Nonlinear Maxwell equations – a variational approach

Version of November 15, 2018

Course at the Karlsruher Institut für Technologie

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We are interested in the propagation of electromagnetic waves is described by the Maxwell equations for the electric field $\mathcal{E}$, the electric displacement field $\mathcal{D}$, the magnetic field $\mathcal{H}$, and the magnetic induction $\mathcal{B}$. These are time-dependent vector fields in a domain $\Omega \subset \mathbb{R}^3$. Given the current intensity $\mathcal{J}$ and the scalar charge density ρ, the *Maxwell equations* in differential form are as follows:

$$\begin{cases} \partial_t \mathcal{B} + \nabla \times \mathcal{E} = 0 & \text{(Faraday's Law)} \\ \nabla \times \mathcal{H} = \mathcal{J} + \partial_t \mathcal{D} & \text{(Ampere's Law)} \\ \operatorname{div}(\mathcal{D}) = \rho & \text{(Gauss' Electric Law)} \\ \operatorname{div}(\mathcal{B}) = 0 & \text{(Gauss' Magnetic Law).} \end{cases}$$

These fields are related by constitutive equations determined by the material. The relation between the electric displacement field and the electric field is given by $\mathcal{D} = \varepsilon\mathcal{E} + \mathcal{P}_{NL}(x, \mathcal{E})$ where $\varepsilon = \varepsilon(x) \in \mathbb{R}^{3\times 3}$ is the (linear) permittivity tensor of the material, and $\mathcal{P}_{NL}$ is the nonlinear part of the polarization. The relation between magnetic field and magnetic induction is $\mathcal{B} = \mu\mathcal{H} - \mathcal{M}$ where $\mu = \mu(x) \in \mathbb{R}^{3\times 3}$ denotes the magnetic permeability tensor and $\mathcal{M}$ the magnetization of the material. In a linear medium one has $\mathcal{P}_{NL} = 0$ leading to the linear Maxwell equations.

Suppose there are no currents, charges nor magnetization, i.e. $\mathcal{J} = 0$, $\rho = 0$, $\mathcal{M} = 0$. Then multiplying Faraday's law with μ^{-1}, taking the curl and using the constitutive relations and Ampere's law leads to the *nonlinear electromagnetic wave equation* of the form

$$\nabla \times (\mu(x)^{-1} \nabla \times \mathcal{E}) + \varepsilon(x)\partial_t^2 \mathcal{E} + \partial_t^2 \mathcal{P}_{NL}(x, \mathcal{E}) = 0 \tag{0.1}$$

for the electric field $\mathcal{E}$. Solving this one obtains $\mathcal{D} = \varepsilon\mathcal{E} + \mathcal{P}_{NL}(x, \mathcal{E})$ by the constitutive relation and $\mathcal{B}$ by time integrating Faraday's law. Finally $\mathcal{H} = \mu^{-1}\mathcal{B}$ is also determined by the constitutive relation.

Equation (0.1) is particularly challenging and in the literature there are several simplifications relying on approximation of the nonlinear electromagnetic wave equation. The most prominent one is the scalar or vector nonlinear Schrödinger equation. In order to justify this approximation one assumes that the term $\nabla(\operatorname{div}(\mathcal{E}))$ in $\nabla \times (\nabla \times \mathcal{E}) = \nabla(\operatorname{div}(\mathcal{E})) - \Delta\mathcal{E}$ is negligible and can be dropped, and that one can use the so-called *slowly varying envelope approximation*. However, this approach may produce non-physical solutions and *our goal* is to find *exact* solutions of the Maxwell equations and develop analytical tools which allow to look for *time-harmonic fields* $\mathcal{E}$ of the form

$$\mathcal{E}(x,t) = u(x)\cos(\omega t) \quad \text{for } x \in \Omega \text{ and } t \in \mathbb{R}$$

with frequency $\omega > 0$. Suppose that the nonlinear polarization is of the form

$$\mathcal{P}_{NL}(x,\mathcal{E}) = \chi\big(x, |u(x)|^2\big)\mathcal{E}$$

i.e. the scalar susceptibility χ depends only on the intensity of $\mathcal{E}$. Then (0.1) reduces to the *curl-curl* equation which is the main subject of our research

$$\nabla \times \big(\mu(x)^{-1}\nabla \times u\big) - V(x)u = f(x,u) \qquad \text{in } \Omega, \tag{0.2}$$

where $f(x,u) := \chi\big(x,|u|^2\big)u$ and $V(x) = \omega^2\varepsilon(x) \in \mathbb{R}^{3\times 3}$. Probably the most common type of nonlinearity in the physics and engineering literature is the *Kerr nonlinearity* $f(x,u) = \chi^{(3)}(x)|u|^2u$. Other examples for f that appear in applications are nonlinearities with *saturation* like $f(x,u) = \chi^{(3)}(x)\frac{|u|^2}{1+|u|^2}u$. The problem (0.2) has a variational nature, i.e. weak solutions correspond to critical points of the functional associated with (0.2). The problem is strongly indefinite and during the lectures we will develop variational tools which allow to find ground state and bound state solutions.

Plan of the lectures

- Nonlinear Dirichlet problem on a bounded domain. Mountain Pass Theorem and Nehari manifold approach.
- General conditions imposed on the nonlinear term. Mountain Pass Theorem vs. Nehari manifold approach.
- Functional and variational setting for the curl-curl equation on a bounded domain.
- The role of the cylindrical symmetry in the curl-curl problems.
- Generalized Nehari manifold approach for strongly indefinite problems. Critical point theory I.
- Generalized Nehari manifold approach for strongly indefinite problems. Critical point theory II.
- Ground state solutions and the multiplicity of bound state solutions.
- Curl-curl equations in $\mathbb{R}^3$ and the lack of compactness.
- Recent results and the list of open problems.

1 Nonlinear Dirichlet problem on a bounded domain. Mountain Pass Theorem and Nehari manifold approach

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ be a Lipschitz domain. Our aim is to find a function $u : \bar{\Omega} \to \mathbb{R}$ such that

$$(\mathcal{DP}) \qquad \begin{cases} -\Delta u + \lambda u = |u|^{p-2}u & \text{for } x \in \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta u(x) = \sum_{i=1}^N \partial_{x_i}^2 u(x)$, $\lambda \in \mathbb{R}$ and $p > 2$.

Suppose that $u \in \mathcal{C}^2(\bar{\Omega})$ satisfies the above equation and let $\varphi \in \mathcal{C}_0^\infty(\Omega)$. Then, in view of Green's theorem we obtain

$$\begin{aligned} 0 &= \int_\Omega (-\Delta u + \lambda u - |u|^{p-2}u)\varphi\, dx = -\int_{\partial\Omega} \frac{\partial u}{\partial \nu}\varphi\, d\sigma + \int_\Omega \nabla u \nabla \varphi\, dx + \int_\Omega \lambda u\varphi - |u|^{p-2}u\varphi\, dx \\ &= \int_\Omega \nabla u \nabla \varphi\, dx + \int_\Omega \lambda u \varphi - |u|^{p-2}u\varphi\, dx, \end{aligned}$$

where $\frac{\partial u}{\partial \nu}(x) = \nabla u(x)\nu(x)$, $\nu(x)$ is the exterior normal $\partial\Omega$, σ is the surface measure.

Definition 1. *Function u is a weak solution to $(\mathcal{DP})$ if*

$$\int_\Omega \nabla u \nabla \varphi\, dx + \int_\Omega \lambda u\varphi - |u|^{p-2}u\varphi\, dx = 0 \tag{1.1}$$

for any $\varphi \in \mathcal{C}_0^\infty(\Omega)$.

