising that the minimizer satisfies $\gamma u = 0$ and thus belongs to $H_0^1(\Omega)$. One can make this rigorous using the Calculus of Variations (Euler-Lagrange equations). Nevertheless it is preferable to work with $H_0^1(\Omega)$ since it does not involve the Trace operator machinery. In particular, no boundary regularity is needed.

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12 Separability

Definition 12.1. *A Banach space X is called separable if it has a countable dense subset.*

We first investigate whether L^p–spaces have this property. It turns out that the case $p = \infty$ is different from $p \in [1,\infty)$. To prove the non-separability of $L^\infty(\Omega)$ we are ging to use the following criterion.

Proposition 12.2. *Let X be a Banach space with an uncountable set of open, non-empty and pairwise disjoint sets. Then X is not separable.*

Beweis:
Denote the uncountable set by $(U_i)_{i\in I}$ and assume for contradiction that $M := \{x_n : n \in \mathbb{N}\}$ is a dense subset of X. Since the sets U_i are open and non-empty and M is dense, there is some $n = n(i) \in \mathbb{N}$ such that $x_{n(i)} \in U_i$. On the other hand $U_i \cap U_j = \emptyset$ for $i \neq j$, so we must have $n(i) \neq n(j)$ for $i \neq j$. This shows that $n : I \to \mathbb{N}$ ist injective, which implies that I is countable, a contradiction. □

Satz 12.3. *Let $\Omega \subset \mathbb{R}^N$ be open and non-empty.*

(i) $L^p(\Omega)$ is separable provided that $1 \le p < \infty$.

(ii) $L^\infty(\Omega)$ is not separable.

Beweis:
We first prove (ii) using the previous proposition. For $x \in \Omega$ choose $r_x > 0$ such that $B_{r_x}(x) \subset \Omega$ and set

$$U_x := \left\{ f \in L^\infty(\Omega) : \|f - \mathbb{1}_{B_{r_x}(x)}\|_\infty < \frac{1}{2} \right\}.$$

Then $(U_x)_{x\in\Omega}$ is an uncountable set of pairwise disjoint open and non-empty sets. The disjointness follows from

$$f \in U_x \cap U_y \quad \Rightarrow \quad \|\mathbb{1}_{B_{r_x}(x)} - \mathbb{1}_{B_{r_y}(y)}\|_\infty < 1 \quad \Rightarrow \quad B_{r_x}(x) = B_{r_y}(y) \quad \Rightarrow \quad x = y.$$

So $L^\infty(\Omega)$ is not separable.

To prove the separability of $L^p(\Omega)$ we use Theorem 4.8 where we proved that $C_0^\infty(\Omega)$ is dense. So it suffices to find a countable subset of $L^p(\Omega)$ that approximates $C_0^\infty(\Omega)$ with respect to the norm in $L^p(\Omega)$. We choose[45]

$$\mathcal{P} := \left\{ p \cdot \mathbb{1}_{\Omega\cap B_M(0)} : p \text{ is a polynomial with rational coefficients and } M \in \mathbb{N} \right\}.$$

So consider any function $\phi \in C_0^\infty(\Omega)$ and $\varepsilon > 0$. Then choose $M \in \mathbb{N}$ such that the support of ϕ is contained in $B_M(0)$. Weierstrass' Approximation Theorem provides a polynomial $\tilde{p}$ such that

$$\|\tilde{p} - \phi\|_{C(\overline{B_M(0)})} \cdot |B_M(0)|^{\frac{1}{p}} < \frac{\varepsilon}{2}.$$

This implies

$$\|\tilde{p} \cdot \mathbb{1}_{\Omega\cap B_M(0)} - \phi\|_{L^p(\Omega)} = \|\tilde{p} - \phi\|_{L^p(\Omega\cap B_M(0))} \le \|\tilde{p} - \phi\|_{C(\overline{B_M(0)})} \cdot |B_M(0)|^{\frac{1}{p}} < \frac{\varepsilon}{2}. \tag{12.1}$$

Since $\tilde{p}$ is a polynomial, we have $\tilde{p}(x) = \sum_{|\alpha|\le n} \tilde{a}_\alpha x^\alpha$ for some $n \in \mathbb{N}$ and $\tilde{a}_\alpha \in \mathbb{R}$. Choosing $a_\alpha \in \mathbb{Q}$ sufficiently close to $\tilde{a}_\alpha$, we obtain for $p(x) := \sum_{|\alpha|\le n} a_\alpha x^\alpha$:

$$\begin{aligned} \|\tilde{p} \cdot \mathbb{1}_{\Omega\cap B_M(0)} - p \cdot \mathbb{1}_{\Omega\cap B_M(0)}\|_{L^p(\Omega)} &\le \|\tilde{p} - p\|_{C(\overline{B_M(0)})} \cdot |B_M(0)|^{\frac{1}{p}} \\ &\le \sum_{|\alpha|\le N} |\tilde{a}_\alpha - a_\alpha| \|x^\alpha\|_{C(\overline{B_M(0)})} |B_M(0)|^{\frac{1}{p}} \\ &< \frac{\varepsilon}{2}. \end{aligned} \tag{12.2}$$

As a consequence of (12.1) and (12.2),

$$\inf_{q\in\mathcal{P}} \|q - \phi\|_{L^p(\Omega)} \le \|p \cdot \mathbb{1}_{\Omega\cap B_M(0)} - \phi\|_{L^p(\Omega)} < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the claim follows. □

[45] This means that for each $p \in \mathcal{P}$ there is $N \in \mathbb{N}$ and $a_\alpha \in \mathbb{Q}$ for multi-indices $0 \le |\alpha| \le N$ such that $p(x) = \sum_{|\alpha|\le N} a_\alpha x^\alpha$. This set is countable!

We draw the consequences for Sobolev spaces. We first note that (finite) product spaces $L^p(\Omega) \times \ldots \times L^p(\Omega)$ are also separable for $1 \le p < \infty$. Defining now[46]

$$\Psi : W^{k,p}(\Omega) \to L^p(\Omega)^K, \quad u \mapsto (\partial^\alpha u)_{0 \le |\alpha| \le k},$$

we find that the subspace $\Psi(W^{k,p}(\Omega))$ of $L^p(\Omega)^K$ is closed. Moreover, Ψ is even isometric by definition of the respective norms, in particular it is injective. We will use the following criterion.

Proposition 12.4. *Let X be a separable Banach space and $\emptyset \neq M \subset X$. Then M is separable*[47].

