Travelling Waves, WS 2015/16

Exercise sheet 2

Exercise 1 (6 Points)

Let \( f \in C^1[0,1] \). Show \( \tilde{p} \in C^1(0,1) \) with \( \tilde{p}(\tilde{v}) < 0 \) for all \( 0 < \tilde{v} < 1 \) solves

\[
\frac{d\tilde{p}(\tilde{v})}{d\tilde{v}} = -c - f(\tilde{v}) \quad \tilde{v} \in (0,1),
\]

\[
\lim_{\tilde{v} \downarrow 0} \tilde{p}(\tilde{v}) = 0, \quad \lim_{\tilde{v} \uparrow 1} \tilde{p}(\tilde{v}) = 0,
\]

if and only if the graph \( \Gamma = \{(\tilde{v}, \tilde{p}(\tilde{v})) : \tilde{v} \in (0,1)\} \) is the orbit of a global solution \( (v,p)^\top \in C^1(\mathbb{R},S) \) of

\[
\frac{d}{d\xi} \begin{pmatrix} v \\ p \end{pmatrix}(\xi) = \begin{pmatrix} p(\xi) \\ -cp(\xi) - f(v(\xi)) \end{pmatrix}, \quad \lim_{\xi \to \infty} \begin{pmatrix} v(\xi) \\ p(\xi) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \lim_{\xi \to -\infty} \begin{pmatrix} v(\xi) \\ p(\xi) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},
\]

that is \( \Gamma = \{(v(\xi), p(\xi))^\top : \xi \in \mathbb{R}\} \). Also show that (2) is not uniquely solvable.

Exercise 2 (6 Points)

In the setting of the proof of Lemma 3.8, e.g. Assumption (3.2) for \( f \) and the construction of \( v^* \) (\( 0 < v^* < 1 \)):

Show that there is \( \hat{c} > 0 \) such that for every \( |c| < \hat{c} \) there exists \( T(c) > 0 \), so that the solution of the initial value problem

\[
\begin{align*}
v' &= p, & v(0) &= v^*, \\
p' &= -cp - f(v), & p(0) &= 0,
\end{align*}
\]

satisfies \( p(t) < 0 \) for \( v(t) > 0, 0 < t \leq T(c) \) and \( v(T(c)) = 0 \).

Hint: Consider the extended system

\[
\begin{align*}
v' &= p, & v(0) &= v_0, \\
p' &= -cp - f(v), & p(0) &= p_0, \\
c' &= 0, & c(0) &= c_0,
\end{align*}
\]

Let \( V(t, v_0, p_0, c_0) := (v(t, (v_0, p_0, c_0)^\top), p(t, (v_0, p_0, c_0)^\top), c(t, (v_0, p_0, c_0)^\top))^\top \) denote the solution of this initial value problem at time \( t \). Let \( \psi : \mathbb{R}^3 \to \mathbb{R} \) be given by

\[
\psi \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x.
\]
Then apply the implicit function theorem to the mapping

\[ H : (T, v_0, p_0, c_0) \mapsto \psi(V(T, v_0, p_0, c_0)). \]

**Exercise 3**

(6 Points)

Let \( f : u \mapsto u(1-u)(u-\alpha). \) For two different values of \( c > 0 \) of your choice explore the behavior of solutions to

\[
\begin{align*}
v' &= p, \\
p' &= -cp - f(v)
\end{align*}
\]

in the neighborhood \([-0.1, 0.1]^2\) of \((0, 0)\) by plotting orbits of at least five solutions with initial values close to \((0,0)\).

You may use matlab and the built-in functions (like ode45) to solve the ODE system.

**Discussion:** Thursday, November 12.