We introduce the Hilbert space

$$H^1(\Omega) = \{u \in L^2(\Omega) : \nabla u \in L^2(\Omega)\}$$

equipped with the inner product

$$\langle u, v \rangle = \int_\Omega \nabla u \nabla v\, dx + \int_\Omega uv\, dx.$$

Then $\|u\| = \sqrt{\langle u, u \rangle}$. Let $H_0^1(\Omega)$ be the closure of $\mathcal{C}_0^\infty(\Omega)$ in $H^1(\Omega)$. If $\partial\Omega$ is smooth and $u \in H^1(\Omega) \cap \mathcal{C}(\bar{\Omega})$, then $u \in H_0^1(\Omega)$ if and only if $u_{|\partial\Omega} = 0$. In general, we define the trace operator $T : H^1(\Omega) \to L^2(\partial\Omega)$ such that $T(u) = u_{|\partial\Omega}$ for $u \in \mathcal{C}^1(\bar{\Omega})$ [15, 27]. Then we show that

$$H_0^1(\Omega) = \{u \in H^1(\Omega) |\; T(u) = 0\}.$$

In view of the Sobolev embedding

$$H_0^1(\Omega) \subset L^q(\Omega) \text{ for } 1 \leq q \leq 2^* = \frac{2N}{N-2}. \tag{1.2}$$

Let $2 < p \leq 2^*$. Let us consider a functional $J : H_0^1(\Omega) \to \mathbb{R}$ give by

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\lambda}{2}\int_\Omega |u|^2\,dx - \frac{1}{p}\int_\Omega |u|^p\,dx. \tag{1.3}$$

Then

$$J'(u)(v) = \lim_{t\to 0}\frac{J(u+tv)-J(u)}{t} = \int_\Omega \nabla u \nabla v\,dx + \lambda\int_\Omega uv\,dx - \int_\Omega |u|^{p-2}uv\,dx \tag{1.4}$$

and $J \in \mathcal{C}^1$.

Problem 2. *Applying the Hölder inequality and the Lebesgue's dominated convergence theorem, show formula* (1.4). *Show the continuity of the Gateaux derivative and infer that* $J \in \mathcal{C}^1$.

Observe that (1.1) holds if and only if $J'(u)(\varphi) = 0$ for any $\varphi \in \mathcal{C}_0^\infty(\Omega)$. Since $\mathcal{C}_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$ we conclude that solutions $u \in H_0^1(\Omega)$ of $(\mathcal{DP})$ correspond to critical points of the functional J, i.e. $J'(u) = 0$.

1.1 Liner case – eigenvalue problem

The following problem $u \in H_0^1(\Omega)$

$$\begin{cases} -\Delta u + \lambda u = 0 & \text{for } x \in \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a (weak) solution if and only if

$$-\lambda \in \sigma(-\Delta) = \{\lambda_i : i = 1,2,3,...\},$$

for some $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq ... \leq \lambda_n \to \infty$. Let e_i be a unit eigenvector with the corresponding eigenvalue λ_i, i.e. $\|e_i\| = 1$ and $-\Delta e_i = \lambda_i e_i$. Each eigenvalue is of finite multiplicity and we can choose the vectors e_i such that $\{e_i\}_{i\geq 1}$ is the orthonormal basis in $H_0^1(\Omega)$, hence $\text{span}\{e_i : i = 1,2,3...\} = H_0^1(\Omega)$ and $\langle e_i, e_j\rangle = \delta_{i,j}$.

Suppose that $\lambda_k \leq -\lambda < \lambda_{k+1}$ for some $k \geq 0$, where $\lambda_0 = 0$. Let

$$\widetilde{X} := \text{span}\{e_1, e_2, ..., e_k\} \tag{1.5}$$

$$X^+ := \text{span}\{e_{k+1}, e_{k+2}, ...\}. \tag{1.6}$$

If $k = 0$ or $\lambda > 0$, then we set $\widetilde{X} = \{0\}$ and $X^+ = H_0^1(\Omega)$.

Proposition 3. *There is a constant $c > 0$ such that*

$$\int_\Omega |\nabla u|^2\,dx + \lambda \int_\Omega |u|^2\,dx \;\geq\; c\|u\|^2 \text{ for any } u \in X^+, \tag{1.7}$$

$$\int_\Omega |\nabla u|^2\,dx + \lambda \int_\Omega |u|^2\,dx \;\leq\; 0 \text{ for any } u \in \widetilde{X}. \tag{1.8}$$

Corollary 4. *If $\lambda > -\lambda_1$, then there are constants $c_1 \geq c_2 > 0$ such that*

$$c_1\|u\|^2 \geq \int_\Omega |\nabla u|^2\,dx + \lambda \int_\Omega |u|^2\,dx \geq c_2\|u\|^2 \quad \text{for } u \in H_0^1(\Omega).$$

1.2 General variational approach and the Montain Pass Theorem

In order to solve a variational problem like $(\mathcal{DP})$ we need to recognize the geometry of J which enable us to find a Palais-Smale sequence.

Suppose that X is a Banach space and $J : X \to \mathbb{R}$ is of class $\mathcal{C}^1$. We say that $(u_n) \subset X$ is a *Palais-Smale sequence* of J at level c if $J'(u_n) \to 0$ and $J(u_n) \to c$. If any Palais-Smale sequence (u_n) contains a convergent subsequence, the we say that the *Palais-Smale condition* is satisfied.

The usual **3-steps approach** (**G-C-S**) in variational problems:

Step (**G**eometry): Find the geometry, which provides a bounded Palais-Smale sequence.
Step (**C**ompactness): Find a convergent subsequence in some topology (e.g. weak or strong).
Step (**S**olution): Show that the (e.g. weak or strong) limit point is a solution.

Theorem 5 (Ambrosetti, Rabinowitz [2,30])**.** *Suppose that X is a Banach space and $J : X \to \mathbb{R}$ is of class $\mathcal{C}^1$ such that*

$$\inf_{\|u\|=r} J(u) > J(0) \geq J(u_0)$$

for some $r > 0$ and $u_0 \in X$ with $\|u_0\| > r$.
a) Then there is a Palais-Smale sequence (u_n) at the mountain pass level c, i.e. $J'(u_n) \to 0$ and $J(u_n) \to c$, where c is the mountain pass level

$$c := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} J(\gamma(t)) > 0,$$

$$\Gamma = \big\{\gamma \in \mathcal{C}([0,1], X) : \gamma(0) = 0 \text{ and } \gamma(1) = v\big\}.$$

b) If J satisfies the Palais-Smale condition, then c is a critical value, i.e. there is $u \in X$ such that $J'(u) = 0$ and $J(u) = c$.

1.3 Mountain Pass geometry, Palais-Smale condition and solutions to $(\mathcal{DP})$

Suppose that $\lambda > -\lambda_1$ and $2 < p < 2^*$. Let $|\cdot|_q$ be the usual Lebesgue norm in $L^q(\Omega)$. In view of Corollary 4

$$J(u) \geq c_2\|u\|^2 - \frac{1}{p}|u|_p^p \geq c_2\|u\|^2 - C\|u\|^p = \|u\|^2(c_2 - C\|u\|^{p-2}).$$

Then we find $r > 0$ such that

$$\inf_{\|u\|=r} J(u) > 0.$$

Moreover for $u \neq 0$ we get

$$J(tu) = t^2\Big(\frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\lambda}{2}\int_\Omega |u|^2\,dx\Big) - t^p\Big(\frac{1}{p}\int_\Omega |u|^p\,dx\Big) \to -\infty$$

as $t \to \infty$.