Beweis:
Let $\{x_n : n \in \mathbb{N}\}$ be dense in X and $x^* \in M$. Then define $y_{n,m} = x^*$ if $B_{1/m}(x_n) \cap M = \emptyset$ and $y_{n,m} \in B_{1/m}(x_n) \cap M$ (arbitrary) otherwise. We claim that $Y := \{y_{n,m} : n, m \in \mathbb{N}\} \subset M$ is dense. Indeed, for all $x \in M$ and $N > 0$ we can find $n = n(N) \in \mathbb{N}$ such that $\|x - x_n\| < \frac{1}{2N}$. Then we have $x \in B_{1/2N}(x_n) \cap M$, so there is $y_{n,2N} \in B_{1/2N}(x_n) \cap M$. Hence,

$$\|x - y_{n,2N}\| \le \|x - x_n\| + \|x_n - y_{n,2N}\| < \frac{1}{2N} + \frac{1}{2N} = \frac{1}{N}.$$

Since $x \in M, N \in \mathbb{N}$ were arbitrary and $y_{n,2N} \in Y$, which is a countable subset of M, we infer that M is separable. □

Korollar 12.5. *Let $\Omega \subset \mathbb{R}^N$ be open and non-empty, $k \in \mathbb{N}$.*

(i) $W^{k,p}(\Omega)$ is separable provided that $1 \le p < \infty$.

(ii) $W^{k,\infty}(\Omega)$ is not separable.

Beweis:
In the case $1 \le p < \infty$ the space $\Psi(W^{k,p}(\Omega))$ is separable as a closed subspace of $L^p(\Omega)^M$ in view of the previous proposition. So if $\tilde{\mathcal{P}} \subset \Psi(W^{k,p}(\Omega))$ is a countable dense subset, then $\mathcal{P} := \{\Psi^{-1}(p) : p \in \tilde{\mathcal{P}}\}$ is countable and dense in $W^{k,p}(\Omega)$.

To prove (ii) we essentially repeat the trick from the proof of Theorem 12.3 (ii). Choose bounded open subsets $\Omega' \subset \mathbb{R}^{N-1}, I \subset \mathbb{R}$ such that $\Omega' \times I \subset \Omega$ and a cutoff function $\chi \in C_0^\infty(\mathbb{R}^N)$ such that $\chi(x) = 1$ for all $x \in \Omega' \times I$. W.l.o.g. $0 \in \overline{I}$. For $z \in I$ we may then choose $r_z > 0$ such that $I_z := (z - r_z, z + r_z) \subset I$ and

$$F_z(x) := \chi(x) \int_0^{x_N} \int_0^{t_1} \ldots \int_0^{t_{k-1}} \mathbb{1}_{I_z}(s)\, ds \ldots dt_{k-1} \quad \text{where } x = (x', x_N) \in \Omega.$$

[46] K can be computed in terms of N and k

[47] We do not insist on the fact that M is a subspace. Note that the only issue is that the approximating countable set has to be a subset of M.

Then $z \in I$ implies[48] $F_z \in W^{k,\infty}(\Omega)$. We define

$$U_z := \left\{ f \in W^{k,\infty}(\Omega) : \|f - F_z\|_{W^{k,\infty}(\Omega)} < \frac{1}{2} \right\} \qquad (z \in I).$$

Then $(U_z)_{z\in I}$ is an uncountable set of pairwise disjoint open and non-empty subsets of $W^{k,\infty}(\Omega)$. The disjointness follows from

$$\begin{aligned} f \in U_{z_1} \cap U_{z_2} \quad &\Rightarrow \quad \|F_{z_1} - F_{z_2}\|_{W^{k,\infty}(\Omega)} < 1 \\ &\Rightarrow \quad \|\partial_N^k (F_{z_1} - F_{z_2})\|_{L^\infty(\Omega)} < 1 \\ &\Rightarrow \quad \|\partial_N^k (F_{z_1} - F_{z_2})\|_{L^\infty(\Omega' \times I)} < 1 \\ &\Rightarrow \quad \|\mathbb{1}_{I_{z_1}} - \mathbb{1}_{I_{z_2}}\|_{L^\infty(I)} < 1 \\ &\Rightarrow \quad I_{z_1} = I_{z_2} \\ &\Rightarrow \quad z_1 = z_2. \end{aligned}$$

(The restriction from Ω to $\Omega' \times I$ is made in order to get rid of χ. If Ω is known to be bounded, then χ may be replaced by 1.) We conclude that $W^{k,\infty}(\Omega)$ is not separable. □

This result and Proposition 12.4 also imply that $W_0^{k,p}(\Omega)$ and other subspaces are separable for $1 \le p < \infty$. Moreover, one can modify the proof in such a way that $W_0^{k,\infty}(\Omega)$ is seen not to be separable.

13 Reflexivity

Let $(X, \|\cdot\|_X)$ be a real Banach space. Then its dual space is defined by

$$X' := \{\phi : X \to \mathbb{R}, \phi \text{ is linear and bounded}\}.$$

Here, a linear functional $\phi : X \to \mathbb{R}$ is called bounded if there is a $C > 0$ such that $|\phi(f)| \le C\|f\|_X$ for all $f \in X$. It is an important fact that linear functionals are bounded if and only if they are continuous. One can show that X' is a Banach space when equipped with the norm

$$\|\phi\|_{X'} = \sup\{|\phi(f)| : f \in X, \|f\|_X = 1\}.$$

As an example, one may consider $X = L^1(\Omega)$ and $\phi : L^1(\Omega) \to \mathbb{R}$, $f \mapsto \int_\Omega f(x)\,dx$. One may check $\|\phi\|_{L^1(\Omega)'} = 1$. Introducing suitable weights or assumptions on Ω one may as well ensure $\phi \in L^p(\Omega)'$ for $1 \le p \le \infty$. Now $X'' := (X')'$ is the dual space of the dual space, called its bidual.

[48] Here $\chi \in C_0^\infty(\mathbb{R}^N)$ implies that all weak partial derivatives of order $\le k$ are bounded on Ω.

The notion of a reflexive space has to do with the nature of X''. More precisely, define $J : X \to X''$ via $(Jf)(g) := g(f)$ for $g \in X'$. This is well-defined because this map is linear, satisfies $J(0) = 0$ and for $f \in X \setminus \{0\}$ we have

$$\|Jf\|_{X''} = \sup_{\|g\|_{X'}=1} |(Jf)(g)| = \sup_{\|g\|_{X'}=1} |g(f/\|f\|_X)| \cdot \|f\|_X \le \sup_{\|g\|_{X'}=1} \|g\|_{X'}\|f\|_X = \|f\|_X.$$

Hence, J is a bounded linear operator. In a course on functional analysis one shows that J is injective (using the Hahn-Banach Theorem), but it need not be surjective.