Theorem 6 (Rellich)**.** *Any bounded sequence in $H_0^1(\Omega)$ contains a convergent subsequence in $L^q(\Omega)$ for $1 \leq q < 2^*$.*

Lemma 7. *J satisfies the Palais-Smale condition.*

Proof. Let (u_n) be a Palais-Smale sequence at level c. Observe that

$$c + o(1)\|u_n\| = J(u_n) - \frac{1}{p}J'(u_n)(u_n) = \Big(\frac{1}{2} - \frac{1}{p}\Big)\int_\Omega |\nabla u|^2 + \lambda|u|^2\,dx \geq \Big(\frac{1}{2} - \frac{1}{p}\Big)c_2\|u_n\|^2,$$

thus $\|u_n\|$ is bounded in $H_0^1(\Omega)$. In view of Rellich theorem there is a subsequence u_{n_k} convergent to $u_0 \in H_0^1(\Omega)$ in $L^p(\Omega)$ and in $L^2(\Omega)$. Moreover

$$J'(u_{n_k})(u_{n_k} - u_0) = \langle u_{n_k}, u_{n_k} - u_0\rangle + (1-\lambda)\int u_{n_k}(u_{n_k} - u_0)\,dx - \int |u_{n_k}|^{p-2}u_n(u_{n_k} - u_0)\,dx.$$

By the Hölder inequality the last two integrals converges to 0. In addition,

$$\langle u_{n_k}, u_{n_k} - u_0\rangle = \|u_{n_k} - u_0\|^2 + \langle u_0, u_{n_k} - u_0\rangle$$

and by the Banach-Alaoglu theorem we may assume that $u_{n_k} \rightharpoonup u_0$ (bounded sequence in weakly sequentially compact in a reflexive space). Hence

$$o(1) = J'(u_{n_k})(u_{n_k} - u_0) = \|u_{n_k} - u_0\|^2 + \langle u_0, u_{n_k} - u_0\rangle = \|u_{n_k} - u_0\|^2 + o(1)$$

and we obtain that $u_{n_k} \to u_0$ in $H_0^1(\Omega)$. □

Theorem 8. *If $\lambda > -\lambda_1$ and $2 < p < 2^*$, then $(\mathcal{DP})$ has a (weak) solution at the mountain pass level.*

Proof. We apply Theorem 5 and Lemma 7. $\square$

Problem 9. *Consider $J : H_0^1(\Omega) \to \mathbb{R}$ such that*

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\lambda}{2}\int_\Omega |u|^2\,dx - \frac{1}{p}\int_\Omega (u)_+^p\,dx,$$

where $(u)_+(x) = \max\{u(x), 0\}$. Show that J has a critical point $u \neq 0$, such that $u(x) \geq 0$ for $x \in \Omega$. Show that u solves $(\mathcal{DP})$.

Problem 10. *Consider the following (defocusing) problem*

$$\begin{cases} -\Delta u + \lambda u = -|u|^{p-2}u & \text{for } x \in \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\lambda > -\lambda_1$. Show that $J : X \to \mathbb{R}$ is strictly convex and coercive, i.e. $J(u) \to \infty$ as $\|u\| \to \infty$. Then J attains the unique global minimum, which is the trivial solution.

2 Nonlinear Dirichlet problem on a bounded domain. Nehari manifold approach

We look for solutions to $(\mathcal{DP})$ with $\lambda > -\lambda_1$ and $2 < p < 2^*$, which minimize J among all nontrivial solutions. Observe that, by Proposition 3

$$\|u\|_\lambda := \Big(\int_\Omega |\nabla u|^2\,dx + \lambda\int_\Omega |u|^2\,dx\Big)^{1/2}$$

defines an equivalent norm in $H_0^1(\Omega)$.

We introduce the Nehari manifold

$$\mathcal{N} = \{u \in H_0^1(\Omega) \setminus \{0\} : J'(u)(u) = 0\}.$$

Lemma 11. *$m := \inf_{\mathcal{N}} J > 0$.*

Proof. If there is $u_n \in \mathcal{N}$ such that $\limsup_{n\to\infty} J(u_n) \leq 0$, then

$$\limsup_{n\to\infty} \Big(\frac{1}{2} - \frac{1}{p}\Big)\int_\Omega |\nabla u_n|^2 + \lambda|u_n|^2\,dx = \limsup_{n\to\infty}\Big(J(u_n) - \frac{1}{p}J'(u_n)(u_n)\Big) \leq 0.$$

Hence $u_n \to 0$. On the other hand, by the Sobolev embedding

$$c_2\|u_n\|^2 \le \int_\Omega |\nabla u_n|^2 + \lambda|u_n|^2\,dx = |u_n|_p^p \le C_1\|u_n\|^p,$$

and we get a contradiction $\|u_n\|^{p-2} \ge c_2/C_1 > 0$. □

Proposition 12. *If $J(u_0) = \inf_{\mathcal{N}} J$ for some $u_0 \in \mathcal{N}$, then $J'(u_0) = 0$. Moreover $\mathcal{N}$ is a manifold of class $\mathcal{C}^1$ diffeomorphic to the unit sphere*

$$\mathcal{S} = \{u \in H_0^1(\Omega) : \|u\| = 1\}.$$

Proof. Let $G(u) = J'(u)(u)$ and take any $v \in H_0^1(\Omega)$ and let $f(s,t) = G(t(u_0 + sv))$. Note that $\partial_t f(0,1) = G'(u)(u) < 0$ and by the implicit function theorem there exists $\delta > 0$ and a $\mathcal{C}^1$ function $t : (-\delta,\delta) \to \mathbb{R}$ such that $t(0) = 1$ and $f(s,t(s)) = 0$ for $s \in (-\delta,\delta)$. Then $\gamma(s) := t(s)(u+sv) \in \mathcal{N}$ defines a differentiable curve passing through u. Observe that $J(\gamma(s))$ attains a minimum at $s = 0$, hence

$$0 = \frac{\partial}{\partial s} J(\gamma(0)) = J'(u_0)(t'(0)u_0 + v) = J'(u_0)(v).$$

Therefore $J'(u_0) = 0$. Now observe that

$$G'(u)(u) = (1-p)\int_\Omega |u|^p\,dx < 0 \quad \text{for } u \in \mathcal{N},$$

hence by the implicit function theorem $\mathcal{N}$ is a $\mathcal{C}^1$-manifold of codimension 1. In fact, for any $u \neq 0$ we find the unique $t = t(u) > 0$ such that

$$J(t(u)u) = \max_{t>0} J(tu).$$

Then $t(u)u \in \mathcal{N}$ and t is of class $\mathcal{C}^1$. On the other hand, $\mathcal{N} \ni u \mapsto \frac{u}{\|u\|} \in \mathcal{S}$ defines the inverse to $t|_{\mathcal{S}}$.

□

Theorem 13. *There is a ground state solution u_0 of $(\mathcal{DP})$, i.e. $u_0 \in \mathcal{N}$ is such that $J'(u_0) = 0$ and*

$$J(u_0) = \inf_{\mathcal{N}} J = \inf\big\{J(u) : u \neq 0 \text{ and } J'(u) = 0\big\}.$$

Proof. Take any minimizing sequence (u_n) such that $J(u_n) \to m > 0$ and $u_n \in \mathcal{N}$. Observe that J is coercive on $\mathcal{N}$, since

$$J(u_n) \ge \Big(\frac{1}{2} - \frac{1}{p}\Big)\int_\Omega |\nabla u|^2 + \lambda|u|^2\,dx \ge \Big(\frac{1}{2} - \frac{1}{p}\Big)c_2\|u_n\|^2.$$

Therefore (u_n) is bounded and we may assume that $u_n \rightharpoonup u_0$ in $H_0^1(\Omega)$ and $u_n \to u_0$ in $L^p(\Omega)$ and in $L^2(\Omega)$. Then

$$c = \lim_{n\to\infty} J(u_n) \geq J(u_0).$$

Since $u_n \in \mathcal{N}$, we get

$$\int_\Omega |\nabla u_n|^2 + \lambda |u_n|^2\, dx = \int_\Omega |u_n|^p\, dx,$$

and

$$\int_\Omega |\nabla u_0|^2 + \lambda |u_0|^2\, dx \leq \int_\Omega |u_0|^p\, dx$$

and $u_0 \neq 0$. Note that we find $t \leq 1$ such that

$$\int_\Omega |\nabla t u_0|^2 + \lambda |t u_0|^2\, dx = \int_\Omega |t u_0|^p\, dx,$$

that is $tu_0 \in \mathcal{N}$. On the other hand

$$\begin{aligned} m &\leq J(tu_0) = t^2\Big(\frac{1}{2} - \frac{1}{p}\Big)\int_\Omega |\nabla u_0|^2 + \lambda |u_0|^2\, dx \leq t^2 \liminf_{n\to\infty}\Big(\frac{1}{2} - \frac{1}{p}\Big)\int_\Omega |\nabla u_n|^2 + \lambda |u_n|^2\, dx \\ &= t^2 \liminf_{n\to\infty} J(u_n) = m \end{aligned}$$

Thus $t = 1$, $u_0 \in \mathcal{N}$ and by Lemma 12 we obtain that $J'(u_0) = 0$. □

3 Mountain Pass Theorem vs. Nehari manifold approach

Theorem 14. *Let X be a Banach space, $J : X \to \mathbb{R}$ is a functional of class $\mathcal{C}^1$ and*

$$\mathcal{N} := \{u \in X \setminus \{0\} : J'(u) = 0\}.$$

Suppose that

$$\inf_{\|u\|=r} J(u) > J(0)$$

for some $r > 0$, and $u_0 \in X$ with $\|u_0\| > r$.
a) Then there is a Palais-Smale sequence (u_n) at the following mountain pass level

$$c = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} J(\gamma(t)) > 0$$

$$\Gamma = \{\gamma \in \mathcal{C}([0,1], X) |\ \gamma(0) = 0,\ \|\gamma(1)\| > r \text{ and } J(\gamma(1)) < 0\}.$$

b) If J satisfies the Palais-Smale condition, then c is a critical value, i.e. there is $u \in X$ such that $J'(u) = 0$ and $J(u) = c$.
c) Suppose that for any $u \in \mathcal{N}$ we have $J(tu) \to -\infty$ as $t \to \infty$. If c is attained by a critical point and $J(u) \geq J(tu)$ for any $t \geq 0$ and $u \in \mathcal{N}$, then $c = \inf_{\mathcal{N}} J$.

Proof. c) Since c is attained by a critical point u_0, we get $c = J(u_0) \geq \inf_{\mathcal{N}} J$. Take any $\varepsilon > 0$ and $u \in \mathcal{N}$ such that $J(u) \leq \inf_{\mathcal{N}} J + \varepsilon$. We find $t_0 > r/\|u\|$ such that $J(t_0 u) < 0$. Consider a path $\gamma(t) := t t_0 u$ for $t \in [0,1]$ and observe that $\gamma \in \Gamma$ and

$$c \leq J(\gamma(t)) \leq J(u) \leq \inf_{\mathcal{N}} J + \varepsilon.$$

Since ε is arbitrary, we infer $c = \inf_{\mathcal{N}} J$. □

Problem 15. *Show that in problem* $(\mathcal{DP})$ *one has* $c = m$*, where* c *is the mountain pass level given in Theorem 14.*

3.1 General nonlinearity

Let us consider the following problem

$$(\mathcal{DP}2) \qquad \begin{cases} -\Delta u + \lambda u = f(x,u) & \text{for } x \in \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\lambda > -\lambda_1$, $N \geq 3$, where f satisfies the following assumptions

(F1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is measurable and continuous in $u \in \mathbb{R}$ for a.e. $x \in \mathbb{R}^N$, and there are $c > 0$ and $2 < p < 2^*$ such that

$$|f(x,u)| \leq c(1 + |u|^{p-1}) \text{ for all } u \in \mathbb{R},\ x \in \Omega.$$

(F2) $f(x,u) = o(|u|)$ uniformly in x as $|u| \to 0$.

We observe that the following functional associated with $(\mathcal{DP}2)$ is given by

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx + \frac{\lambda}{2}\int_\Omega |u|^2\, dx - \int_\Omega F(x,u)\, dx$$

and has the mountain pass geometry as in Theorem 5. Hence we may find a Palais-Smale sequence. In order to show the boundedness of the sequence the following condition has been introduced by Ambrosetti and Rabinowitz [2].

(AR) There is $\gamma > 2$ such that $f(x,u)u \geq \gamma F(x,u) > 0$ for $u \neq 0$.

Similarly as in Lemma 7 we estimate

$$J(u_n) - \frac{1}{\gamma} J'(u_n)(u_n)$$

and show the boundedness and show the Palais-Smale condition is satisfied.

Theorem 16. *There is a weak solution to ($\mathcal{DP}2$) provided that (F1)-(F2) and (AR) are satisfied.*

In order to use the Nehari approach we assume the following conditions:

(F3) $F(x,u)/|u|^2 \to \infty$ uniformly in x as $|u| \to \infty$, where F is the primitive of f with respect to u.

(F4) $u \mapsto f(x,u)/|u|$ is strictly increasing on $(-\infty, 0)$ and $(0, \infty)$.

Note that $f(x,u) = u\ln(1+|u|^{p-2})$ satisfies (F1)-(F4), but (AR) does not hold. On the other hand, (AR) implies only (F3) and one can easy provide examples which satisfy (AR) and do not satisfy the monotonicity condition (F4). Since J need not to be of class $\mathcal{C}^2$, so $\mathcal{N}$ need not to be of class $\mathcal{C}^1$ and we cannot directly minimize on $\mathcal{N}$. We apply approach due to Szulkin and Weth [28] based on the observation that the Nehari manifold $\mathcal{N}$ is homemorphic to the unit sphere $\mathcal{S}$ where we may apply a critical point theory.

3.2 Nehari manifold in the abstract setting

Let X be a Hilbert space with the norm $\|\cdot\|$. We consider a functional $J : X \to \mathbb{R}$ of the following form

$$J(u) = \frac{1}{2}\|u\|^2 - I(u),$$

where $I : X \to \mathbb{R}$ is of $\mathcal{C}^1$ class. In this case the Nehari manifold is given by

$$\begin{aligned} \mathcal{N} &:= \{u \in X \setminus \{0\} \ : \ J'(u)(u) = 0\} \\ &= \{u \in X \setminus \{0\} \ : \ \|u\|^2 = I'(u)(u)\}. \end{aligned}$$

Now we formulate the main result of this section.

Theorem 17 ([12]). *Suppose that the following conditions hold:*

(J1) there is $r > 0$ such that $a := \inf_{\|u\|=r} J(u) > J(0) = 0$;

(J2) there is $q \geq 2$ such that $I(t_n u_n)/t_n^q \to \infty$ for any $t_n \to \infty$ and $u_n \to u \neq 0$ as $n \to \infty$;

(J3) for $t \in (0,\infty) \setminus \{1\}$ and $u \in \mathcal{N}$

$$\frac{t^2-1}{2} I'(u)(u) - I(tu) + I(u) < 0;$$

(J4) J is coercive on $\mathcal{N}$.

Then $\inf_{\mathcal{N}} J > 0$ *and there exists a bounded minimizing sequence for* J *on* $\mathcal{N}$*, i.e. there is a sequence* $(u_n) \subset \mathcal{N}$ *such that* $J(u_n) \to \inf_{\mathcal{N}} J$ *and* $J'(u_n) \to 0$.

Observe that condition (J2) implies that for any $u \neq 0$ there is $t > 0$ such that $\mathcal{J}(tu) < 0$, hence taking into account also (J1) we easily check that J has the classical mountain pass geometry [2, 30] and we are able to find a Palais-Smale sequence. However we do not know whether it is a bounded sequence and contained in $\mathcal{N}$. In order to get the boundedness we assume the coercivity in (J4), which is, in applications, a weaker requirement than the classical Ambrosetti-Rabinowitz condition; see e.g. [28].