Definition 13.1. *A Banach space X is called reflexive if $J : X \to X''$ is surjective.*

At first sight it is entirely unclear why such a seemingly artificial property should be important and even if so, how it can be checked. To get a better feeling for this property:

- $L^p(\Omega)$ is reflexive if and only if $1 < p < \infty$ (see below)
- $W^{k,p}(\Omega)$ is reflexive if and only if $1 < p < \infty$ (see below)
- Hilbert spaces are reflexive (Riesz' Representation Theorem)
- The sequence spaces l^1, l^∞, c_0 are not reflexive, $C([0,1])$ is not reflexive.

Its importance comes from the fact that in reflexive Banach spaces, bounded sequences have "weakly convergent" subsequences, see Corollary 13.8 below. This is the true generalization of the Bolzano-Weierstraß Theorem to infinite-dimensional Banach spaces. This result is the standard tool to prove the existence of minimizers of functionals in the Calculus of Variations. In fact, the minimizers are in most cases constructed as the weak limits of suitable bounded minimizing sequences for a given functional. The latter is often nonlinear, but one can check that a property called "weak lower-semicontinuity" is sufficient. The reflexivity of the space may be checked by proving the uniform convexity of its norm.

End Lec 17

Definition 13.2 (Uniform Convexity). *A normed vector space $(X, \|\cdot\|_X)$ is called uniformly convex if*

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x, y \in X \qquad \Big(\|x\| = \|y\| = 1, \|x - y\| \ge \varepsilon \quad \Rightarrow \quad \|\frac{x+y}{2}\| \le 1 - \delta \Big)$$

Lemma 13.3. *Let $1 < p < \infty$ and $\Omega \subset \mathbb{R}^N$. Then $(L^p(\Omega), \|\cdot\|_p)$ is uniformly convex.*

The same is true for the spaces $L^p(\Omega)^K$, $1 < p < \infty$, $K \in \mathbb{N}$, and subspaces thereof. A proof can be found in [1, p.41-45]. The proof of the following result is given in the Appendix.

Satz 13.4 (Milman-Pettis). *Assume that $(X, \|\cdot\|_X)$ is a uniformly convex Banach space. Then X is reflexive.*

Korollar 13.5. *Let $1 < p < \infty$ and $\Omega \subset \mathbb{R}^N$. Then $(L^p(\Omega), \|\cdot\|_p)$ and $(W^{k,p}(\Omega), \|\cdot\|_{k,p})$ are reflexive Banach spaces.*

Beweis:
The statement for $L^p(\Omega)$ is a direct consequence of the previous two results. Moreover, choosing $\Psi : W^{k,p}(\Omega) \to L^p(\Omega)^K$ as before, one finds that $\Psi(W^{k,p}(\Omega))$ is a closed subspace of the uniformly convex Banach space $L^p(\Omega)^K$. Hence, $(\Psi(W^{k,p}(\Omega)), \|\cdot\|_{L^p(\Omega)^K})$ is uniformly convex. Since Ψ is a linear isometry[49], this implies that $(W^{k,p}(\Omega), \|\cdot\|_{k,p})$ is uniformly convex and hence reflexive by the Milman-Pettis Theorem. □

We mention that closed subspaces of reflexive Banach spaces are again reflexive, see [2, Proposition 3.20]. In particular, this is true for $W_0^{k,p}(\Omega)$ for $k \in \mathbb{N}, 1 \le p < \infty$. We finally provide the main motivation why one cares about the reflexivity of Banach spaces, notably of $W^{k,p}(\Omega)$. To this end we introduce the following notions of convergence.

Definition 13.6. *Let X be a Banach space with dual space X'.*

(i) (Weak-$\star$-convergence) A sequence $(f_k)_{k\in\mathbb{N}} \subset X'$ is said to converge to $f \in X'$ in the weak-$\star$-sense (i.e., pointwise), written $f_k \rightharpoonup^\star f$, if $f_k(x) \to f(x)$ as $k \to \infty$ for all $x \in X$.

(ii) (Weak convergence) A sequence $(x_k)_{k\in\mathbb{N}} \subset X$ is said to converge weakly to $x \in X'$, written $x_k \rightharpoonup x$, if $f(x_k) \to f(x)$ as $k \to \infty$ for all $f \in X'$.

A detailed discussion about weak topologies and weak convergence is beyond the scope of this course given that we are not interested in abstract functional analysis. Instead some examples:

- Weak convergence in finite-dimensional spaces is equivalent to norm-convergence.
- Weak limits are uniquely determined if they exist.
- $x_n \to x$ implies $x_n \rightharpoonup x$, but in general not vice versa. For instance, if $I \subset \mathbb{R}$ is non-empty and $u_k(x) := \sin(kx)$, then $u_k \rightharpoonup 0$ in $L^2(I)$ but $u_k \not\to 0$.

So weak convergence is indeed a weaker notion of convergence and thus easier to get compared to norm convergence. This relaxation is important due to the following two results from functional analysis, which fail completely if convergence in norm is considered instead[50].

[49] This means $\|\Psi(u)\|_{L^p(\Omega)^K} = \|u\|_{W^{k,p}(\Omega)}$ for all $u \in W^{k,p}(\Omega)$.

[50] Recall: The unit ball of any(!) infinite-dimensional Banach space contains sequences without convergent subsequences (w.r.t. norm-convergence).

Satz 13.7 (Banach-Alaoglu)**.** *Let X be a separable Banach space. Then every bounded sequences in X' has a weak-*-convergent subsequence.*

Beweis:
We mimick the proof of the Ascoli-Arzelà Theorem. Let $M := \{x_n : n \in \mathbb{N}\}$ be dense in X and let $(f_k)_{k\in\mathbb{N}}$ be any bounded sequence in X', w.l.o.g. $\|f_k\|_{X'} \le 1$. One successively defines subsequences of (f_k) such that, after suitable relabeling the sequence at each step,

- $(f_k(x_1))$ converges,
- $(f_k(x_1)), (f_k(x_2))$ converge,
- etc.