Remark 18. *a) In order to get (J3) it is sufficient to check*

$$(1-t)\left(tI'(u)(u) - I'(tu)(u)\right) > 0 \quad \text{for any } t \in (0,\infty)\setminus\{1\} \text{ and } u \text{ such that } I'(u)(u) > 0. \tag{3.1}$$

Indeed, let us consider $t \in (0,\infty)\setminus\{1\}$, $u \in \mathcal{N}$ *and*

$$\varphi(t) = \frac{t^2-1}{2} I'(u)(u) - I(tu) + I(u). \tag{3.2}$$

Then $I'(u)(u) = \|u\|^2 > 0$, $\varphi(1) = 0$, $\varphi'(t) = tI'(u)(u) - I'(tu)(u) > 0$ *for* $t < 1$ *and* $\varphi'(t) < 0$ *for* $t > 1$*. Therefore* $\varphi(t) < \varphi(1) = 0$.
b) Observe that (J3) is equivalent to the following condition: $u \in \mathcal{N}$ *is the unique maximum point of* $(0,+\infty) \ni t \mapsto J(tu) \in \mathbb{R}$*. Indeed, note that for* $u \in \mathcal{N}$

$$J(tu) = J(u) + \big(J(tu) - J(u) - \frac{t^2-1}{2} J'(u)(u)\big) = J(u) + \varphi(t) < J(u) \tag{3.3}$$

if and only if $\varphi(t) < 0$.

Proof of Theorem 17. For a given $u \neq 0$ we consider a map $\varphi : [0,+\infty) \to \mathbb{R}$ defined by $\varphi(t) = J(tu) - J(u)$ for $t \in [0,+\infty)$. Note that from (3.3), $\varphi(t)$ is given by (3.2) provided that $u \in \mathcal{N}$. In view of (J1)-(J2) we obtain

$$\varphi(0) = -J(u) < \varphi\Big(\frac{r}{\|u\|}\Big) \text{ and } \varphi(t) \to -\infty \quad \text{as } t \to \infty.$$

Therefore there is a maximum point $t(u) > 0$ of φ which is a critical point of φ, i.e. $J'(t(u)u)(u) = 0$ and $t(u)u \in \mathcal{N}$. In view of Remark 18 b) we infer that for any $u \neq 0$ there is an unique critical point $t(u) > 0$ of φ, i.e. $t(u)u \in \mathcal{N}$. Let $\hat{m} : X \setminus \{0\} \to \mathcal{N}$ be a map given by $\hat{m}(u) = t(u)u$ for $u \neq 0$. We are going to show that $\hat{m}$ is continuous. Take $u_n \to u_0 \neq 0$ and denote $t_n = t(u_n)$ for $n \geq 0$, so that $\hat{m}(u_n) = t_n u_n$. Observe that if $t_n \to \infty$ then by (3.3) and (J2)

$$o(1) = J(u_n)/t_n^q \leq J(\hat{m}(u_n))/t_n^q = \frac{1}{2}\|u_n\|^2 t_n^{2-q} - I(t_n u_n)/t_n^q \to -\infty$$

and we get a contradiction. Therefore we may assume that $t_n \to t_0 \geq 0$. Again by (3.3)

$$J(t(u_0)u_0) \geq J(t_0 u_0) = \lim_{n\to\infty} J(t_n u_n) \geq \lim_{n\to\infty} J(t(u_0)u_n) = J(t(u_0)u_0)$$

we get $t_0 = t(u_0)$, which completes the proof of continuity of $\hat{m}$. Thus $m = \hat{m}|_{\mathcal{S}}$, where $\mathcal{S} := \{u \in X : \|u\| = 1\}$ is the unit sphere in X, is a homeomorphism. Indeed, the inverse function is given by $m^{-1}(v) = v/\|v\|$ for $v \in \mathcal{N}$. Therefore

$$c := \inf_{u\in\mathcal{S}} (J \circ m)(u) = \inf_{u\in\mathcal{N}} J(u) \geq \inf_{u\in\mathcal{N}} J\left(\frac{r}{\|u\|}u\right) \geq a > 0.$$

Now, arguing as in [28][Proposition 2.9], we show that $J \circ \hat{m}$ is of $\mathcal{C}^1$-class. Moreover for $w, z \in X$, $w \neq 0$ we have

$$(J \circ \hat{m})'(w)z = \frac{\|\hat{m}(w)\|}{\|w\|} J'(\hat{m}(w))z.$$

Indeed, let $w \in X \setminus \{0\}$ and $u = \hat{m}(w) \in \mathcal{N}$. Therefore $t(w) = \frac{\|u\|}{\|w\|}$ and $u = \frac{\|u\|}{\|w\|}w$. Let $z \in X$ and choose $\delta > 0$ so small that

$$w_t := w + tz \in X \setminus \{0\}$$

for $|t| < \delta$. Let $u_t = \hat{m}(w_t)$. Therefore $u_t = s_t w_t$ for $|t| < \delta$ and some $s_t > 0$, $s_0 = \frac{\|u\|}{\|w\|}$. The function $v \mapsto \hat{m}(v)$ is continuous, thus

$$(-\delta, \delta) \ni t \mapsto s_t \in \mathbb{R}$$

is also continuous. From the mean value theorem, there is $\tau_t \in (0, 1)$ such that

$$\begin{aligned}(J \circ \hat{m})(w_t) - (J \circ \hat{m})(w) &= J(u_t) - J(u) = J(s_t w_t) - J(s_0 w) \leq \\ &\leq J(s_t w_t) - J(s_t w) = J'(s_t[w + \tau_t(w_t - w)])s_t(w_t - w).\end{aligned}$$

Since $s_t \to s_0$ as $t \to 0$, we get

$$(J \circ \hat{m})(w_t) - (J \circ \hat{m})(w) \leq s_0 J'(u)tz + o(t) \text{ as } t \to 0.$$

In a similar way we have

$$(J \circ \hat{m})(w_t) - (J \circ \hat{m})(w) \geq s_0 J'(u)tz + o(t) \text{ as } t \to 0$$

Therefore

$$(J \circ \hat{m})'(w)z = \lim_{t \to 0} \frac{(J \circ \hat{m})(w_t) - (J \circ \hat{m})(w)}{t} = s_0 J'(u)z.$$

We see that $(J \circ \hat{m})'(w)z$ is linear and countinuous in z, and it is continouous in w, thus $J \circ \hat{m}$ is of $\mathcal{C}^1$ class. In view of the Ekeland variational principle [30][Theorem 8.5] we find a minimizing sequence $(v_n) \subset \mathcal{S}$ for $J \circ m$ such that $(J \circ m)'(v_n) \to 0$. Then take $u_n = m(v_n) \in \mathcal{N}$ and observe that $J'(u_n)(v_n) = 0$ and

$$(J \circ m)'(v_n)(z) = \|u_n\| J'(u_n)(z) = \|u_n\| J'(u_n)(z + t v_n).$$

for any $z \in T_{v_n}\mathcal{S}$ and $t \in \mathbb{R}$, where $T_{v_n}\mathcal{S}$ stands for the tangent space $\mathcal{S}$ at v_n. Therefore

$$\|(J \circ m)'(v_n)\| = \sup_{z \in T_{v_n}\mathcal{S},\ \|z\|=1} (J \circ m)'(v_n)(z) = \|u_n\| \|J'(u_n)\|.$$

Since $u_n \in \mathcal{N}$, we have $\|u_n\| \geq \eta$ for some $\eta > 0$. The coercivity of J implies that $\sup_n \|u_n\| < \infty$. Hence (u_n) is a bounded minimizing sequence for J on $\mathcal{N}$ such that $J'(u_n) \to 0$. □

3.3 Ground state solutions

Theorem 19. *There is a weak solution u_0 to $(\mathcal{DP}2)$ such that $J(u_0) = \inf_{\mathcal{N}} J$ provided that (F1)-(F4) are satisfied.*

Proof. We easy check conditions (J1)-(J4) and in view of Theorem 17 we obtain a bounded Palais-Smale sequence. It contains a convergent subsequence due to Rellich theorem. □

4 Maxwell equations on a bounded domain

We look for a vector field $u : \Omega \to \mathbb{R}^3$ satisfying the following equation

$$(\mathcal{MP}) \qquad \nabla \times \big(\mu(x)^{-1} \nabla \times u\big) - V(x)u = f(x,u) \qquad \text{in } \Omega$$

derived from (0.1) in the time-harmonic case. Observe that the nonlinearity is a gradient: $f(x,u) = \nabla_u F(x,u)$ with $F(x,u) = \frac{1}{2}\psi\big(x, |u|^2\big)$ where $\psi(x,s) = \int_0^s \chi(x,r)dr$. Solutions of $(\mathcal{MP})$ are critical points of the functional

$$J(u) = \frac{1}{2}\int_\Omega \langle \mu(x)^{-1} \nabla \times u, \nabla \times u \rangle \, dx - \frac{1}{2}\int_\Omega \langle V(x)u, u \rangle \, dx - \int_\Omega F(x,u)\, dx \tag{4.1}$$

defined on an appropriate subspace X of $H_0(\mathrm{curl};\Omega)$ such that $F(x,u)$ and $\langle V(x)u,u\rangle$ are integrable. The precise definition of the domain of J will be given in the next subsection.