This procedure yields a subsequence that converges in each point of M. Define

$$f : M \to \mathbb{R},\ y \mapsto \lim_{k\to\infty} f_k(y).$$

Then $f : M \to \mathbb{R}$ is Lipschitz-continuous because of

$$|f(x_i) - f(x_j)| = |\lim_{k\to\infty} f_k(x_i - x_j)| \le \|x_i - x_j\|.$$

Here we used $\|f_k\|_{X'} \le 1$ for all $k \in \mathbb{N}$. Hence, by density of M, the map

$$F : X \to \mathbb{R}, \quad x \mapsto \lim_{\substack{y\to x,\\ y\in M}} f(y)$$

is a well-defined bounded linear functional (i.e., $F \in X'$) with $\|F\| \le \|f\| \le 1$. It remains to check $f_k \rightharpoonup^\star F$.

To this end choose $x \in X$ and $\varepsilon > 0$ arbitrary. First take $x_i \in M$ such that $\|x_i - x\|_X \le \frac{\varepsilon}{3}$. Then we can find k_0 sufficiently large such that

$$\sup_{k\ge k_0} |f_k(x_i) - F(x_i)| \le \frac{\varepsilon}{3}.$$

But then, for all $k \ge k_0$,

$$\begin{aligned}
|f_k(x) - F(x)| &\le |f_k(x - x_i)| + |f_k(x_i) - F(x_i)| + |F(x_i - x)| \\
&\le \|f_k\|\|x - x_i\| + \frac{\varepsilon}{3} + \|F\|\|x_i - x\| \\
&\le 2\|x - x_i\| + \frac{\varepsilon}{3} \\
&\le \varepsilon.
\end{aligned}$$

This proves $f_k(x) \to F(x)$ as $k \to \infty$ for any given $x \in X$, which is the claim. □

Korollar 13.8. *Let X be a*[51] *(separable and) reflexive Banach space, then every bounded seqeunce in X has a weakly convergent subsequence.*

Beweis:
Let (x_n) be a bounded sequence in X. Then $(Jx_n)_{n\in\mathbb{N}}$ is a bounded sequence in $X'' = (X')'$. Since X' is separable[52], Theorem 13.7 provides a subsequence $(J(x_{n_j}))_{j\in\mathbb{N}}$ such that $J(x_{n_j}) \rightharpoonup^\star T$ in $(X')'$. This means

$$\lim_{j\to\infty} J(x_{n_j})(f) = T(f) \qquad \text{for all } f \in X'.$$

Since X is reflexive, we have $T = J(x)$ for some $x \in X$. So we find for any given $f \in X'$

$$f(x_{n_j}) = J(x_{n_j})(f) \to T(f) = J(x)(f) = f(x) \quad (j \to \infty).$$

In other words, $x_{n_j} \rightharpoonup x$ as $j \to \infty$. □

Finally, let us mention that $L^1(\Omega), L^\infty(\Omega)$ are not reflexive whenever $\Omega \subset \mathbb{R}^N$ is a nontrivial open set, see [2, p.101f]. This property carries over to the Sobolev spaces $W^{k,1}(\Omega), W^{k,\infty}(\Omega)$ for $k \in \mathbb{N}$.

[51] This remains true for non-separable Banach spaces, because one may then consider $\tilde{X} := \overline{\operatorname{span}\{x_n : n \in \mathbb{N}\}}$ instead. This space is now separable by construction and reflexive as a closed subspace of a reflexive Banach space.

[52] It is a general fact that Y' separable implies Y separable. So the separability of $X \simeq X''$ implies the separability of X'.

Notation and conventions

- All sets and functions are Lebesgue-measurable
- $B_r(x_0) = \{x \in \mathbb{R}^N : |x - x_0| < r\}$ = the open ball around $x_0 \in \mathbb{R}^N$ with radius $r > 0$
- $|A|$ = Lebesgue measure of a (Lebesgue-measurable) set $A \subset \mathbb{R}^N$
- $\omega_N := |B_1(0)|$ the volume of the unit ball in $\mathbb{R}^N$

Appendix

13.1 The Riesz-Fischer Theorem

We recall the Riesz-Fischer Theorem that establishes the completeness of $L^p(\Omega)$ for $1 \le p \le \infty$. As a byproduct, it gives useful additional information about subsequences of (convergent) Cauchy sequences in $L^p(\Omega)$.

Satz 13.1 (Riesz-Fischer [6,20]). *Assume that $\Omega \subset \mathbb{R}^N$ and $1 \le p \le \infty$. Then $(L^p(\Omega), \|\cdot\|_p)$ is complete. Additionally, for any Cauchy sequence $(u_n) \subset L^p(\Omega)$ there is a subsequence $(u_{n_k}) \subset (u_n)$ and $w \in L^p(\Omega)$ such that $|u_{n_k}| \le w$ and (u_{n_k}) converges pointwise almost everywhere to its $L^p(\Omega)$-limit.*

Beweis:
We only prove the claim for $1 \le p < \infty$. Let (u_n) be a Cauchy sequence. Choose a subsequence (u_{n_k}) such that

$$\|u_{n_k} - u_{n_{k+1}}\|_p \le 2^{-k} \qquad (k \in \mathbb{N})$$

Then define

$$w := |u_{n_1}| + \sum_{k=1}^{\infty} |u_{n_k} - u_{n_{k+1}}|.$$

We then have

$$|u_{n_k}| \le |u_{n_1}| + \sum_{j=1}^{k-1} |u_{n_j} - u_{n_{j+1}}| \le w \quad \text{for all } k \in \mathbb{N}.$$

Moreover, the Monotone Convergence Theorem implies

$$\|w\|_p = \||u_{n_1}| + \sum_{j=1}^{\infty} |u_{n_j} - u_{n_{j+1}}|\|_p$$

$$
\begin{aligned}
&= \lim_{m\to\infty} \| |u_{n_1}| + \sum_{j=1}^{m} |u_{n_j} - u_{n_{j+1}}| \|_p \\
&\le \liminf_{m\to\infty} \|u_{n_1}\|_p + \sum_{j=1}^{m} \|u_{n_j} - u_{n_{j+1}}\|_p \\
&\le \|u_{n_1}\|_p + \sum_{k=1}^{\infty} 2^{-k} < \infty,
\end{aligned}
$$

In particular, $|u_{n_1}| + \sum_{j=1}^{\infty} |u_{n_j} - u_{n_{j+1}}| \le w$ is finite almost everyhwere. So $(u_{n_j}(x))$ is a Cauchy sequence for almost all $x \in \Omega$. Since $(\mathbb{R}, |\cdot|)$ is complete, this subseqeunce converges pointwise almost everywhere to some measurable function u satisfying $|u(x)| \le w(x)$ almost everywhere. This proves the "additionally"-part that we claimed to hold.

Let's prove $u_{n_k} \to u$ in $L^p(\Omega)$. The Dominated Convergence Theorem gives

$$
\lim_{k\to\infty} \|u_{n_k} - u\|_p^p = \lim_{k\to\infty} \int_\Omega |u_{n_k}(x) - u(x)|^p \, dx = 0.
$$

Here we used $u_{n_k} - u \to 0$ pointwise almost everyhwere and $|u_{n_k} - u| \le 2w \in L^p(\Omega)$. We have to pshow rove that convergence actually holds for the full sequence, which is known to be a Cauchy sequence. For any given $\varepsilon > 0$ choose $k = k(\varepsilon)$ such that

$$
\|u_{n_k} - u_l\|_p \le \frac{\varepsilon}{2} \quad (l \ge n_k), \qquad \|u_{n_k} - u\|_p \le \frac{\varepsilon}{2}.
$$

It follows

$$
\|u - u_l\|_p \le \|u - u_{n_k}\|_p + \|u_{n_k} - u_l\|_p \le \varepsilon \qquad \text{for all } l \ge n_k,
$$

which is all we had to prove. □

13.2 Whitney's Covering Lemma

Lemma 13.2. *Let $\Omega \subset \mathbb{R}^N$ be open and $\emptyset \subsetneq \Omega \subsetneq \mathbb{R}^N$. Then there are closed almost disjoint dyadic cubes $W_1, W_2, \ldots$ with the following properties*

- *(I)* $\bigcup_{j\in\mathbb{N}} W_j = \Omega$,
- *(II)* $\operatorname{diam}(W_j) \le \operatorname{dist}(W_j, \Omega^c) \le 4\operatorname{diam}(W_j)$ *for all* $j \in \mathbb{N}$.
- *(III)* $\overline{W}_i \cap \overline{W}_j \ne \emptyset$ *implies* $\frac{1}{4}\operatorname{diam}(W_i) \le \operatorname{diam}(W_j) \le 4\operatorname{diam}(W_i)$,
- *(IV)* $\#\{i \in \mathbb{N} : \overline{W_i} \cap \overline{W_j} \ne \emptyset\} \le 12^N$ *for all* $j \in \mathbb{N}$.

Furthermore, for any fixed $\kappa \in (0, \frac{1}{4})$ there are $\phi_1, \phi_2, \ldots \in C_0^\infty(\mathbb{R}^N)$ such that

- *(V)* $0 \le \phi_j \le 1$, $\phi_j(x) = 1$ *for* $x \in W_j$ *and* $\phi_j(x) = 0$ *for* $\operatorname{dist}(x, W_j) \ge \kappa \operatorname{diam}(W_j)$. *(In particular, $\phi_j(x) \ne 0$ and $x \in W_i$ implies $\overline{W_i} \cap \overline{W_j} \ne \emptyset$.*

(VI) $|\partial^\alpha \phi_j(x)| \le C_\alpha \operatorname{diam}(W_j)^{-|\alpha|}$ *for all* $\alpha \in \mathbb{N}_0^N$.

Beweis:
For $k \in \mathbb{Z}$ we define the k-th dyadic mesh as follows:

$$W \in \mathcal{S}_k \quad \Leftrightarrow \quad W = \{2^{-k}(z+w) : w \in [0,1]^N\} \quad \text{for some } z \in \mathbb{Z}^N.$$

So elements of $\mathcal{S}_k$ for $|k|$ large and $k < 0$ are large dyadic cubes whereas the cubes from $\mathcal{S}_k$ for large k are small ones (to approximate the fine structures of Ω close to the potentially complicated boundary). We use

$$\Omega = \bigcup_{k\in\mathbb{Z}} \Omega_k \qquad \text{where } \Omega_k := \left\{x \in \Omega : 2^{1-k}\sqrt{N} < \operatorname{dist}(x,\Omega^c) \le 2^{2-k}\sqrt{N}\right\} \tag{13.1}$$

and define the collection of all dyadic cubes as follows:

$$\mathcal{F} := \bigcup_{k\in\mathbb{Z}} \mathcal{F}_k, \qquad \text{where } \mathcal{F}_k := \{W \in \mathcal{S}_k : W \cap \Omega_k \ne \varnothing\}. \tag{13.2}$$

The set $\mathcal{F}$ is countable as a countable union of countable sets. Then one can check

$$\Omega_k \subset \bigcup_{W\in\mathcal{S}_k} W \subset \Omega. \tag{13.3}$$

Due to $\Omega \ne \mathbb{R}^N$ we can attribute to each cube $W \in \mathcal{F}$ its uniquely determined "ancestor" cube $\hat{W} \in \mathcal{F}, \hat{W} \supset W$ that is maximal w.r.t inclusion, set[53]

$$\{W_1, W_2, \ldots\} := \{\hat{W} : W \in \mathcal{F}_k\}$$

For $j \in \mathbb{N}$ we define $k_j \in \mathbb{Z}$ by $W_j \in \mathcal{F}_{k_j}$ and the basepoint $z_j \in \mathbb{Z}^N$ by $W_j = 2^{-k_j}(z_j + [0,1]^N)$.

Proof of (I): Let $x \in \Omega$. By (13.1) there is some $k \in \mathbb{Z}$ such that $x \in \Omega_k$. By (13.3) there is $W \in \mathcal{F}_k$ such that $x \in \Omega_k \cap W$. Then $\operatorname{diam}(W) = \sqrt{N}2^{-k} < \operatorname{dist}(x,\Omega^c)$ by (13.1) implies $W \subset \Omega$ and thus

$$x \in W \subset \hat{W} \subset \bigcup_{j\in\mathbb{N}} W_j.$$

Proof of (II): $W_j \in \mathcal{F}_{k_j} \subset \mathcal{S}_{k_j}$ implies $\operatorname{diam}(W_j) = 2^{-k_j}\sqrt{N}$. By definition of $\mathcal{F}_{k_j}$ we may choose $x \in W_j \cap \Omega_{k_j}$, whence

$$\operatorname{dist}(W_j,\Omega^c) \le \operatorname{dist}(x,\Omega^c) \le 2^{2-k_j}\sqrt{N} = 4\operatorname{diam}(W_j).$$

[53] This is done in order not to count subcubes as new cubes, so $[0,1]\times[0,1]$ should not be added to the list of cubes if $[0,2]\times[0,2]$ is already there. We want to have almost disjoint cubes!