Let us mention already at this point a major difficulty when dealing with this equation. If $u = \nabla\phi$ is a gradient then $\nabla \times u = 0$, hence the differential operator in $(\mathcal{MP})$ has an infinite-dimensional kernel and J has no longer the mountain pass geometry.

The above mentioned difficulty that the curl operator $\nabla\times$ has an infinite-dimensional kernel is of course also present in the variational approach. One of the consequences is that the functional is strongly indefinite, i.e. Morse indices of critical points will be infinite. Another consequence is that the Palais-Smale condition does not hold. And a third difficulty is that the derivative $J' : X \to X^*$ is not weak-to-weak* continuous even when the growth of F is subcritical. Thus even if J has a linking geometry in the spirit of Benci and Rabinowitz [4], the problem cannot be treated by standard variational methods for strongly indefinite functionals as in [4, 6, 19].

Probably the most common type of nonlinearity in the physics and engineering literature is the Kerr nonlinearity

$$f(x,u) = \chi^{(3)}(x)|u|^2u. \tag{4.2}$$

Other examples for f that appear in applications are nonlinearities with saturation like

$$f(x,u) = \chi^{(3)}(x)\frac{|u|^2}{1+|u|^2}u, \tag{4.3}$$

or cubic-quintic nonlinearities like

$$f(x,u) = \chi^{(3)}(x)|u|^2u - \chi^{(5)}(x)|u|^4u. \tag{4.4}$$

We refer the reader to [23, 25, 26] and the references therein for these and further examples.

When Ω has a boundary then boundary conditions depend of course on the material characteristics of the complement $\mathbb{R}^3 \setminus \overline{\Omega}$. In this lecture we shall only consider the case of Ω being surrounded by a perfectly conducting medium which leads to the so-called metallic boundary condition

$$\nu \times u = 0 \qquad \text{on } \partial\Omega \tag{$\mathcal{BC}$}$$

where $\nu : \partial\Omega \to \mathbb{R}^3$ is the exterior normal.

4.1 Autonomous curl-curl problem

We look for solutions $u : \Omega \to \mathbb{R}^3$ of the problem

$$(\mathcal{MP}2) \qquad \nabla \times (\nabla \times u) + \lambda u = |u|^{p-2}u \qquad \text{in } \Omega$$

with the boundary condition $(\mathcal{BC})$. Let us consider

$$W^p(\mathrm{curl};\Omega) := \{E \in L^p(\Omega,\mathbb{R}^3) : \nabla \times E \in L^2(\Omega,\mathbb{R}^3)\}$$

which is a Banach space with the norm

$$\|E\|_{W^p(\mathrm{curl};\Omega)} := \left(|E|_p^2 + |\nabla \times E|_2^2\right)^{1/2},$$

where $|\cdot|_q$ denotes the L^q-norm. We look for solutions of $(\mathcal{MP}2)$ in the closure

$$X := W_0^p(\mathrm{curl};\Omega)$$

of $\mathcal{C}_0^\infty(\Omega,\mathbb{R}^3)$ in $W^p(\mathrm{curl};\Omega)$ and

$$J(u) = \frac{1}{2}\int_\Omega |\nabla \times u|^2\,dx + \frac{\lambda}{2}\int_\Omega |u|^2\,dx - \frac{1}{p}\int_\Omega |u|^p\,dx$$

is a well-defined functional of class $\mathcal{C}^1$. Observe that X contains gradient fields $\nabla\varphi$ for $\varphi \in W_0^{1,p}(\Omega)$ and we cannot expect any compact embedding of X into L^q space.

In the linear case $W_0^2(\mathrm{curl};\Omega)$ is the Hilbert space denoted by $H_0(\mathrm{curl};\Omega)$; see [14, 16, 22], and the following subspace pays a crucial role

$$\mathcal{V} := \left\{v \in H_0(\mathrm{curl};\Omega) : \int_\Omega \langle v,\varphi\rangle\,dx = 0 \text{ for every } \varphi \in \mathcal{C}_0^\infty(\Omega,\mathbb{R}^3) \text{ with } \nabla\times\varphi = 0\right\}.$$

Taking $\varphi = \nabla\psi$ for $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^3)$ we see that if $v \in \mathcal{V}$, then $\mathrm{div}\,(v) = 0$. Hence

$$\mathcal{V} \subset \left\{E \in H_0(\mathrm{curl};\Omega) : \mathrm{div}\,(E) \in L^2(\Omega,\mathbb{R}^3)\right\} =: X_N(\Omega).$$

In view of [1, 13], we know that $X_N(\Omega)$ embeds continuously into $H^s(\Omega,\mathbb{R}^3)$ for some $s \in [1/2, 1]$, but if, in addition Ω is convex or has $\mathcal{C}^{1,1}$-boundary, then $X_N(\Omega)$ embeds into $H^1(\Omega,\mathbb{R}^3)$. Hence, from now on, we assume that Ω is convex or has $\mathcal{C}^{1,1}$-boundary.

4.2 Linear case

Observe that $H_0(\mathrm{curl};\Omega)$ is a Hilbert space with the inner product

$$\langle \nabla \times u_1, \nabla \times u_2\rangle = \int_\Omega \langle \nabla \times u_1, \nabla \times u_2\rangle + \langle u_1, u_2\rangle\,dx.$$

Observe that elements of $H_0(\mathrm{curl};\Omega)$ need not be zero on the boundary. In fact, for $u \in H_0^1(\Omega)$ we claim that $\nabla u \in H_0(\mathrm{curl};\Omega)$. There exists $\phi_n \in \mathcal{C}_0^\infty(\Omega)$ converging towards u in $H^1(\Omega)$ and such that $\nabla\phi_n$ converges towards ∇u in $L^2(\Omega,\mathbb{R}^3)$. Then $\nabla\phi_n$ converges towards ∇u in $W^2(\mathrm{curl};\Omega)$ because the curl of gradient fields is 0. Vector fields $u \in H_0(\mathrm{curl};\Omega)$ satisfy the boundary condition $\nu \times u = 0$ on $\partial\Omega$ in the weak sense.

Next we discuss the Helmholtz decomposition. The space

$$\mathcal{V}_0 = \left\{ v \in H_0(\mathrm{curl};\Omega) : \int_\Omega \langle v, \phi\rangle\, dx = 0 \text{ for every } \phi \in \mathcal{C}_0^\infty(\Omega,\mathbb{R}^3) \text{ with } \nabla\times\phi = 0 \right\}$$

consists of vector fields $v \in H_0(\mathrm{curl};\Omega)$ such that v is divergence-free in the distributional sense. The space

$$\mathcal{W}_0 = \left\{ w \in H_0(\mathrm{curl};\Omega) : \int_\Omega \langle w, \nabla\times\phi\rangle = 0 \text{ for all } \phi \in \mathcal{C}_0^\infty(\Omega,\mathbb{R}^3) \right\}$$

consists of curl-free vector fields in $H_0(\mathrm{curl};\Omega)$, in the distributional sense. Since for every $\phi \in \mathcal{C}_0^\infty(\Omega;\mathbb{R}^3)$ the linear map

$$u \mapsto \int_\Omega \langle u, \nabla\times\varphi\rangle dx$$

is continuous on $H_0(\mathrm{curl};\Omega)$, the space $\mathcal{W}_0$ is a closed complement of $\mathcal{V}_0$ in $H_{V,0}(\mathrm{curl};\Omega)$, hence there is a Helmholtz type decomposition

$$H_0(\mathrm{curl};\Omega) = \mathcal{V}_0 \oplus \mathcal{W}_0. \tag{4.5}$$

Therefore any $u \in H_0(\mathrm{curl};\Omega)$ can be decomposed as $u = v + w$ with $v \in \mathcal{V}_0$ and $w \in \mathcal{W}_0$, where v is divergence-free and w is curl-free.

Lemma 20. *The curl-curl source eigenvalue problem*

$$\begin{cases} \nabla\times(\nabla\times v) = \lambda v & \text{in } \Omega, \\ \nu\times v = 0 & \text{on } \partial\Omega \\ v \in \mathcal{V}_0 \end{cases} \tag{4.6}$$

has a discrete sequence $0 < \lambda_1 < \lambda_2 < \lambda_3 < \dots$ *of Maxwell eigenvalues with eigenspaces of finite multiplicity. The quadratic form* $Q : \mathcal{V}_0 \to \mathbb{R}$ *defined by*

$$Q(v) := \int_\Omega |\nabla\times v|^2 + \lambda|v|^2\, dx, \tag{4.7}$$

is positive definite on the sum $\mathcal{V}^+ \subset \mathcal{V}_0$ *of the eigenspaces associated to the eigenvalues* $\lambda_k > -\lambda$ *and it is negative semi-definite on the sum* $\widetilde{\mathcal{V}} \subset \mathcal{V}_0$ *of the eigenspaces associated to the eigenvalues* $\lambda_k \le -\lambda$.

Proof. Since the space $\mathcal{V}_0$ embeds compactly into $L^2(\Omega,\mathbb{R}^3)$. The lemma follows immediately by classical arguments. [14, 22]. □

4.3 Linking geometry of the curl-curl problem

Since $\mathcal{V}_0$ embeds continuously in $L^p(\Omega, \mathbb{R}^3)$, we observe that $\mathcal{V} := \mathcal{V}_0 \cap X = \mathcal{V}_0$ and we obtain the Helmholz decomposition

$$X = \mathcal{V} \oplus \mathcal{W}$$

with $\mathcal{W} := \mathcal{W}_0 \cap X$. We introduce an equivalent norm in X

$$\|u + w\| := \left(|\nabla \times v|_2^2 + |w|_p^2\right)^{1/2}.$$

Suppose that $\lambda > -\lambda_1$. Due to the infinite dimensional kernel $\mathcal{W}$, the functional J is unbounded from above and from below, even on subspaces of finite codimension and its critical points may have infinite Morse index. Therefore the problem has the strongly indefinite nature. If

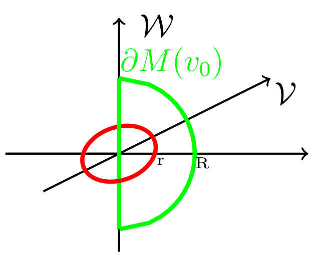
$$\tilde{\lambda} := \mu(\Omega)^{-\frac{p-2}{p}} p^{-\frac{2}{p}} \inf_{v \in \mathcal{V}\colon |v|_p = 1} \int_\Omega |\nabla \times v|^2 \, dx > 0,$$

then J has the linking geometry for $\lambda < \tilde{\lambda}$ in the spirit of Benci and Rabinowitz [4, 24]. E.g. if $\lambda \in (-\lambda_1, \tilde{\lambda})$, then there exist $v_0 \in \mathcal{V}$ and $R > r > 0$ such that

$$\sup_{\partial M(v_0)} J \leq 0 = J(0) < \inf_{\|v\| = r} J \tag{4.8}$$

where

$$\partial M(v_0) \;\; := \;\; \big\{E = tv_0 + w \in X : w \in \mathcal{W},\ (\|E\| = R, t \geq 0) \text{ or } (\|E\| \leq R, t = 0)\big\}.$$



However we cannot apply linking results e.g. [4, 6, 17, 19, 24] in order to get a Palais-Smale sequence, since J' is not (sequentially) weak-to-weak* continuous, i.e. the weak convergence $E_n \rightharpoonup E$ in X does not imply that $J'(E_n) \overset{*}{\rightharpoonup} J'(E)$ in X^*. Moreover, even if we find somehow a bounded Palais-Smale sequence $E_n \rightharpoonup E$ we do not know whether E is a critical point of J. This is caused, again, by the lack of weak-to-weak* continuity of J'.

4.4 Critical point theory – mountain pass approach

We recall an abstract setting from [9]. Let X be a reflexive Banach space with norm $\|\cdot\|$ and with a topological direct sum decomposition $X = X^+ \oplus \widetilde{X}$, where X^+ is a Hilbert space with a scalar product. For $u \in X$ we denote by $u^+ \in X^+$ and $\widetilde{u} \in \widetilde{X}$ the corresponding summands so that $u = u^+ + \widetilde{u}$. We may assume that $\langle u, u\rangle = \|u\|^2$ for any $u \in X^+$ and that $\|u\|^2 = \|u^+\|^2 + \|\widetilde{u}\|^2$. The topology $\mathcal{T}$ on X is defined as the product of the norm topology in X^+ and the weak topology in $\widetilde{X}$. Thus $u_n \xrightarrow{\mathcal{T}} u$ is equivalent to $u_n^+ \to u^+$ and $\widetilde{u}_n \rightharpoonup \widetilde{u}$.

Let J be a functional on X of the form

$$J(u) = \frac{1}{2}\|u^+\|^2 - I(u) \quad \text{for } u = u^+ + \widetilde{u} \in X^+ \oplus \widetilde{X}. \tag{4.9}$$

The set

$$\mathcal{M} := \{u \in X : J'(u)|_{\widetilde{X}} = 0\} = \{u \in X : I'(u)|_{\widetilde{X}} = 0\}, \tag{4.10}$$

obviously contains all critical points of J. Suppose the following assumptions hold.

(I1) $I \in \mathcal{C}^1(X, \mathbb{R})$ and $I(u) \geq I(0) = 0$ for any $u \in X$.
(I2) I is $\mathcal{T}$-sequentially lower semicontinuous: $u_n \xrightarrow{\mathcal{T}} u \quad \Longrightarrow \quad \liminf I(u_n) \geq I(u)$
(I3) If $u_n \xrightarrow{\mathcal{T}} u$ and $I(u_n) \to I(u)$ then $u_n \to u$.
(I4) $\|u^+\| + I(u) \to \infty$ as $\|u\| \to \infty$.
(I5) If $u \in \mathcal{M}$ then $I(u) < I(u + v)$ for every $v \in \widetilde{X} \setminus \{0\}$.

Clearly (I5) is satisfied for a strictly convex functional I. Observe that for any $u \in X^+$ we find $m(u) \in \mathcal{M}$ which is the unique global maximum of $\mathcal{J}|_{u+\widetilde{X}}$. Note that m need not be $\mathcal{C}^1$, and $\mathcal{M}$ need not be a differentiable manifold, because I' is only required to be continuous. In order to apply classical critical point theorems like the mountain pass theorem to $J \circ m$ we need the additional assumptions.