On the other hand, the triangle inequality gives

$$\operatorname{dist}(W_j,\Omega^c) \overset{(13.1)}{\geq} \operatorname{dist}(x,\Omega^c) - \operatorname{diam}(W_j) \geq 2^{1-k_j}\sqrt{N} - 2^{-k_j}\sqrt{N} = \operatorname{diam}(W_j).$$

So (II) is proved.

Proof of (III): Assume $\overline{W}_i \cap \overline{W}_j \neq \varnothing$. Then the triangle inequality gives

$$\operatorname{diam}(W_j) \overset{(II)}{\leq} \operatorname{dist}(W_j,\Omega^c) \leq \operatorname{dist}(W_i,\Omega^c) + \operatorname{diam}(W_i) \overset{(II)}{\leq} 5\operatorname{diam}(W_i).$$

But the quotients of the diameters is necessarily of the form 2^m where $m \in \mathbb{Z}$, so we conclude $\operatorname{diam}(W_j) \leq 4\operatorname{diam}(W_i)$. Interchanging the roles of i, j gives the other inequality and (III) is proved.

Proof of (IV): Assume $\overline{W}_i \cap \overline{W}_j \neq \varnothing$. Then $W_i \in \mathcal{S}_{k_i}, W_j \in \mathcal{S}_{k_j}$ implies $\frac{\operatorname{diam}(W_i)}{\operatorname{diam}(W_j)} = 2^{-k_i+k_j}$ and (III) gives $2^{-k_i+k_j} \in \{\frac{1}{4}, \frac{1}{2}, 1, 2, 4\}$. If z_i, z_j are the basepoints of these cubes, we get for all $l \in \{1,\dots,N\}$

$$2^{k_i}((z_i)_l + \alpha_l) = 2^{k_j}((z_j)_l + \beta_l) \quad \text{where } \alpha_l, \beta_l \in \{0,1\}.$$

Hence,

$$\begin{aligned}\#\{i \in \mathbb{N} : \overline{W_i} \cap \overline{W_j} \neq \varnothing\} &= \#\{i \in \mathbb{N} : 2^{k_i-k_j}((z_i)_l + \alpha_l) - \beta_l = (z_j)_k \text{ for some } \alpha_l, \beta_l \in \{0,1\}, l = 1,\dots,N\} \\ &\leq (5 \cdot 2 + 2)^N = 12^N.\end{aligned}$$

Proof of (V),(VI): Choose $0 < \kappa < \frac{1}{4}$ and $\phi \in C_0^\infty(\mathbb{R}^N)$ such that

$$\phi(x) = 1 \quad \text{for } x \in [0,1]^N, \qquad \phi(x) = 0 \quad \text{if } \operatorname{dist}(x,[0,1]^N) \geq \kappa\sqrt{N}.$$

(You may deduce the existence of such a function from Theorem 4.3.) If $W_j = 2^{-k_j}(z_j + [0,1]^N)$, then we set

$$\phi_j(x) := \phi(2^{k_j}x - z_j) \qquad (x \in \mathbb{R}^N)$$

Then we get $\phi_j(x) = 1$ for $x \in W_j$ as well as $\phi_j(x) = 0$ for $\operatorname{dist}(x,W_j) \geq \kappa\sqrt{N}2^{-k_j} = \kappa\operatorname{diam}(W_j)$. In particular,

$$\operatorname{supp}(\phi_j) \subset \bigcup_{\overline{W}_i \cap \overline{W}_j \neq \varnothing} W_i$$

Furthermore,

$$|(\partial^\alpha \phi_j)(x)| \leq 2^{k_j|\alpha|}\|\partial^\alpha \phi\|_\infty \leq C_\alpha \operatorname{diam}(W_j)^{-|\alpha|}.$$

□

13.3 A technical fact about Lipschitz domains

Proposition 13.3 (see Proposition 11.2)**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain with unit outer normal vector field $\nu : \partial\Omega \to \mathbb{R}^N$ and surface measure σ. Then there is a smooth vector field $F \in C^\infty(\overline{\Omega}; \mathbb{R}^N)$ such that $F(x) \cdot \nu(x) \geq 1$ for σ-almost all $x \in \partial\Omega$ and there is $t^* > 0$ such that $x + tF(x) \in \overline{\Omega}^c$ for $0 < t < t^*$, $x + tF(x) \in \Omega$ for $-t^* < t < 0$ for all $x \in \partial\Omega$.*

13.4 Tietze's Extension Theorem

In the following let (X, d) be a metric space and

$$\operatorname{dist}(x, A) := \inf_{a \in A} d(x, a)$$

measures the distance of $x \in X$ to a given subset $A \subset X$.

Proposition 13.4. *Let $A \subset X$. Then $x \mapsto \operatorname{dist}(x, A)$ is Lipschitz-continuous with Lipschitz constant 1.*

Beweis:
For any given $a \in A$ and $x, y \in X$ we use $|d(x, a) - d(y, a)| \leq d(x, y)$. It implies

$$\operatorname{dist}(x, A) \leq d(x, y) + d(y, a), \qquad \operatorname{dist}(y, A) \leq d(y, x) + d(x, a).$$

Taking the infimums with respect to $a \in A$ gives $|\operatorname{dist}(x, A) - \operatorname{dist}(y, A)| \leq d(x, y)$ and the claim follows □

Lemma 13.5 (Urysohn's Lemma)**.** *Let $A, B \subset X$ be closed disjoint subsets. Then there is a continuous function $f : X \to [0, 1]$ such that $f|_A = 1$ and $f|_B = 0$.*

Beweis:
Choose $f(x) := \frac{\operatorname{dist}(x,B)}{\operatorname{dist}(x,A)+\operatorname{dist}(x,B)}$. □

Satz 13.6 (Tietze's Extension Theorem)**.** *Let $A \subset X$ be closed and $f : A \to \mathbb{R}$ stetig. Then there is a continuous function $F : X \to [\inf_A f, \sup_A f]$ such that*

$$F|_A = f \quad \textit{and} \quad \sup_{x \in X} |F(x)| \leq \sup_{x \in A} |f(x)|.$$

Beweis:
We may w.l.o.g. assume $\inf_A f = -1, \sup_A f = 1$, for otherwise consider the continuous function

$$x \mapsto \frac{1}{d}\arctan(f(x) - c) \quad \text{where } c := \frac{1}{2}(\sup_A f + \inf_A f),\ d := \arctan(\frac{1}{2}(\sup_A f - \inf_A f)).$$

(In the exceptional case $\sup_A f - \inf_A f = \infty$ define $d = \frac{\pi}{2}$, in the case $\sup_A f - \inf_A f = 0$ simply extend by a constant function.)

We first construct a continuous function $g_0 : X \to \mathbb{R}$ such that

$$|f(x) - g_0(x)| \le \frac{2}{3} \quad \forall x \in A \qquad |g_0(x)| \le \frac{1}{3} \quad \forall x \in X.$$

In fact, Urysohn's Lemma provides a function $h : X \to [0,1]$ satisfying $h|_{\{f \le -\frac{1}{3}\}} = 0$ and $h|_{\{f \ge \frac{1}{3}\}} = 1$. Then $g_0(x) := \frac{1}{3}h(x) - \frac{1}{3}$ has the desired properties because of $|g_0(x)| \le \frac{1}{3}$ and

$$\begin{aligned}
\text{Case } -1 \le f(x) \le -\frac{1}{3}: &\qquad |f(x) - g_0(x)| = |f(x) + \frac{1}{3}| \le \frac{2}{3} \\
\text{Case } -\frac{1}{3} \le f(x) \le \frac{1}{3}: &\qquad |f(x) - g_0(x)| = |f(x)| + |g(x)| \le \frac{2}{3} \\
\text{Case } \frac{1}{3} \le f(x) \le 1: &\qquad |f(x) - g_0(x)| = |f(x) - \frac{1}{3}| \le \frac{2}{3}.
\end{aligned}$$

Next apply this preliminary result to the continuous function $\tilde{f} := \frac{3}{2}(f - g_0) : X \to [0,1]$. We thus obtain a continuous function $\tilde{g}_1 : X \to [0, \frac{1}{3}]$ such that $|\tilde{f}(x) - \tilde{g}_1(x)| \le \frac{2}{3}$ for all $x \in A$. Defining $g_1 := \frac{2}{3}\tilde{g}_1$ we obtain

$$|f(x) - g_0(x) - g_1(x)| \le \left(\frac{2}{3}\right)^2 \qquad \text{for all } x \in A.$$

Inductively we obtain a sequence of continuous functions (g_n) that satisfy

$$|g_n(x)| \le \frac{1}{3}\left(\frac{2}{3}\right)^n \quad \forall x \in X, \qquad |f(x) - \sum_{i=0}^{n} g_i(x)| \le \left(\frac{2}{3}\right)^{n+1} \quad \forall x \in A \tag{13.4}$$

Define $F(x) := \sum_{i=0}^{\infty} g_i(x)$. This series converges absolutely, so F is continuous with $|F(x)| \le 1$ and $F|_A = f$ follows from (13.4). □

13.5 The Milman-Pettis Theorem

We now provide a proof of Theorem 13.4 that goes back to [15, 18].

Proposition 13.7. *Let X be a normed space and $f, f_1, \dots, f_n \in X'$. Then f is a linear combination of $f_1, \dots, f_n$ if and only if $\bigcap_{j=1}^n \ker(f_j) \subset \ker(f)$.*

Beweis:
We assume w.l.o.g. that $\{f_1, \dots, f_n\}$ is linearly independent, in particular $f_j \neq 0$ for $j = 1, \dots, n$.

Assume first $f = \sum_{j=1}^n \alpha_j f_j$. Then, for all $x \in \bigcap_{j=1}^n \ker(f_j)$, we have

$$f(x) = \sum_{j=1}^n \alpha_j f_j(x) = 0,$$

hence $x \in \ker(f)$. So this direction is trivially true.

Now assume $\bigcap_{j=1}^n \ker(f_j) \subset \ker(f)$ and our aim is to show $f = \alpha_1 f_1 + \ldots + \alpha_n f_n$ for some $n \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_n \in \mathbb{R}$. We proceed inductively and start with the case $n = 1$.
In that case we have $\ker(f_1) \subset \ker(f)$. Since $f \neq 0$ the spaces $\ker(f), \ker(f_1)$ both have codimension 1[54], we infer $\ker(f_1) = \ker(f)$. Choosing $x^* \in X$ such that $f_1(x^*) \neq 0$ we obtain $f - \frac{f(x^*)}{f_1(x^*)} f_1 \equiv 0$, which settles the case $n = 1$.
Now assume that the claim has been proved for up to n linearly independent functionals and let $f_1, \dots, f_{n+1}$ be given as required. To apply the induction hypothesis define for $j = 1, \dots, n$ the functionals $g_j := f_j|_{\ker(f_{n+1})}$ and $g := f|_{\ker(f_{n+1})}$ on the space $\ker(f_{n+1})$. Then

$$\bigcap_{j=1}^n \ker(g_j) = \bigcap_{j=1}^n (\ker(f_j) \cap \ker(f_{n+1})) \subset \ker(f) \cap \ker(f_{n+1}) = \ker(g).$$

Hence, on $\ker(f_{n+1})$, $g = \sum_{j=1}^n \alpha_j g_j$ for some $\alpha_1, \dots, \alpha_n \in \mathbb{R}$. This implies

$$x \in \ker(f_{n+1}) \quad \Rightarrow \quad f(x) = g(x) = \sum_{j=1}^n \alpha_j g_j(x) = \sum_{j=1}^n \alpha_j f_j(x =) \quad \Rightarrow \quad x \in \ker\left(f - \sum_{j=1}^n \alpha_j f_j\right).$$

The induction hypothesis gives $f - \sum_{j=1}^n \alpha_j f_j = \alpha_{n+1} f_{n+1}$ for some $\alpha_{n+1} \in \mathbb{R}$. This proves the claim. □

Satz 13.8 (Helly). *Let X be a normed space and $f_1, \dots, f_n \in X'$, $c_1, \dots, c_n \in \mathbb{R}$. Then the following statements are equivalent:*

(i) There is $x \in X$ such that $f_j(x) = c_j$ for $j = 1, \dots, n$.

(ii) There is an $M > 0$ such that $|\alpha_1 c_1 + \ldots + \alpha_n c_n| \le M \|\alpha_1 f_1 + \ldots + \alpha_n f_n\|$ for all $\alpha_1, \dots, \alpha_n \in \mathbb{R}$.

[54] More precisely: Choose $x^* \in X$ with $f(x^*) = 1$. If $x \in \ker(f)$ is arbitrary, then $x - f_1(x)x^* \in \ker(f_1)$. By assumption, $x - f_1(x)x^* \in \ker(f)$, which is impossible in case $f_1(x) = 0$, so we get $x \in \ker(f_1)$. This proves $\ker(f) = \ker(f_1)$.

If (ii) is satisfied, then x can be chosen as in (i) with $\|x\| \leq M$ if $\dim(X) < \infty$ or $\|x\| \leq M + \varepsilon, \varepsilon > 0$ if $\dim(X) = \infty$.

Beweis:
(i)→(ii): If such an x exists, then

$$|\alpha_1 c_1 + \ldots + \alpha_n c_n| = |(\alpha_1 f_1 + \ldots + \alpha_n f_n)(x)| \leq \|\alpha_1 f_1 + \ldots + \alpha_n f_n\| \|x\|,$$

so we may choose $M := \|x\|$.

(ii)→ (i): We assume w.l.o.g. that $\{f_1, \ldots, f_n\}$ is linearly independent. Define $T(x) = (f_1(x), \ldots, f_n(x))$ for $x \in \mathbb{R}^n$. Since f_k is not a linear combination of the other $n-1$ functionals, Proposition 13.7 implies $\bigcap_{j \neq k} \ker(f_j) \notin \ker(f_k)$. In particular there is $y_k \in \bigcap_{j \neq k} \ker(f_j)$ such that $f_k(y_k) = 1$. This implies $T(y_k) = e_k$ for all $k \in \{1, \ldots, n\}$ and hence T is surjective. For any $c \in \mathbb{R}^n \setminus \{0\}$ we may thus find $y \in X$ such that

$$(f_1(y), \ldots, f_n(y)) = T(y) = (c_1, \ldots, c_n), \qquad y \notin \bigcap_{j=1}^{n} \ker(f_j).$$

It therefore remains to find $x \in y + \bigcap_{j=1}^n \ker(f_j)$ such that $\|x\|$ can be chosen as required.

In fact, a corollary of the Hahn-Banach Theorem provides a bounded linear functional $f \in X'$ with $\|f\| = 1$ und $f(y) = \operatorname{dist}(y, \bigcap_{j=1}^n \ker(f_j))$ and $f|_{\bigcap_{j=1}^n \ker(f_j)} \equiv 0$. This implies $\bigcap_{j=1}^n \ker(f_j) \subset \ker(f)$. Then Proposition 13.7 implies $f = \sum_{j=1}^n \alpha_j f_j$ for some $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and our assumption implies

$$\operatorname{dist}(y, \bigcap_{j=1}^{n} \ker(f_j)) = f(y) = \sum_{j=1}^{n} \alpha_j f_j(y) = \sum_{j=1}^{n} \alpha_j c_j \leq M \| \sum_{j=1}^{n} \alpha_j f_j \| = M\|f\| = M.$$

We may thus find $z \in \bigcap_{j=1}^n \ker(f_j)$ such that $\|y - z\| \leq M$ if $\dim(X) < \infty$ and $\leq M + \varepsilon$ if $\dim(X) = \infty$. So the claim follows for $x := y - z$. □

Satz 13.9 (Milman (1938), Pettis (1939)). *Let X be a uniformly convex Banach space. Then X is reflexive.*

Beweis:
Let $F \in X''$ be arbitrary with $\|F\| = 1$. We have to construct $x \in X$ with $F = Jx$, which will be achieved with Helly's Theorem.

By definition of the norm in X'' there is a normed sequence $(f_n) \subset X'$ such that $F(f_n) > 1 - \frac{1}{n}$. We then apply Helly's Theorem to $c_j := F(f_j), j = 1, \ldots, n$. In view of

$$|\alpha_1 c_1 + \ldots + \alpha_n c_n| = F(\alpha_1 f_1 + \ldots + \alpha_n f_n) \leq \|F\| \|\alpha_1 f_1 + \ldots + \alpha_n f_n\|$$

we find $x_n \in X$ satisfying

$$\|x_n\| \le 1 + \frac{1}{n} \qquad f_k(x_n) = F(f_k) \text{ for all } k = 1, \dots, n.$$

This implies $1 - \frac{1}{n} < F(f_k) = f_k(x_n) \le \|x_n\| \le 1 + \frac{1}{n}$ and thus for $m \ge n$

$$2 - \frac{2}{n} \le F(f_n) + F(f_n) = f_n(x_n) + f_n(x_m) = f_n(x_n + x_m) \le \|x_n + x_m\| \le 2 + \frac{2}{n}.$$

Hence,

$$\lim_{n\to\infty} \|\frac{x_n + x_m}{2}\| = 1, \qquad \lim_{n\to\infty} \|x_n\| = 1.$$

Uniform convexity implies (argue by contradiction)

$$\sup_{m\ge n} \|x_n - x_m\| \to 0 \qquad (n \to \infty).$$

Hence, (x_n) is a Cauchy sequence in X and thus converges to some $x \in X$. This proves the existence of $x \in X$ such that

$$\|x\| = 1 \quad \text{and} \quad F(f_k) = \lim_{m\to\infty} f_k(x_m) = f_k(x).$$

This element $x \in X$ is uniquely determined. Indeed, if $\tilde{x} \in X$ is another such element, then the sequence $(x_n) := (x, \tilde{x}, x, \tilde{x}, \dots)$ satisfies the conditions $\|x_n\| \le 1 + \frac{1}{n}$ and $f_k(x_n) = F(f_k)$ for all $k = 1, \dots, n$ just as above. We have seen that this implies that (x_n) is Cauchy, so the sequence converges. But this implies $x = \tilde{x}$, which proves the uniqueness.

Now fix any $f \in X', \|f\|_{X'} = 1$ and define $f_0 := f$ and consider the sequence $(f_k)_{k\in\mathbb{N}_0}$ for f_k as above. Helly's Theorem yields a sequence (y_n) with

$$\|y_n\| \le 1 + \frac{1}{n} \qquad f_k(y_n) = F(y_k) \text{ for all } k = 0, \dots, n,$$

which is Cauchy by the arguments presented above. Hence (y_n) converges and the uniqueness property proved above implies $y_n \to x$ as $n \to \infty$. But this implies

$$F(f) = F(f_0) = \lim_{n\to\infty} f_0(y_n) = f_0(x) = f(x).$$

Since x is independent of f and $f \in X', \|f\|_{X'} = 1$ was arbitrary, we conclude

$$F(f) = f(x) \quad \text{for all } f \in X'.$$

Hence $F = Jx$. Since $F \in X''$ was arbitrary, we conclude that $J : X \to X''$ is surjective, i.e., X is reflexive. □

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