(I6) There exists $r > 0$ such that $a := \inf_{u \in X^+, \|u\| = r} J(u) > 0$.
(I7) $I(t_n u_n)/t_n^2 \to \infty$ if $t_n \to \infty$ and $u_n^+ \to u^+ \neq 0$ as $n \to \infty$.

Theorem 21 ([9]). *Suppose $J \in \mathcal{C}^1(X, \mathbb{R})$ satisfies (I1)-(I7). Then*

$$c_{\mathcal{M}} := \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} J(\gamma(t)) \geq a > 0,$$

where $\Gamma = \{\gamma \in \mathcal{C}^0([0,1], \mathcal{M}) : \gamma(0) = 0,\ \|\gamma(1)^+\| > r,\ \text{and } J(\gamma(1)) < a\}$. Moreover

a) J has a $(PS)_{c_{\mathcal{M}}}$-sequence (u_n) in $\mathcal{M}$, i.e. $J'(u_n) \to 0$ and $J(u_n) \to c$.

b) If, in addition, J satisfies the $(PS)^{\mathcal{T}}_{c_{\mathcal{M}}}$-condition in $\mathcal{M}$, i.e. every $(PS)_c$-sequence (u_n) for the unconstrained functional and such that $u_n \in \mathcal{M}$ has a subsequence which converges in the $\mathcal{T}$-topology, then $c_{\mathcal{M}}$ is achieved by a critical point of J.

c) If J satisfies the $(PS)^{\mathcal{T}}_c$-condition in $\mathcal{M}$ for every c and if J is even then it has an unbounded sequence of critical values.

Proof. Recall that for any $u \in X^+$ there is a unique $m(u) \in \mathcal{M}$ with $m(u)^+ = u$. We claim that:

(i) $m : X^+ \to \mathcal{M}$ is a homeomorphism with inverse $\mathcal{M} \ni u \mapsto u^+ \in X^+$.

(ii) $J \circ m : X^+ \to \mathbb{R}$ is $\mathcal{C}^1$.

(iii) $(J \circ m)'(u) = J'(m(u))|_{X^+} : X^+ \to \mathbb{R}$ for every $u \in X^+$.

(iv) $(u_n)_n \subset X^+$ is a Palais-Smale sequence for $J \circ m$ if, and only if, $(m(u_n))_n$ is a Palais-Smale sequence for J in $\mathcal{M}$.

(v) $u \in X^+$ is a critical point of $J \circ m$ if, and only if, $m(u)$ is a critical point of J.

(vi) If J is even, then so is $J \circ m$.

Now we prove these statements.
(i) Let $u_n \to u_0$ in X^+ and $m(u_n) = u_n + v_n$, where $v_n \in \widetilde{X}$ for all $n \geq 0$. In view of (I5) one has

$$I(m(u_n)) \leq I(u_n) \leq I(u_0) + 1 \tag{4.11}$$

for almost all n. Now (I4) implies that v_n is bounded, so we may assume that $v_n \rightharpoonup v_0$. As a consequence of (I2) and (I5) we deduce

$$I(m(u_0)) \leq I(u_0 + v_0) \leq \liminf I(m(u_n)) \leq \liminf I(u_n + (m(u_0) - u_0)) = I(m(u_0)).$$

Finally, using (I2) and (I3) we obtain $m(u_n) \to m(u_0) = u_0 + v_0$.
(ii) Let $u, v \in X^+$ and $h \in \mathbb{R}$. Let $m(u + hv) = u + hv + \widetilde{u}(h)$ for some $\widetilde{u}(h) \in \widetilde{X}$. Observe that by (I5) and by the mean value theorem

$$\begin{aligned} I(m(u+hv)) - I(m(u)) &\geq I(u + hv + \widetilde{u}(h)) - I(u + \widetilde{u}(h)) \\ &= I'(\theta_1(h))(hv) \end{aligned}$$

for some $\theta_1(h) \to u + \widetilde{u}(0)$ as $h \to 0$. Similarly we have

$$\begin{aligned} I(m(u+hv)) - I(m(u)) &\leq I(u + hv + \widetilde{u}(0)) - I(u + \widetilde{u}(0)) \\ &= I'(\theta_2(h))(hv) \end{aligned}$$

for some $\theta_2(h) \to u + \widetilde{u}(0)$ as $h \to 0$. Thus we obtain

$$(I \circ m)'(u)(v) = \lim_{h\to 0} \frac{I(m(u+hv)) - I(m(u))}{h} = I'(m(u))(v). \tag{4.12}$$

Using (i) it follows that $(I \circ m)'(u)$ is continuous, therefore $I \circ m$ and $J \circ m$ are of class $\mathcal{C}^1$ and (ii) holds.

Observe that (iii) follows from $(I \circ m)'(u) = I'(m(u))$ and from the form of J given in (4.9). Finally, (iv), (v) and (vi) are easy consequences of the definition of m.

Next we prove that $J \circ m$ has the classical mountain pass geometry. Assumption (I6) implies

$$J \circ m(u) \geq J(u) \geq a > 0 \text{ if } \|u\| = r. \tag{4.13}$$

In order to see for $0 \neq u \in X^+$ that

$$J \circ m(tu) = \frac{1}{2}\|m(tu)^+\|^2 - I(m(tu)) \to -\infty \text{ as } t \to \infty \tag{4.14}$$

write $m(tu) = tu + \widetilde{u}_t$ with $\widetilde{u}_t \in \widetilde{X}$, and set $u_t = u + \frac{1}{t}\widetilde{u}_t = \frac{1}{t}m(tu)$. Then

$$\frac{1}{t^2}I(m(tu)) = \frac{1}{t^2}I(tu_t) \to \infty \quad \text{as } t \to \infty$$

by (I7). The mountain pass condition (4.14) follows immediately. Setting

$$\Sigma := \{\sigma \in \mathcal{C}([0,1], X^+) : \sigma(0) = 0,\ \|\sigma(1)^+\| > r \text{ and } J \circ m(\sigma(1)) < 0\}$$

the mountain pass value for $J \circ m$ is given by:

$$c_{\mathcal{M}} = \inf_{\sigma\in\Sigma} \sup_{t\in[0,1]} J \circ m(\sigma(t)) \geq a > 0.$$

In view of the mountain pass theorem and using (iv), there exists a $(PS)_{c_{\mathcal{M}}}$-sequence $(u_n)_n$ for J in $\mathcal{M}$, which proves a).

In order to prove b) we consider a $(PS)_c$-sequence $(u_n)_n \subset X^+$ for $J \circ m$. Then $(m(u_n))_n$ is a Palais-Smale sequence for J in $\mathcal{M}$ by (iv), hence $m(u_n) \xrightarrow{\mathcal{T}} v$ after passing to a subsequence. This implies $u_n = m(u_n)^+ \to v^+$ and we have proved:

(vii) If J satisfies the $(PS)_c^{\mathcal{T}}$-condition in $\mathcal{M}$ for some c then $J \circ m$ satisfies the $(PS)_c$-condition.

Next observe that if J satisfies the $(PS)_{c_{\mathcal{M}}}^{\mathcal{T}}$-condition in $\mathcal{M}$ then $c_{\mathcal{M}}$ is achieved by a critical point $u \in X^+$ of $J \circ m$, hence $m(u) \in \mathcal{M}$ is a critical point of J with $J(m(u)) = c_{\mathcal{M}}$. This implies b).

c) follows from the classical symmetric mountain pass theorem. The condition (4.14) implies that for every finite-dimensional subspace $Y \subset X^+$ there exists $R = R(Y) > 0$ such that $J \circ m \leq 0$ on $Y \setminus B_R Y$. Therefore together with (4.13) and the Palais-Smale condition $J \circ m$ satisfies the hypotheses of [24, Theorem 9.12], hence it possesses an unbounded sequence of critical values. $\square$

4.5 Bound states solutions

Theorem 22. *Problem ($\mathcal{MP}2$) has infinitely many solutions for $-\lambda_1 < \lambda < 0$.*

Proof. We show that

a) I is of class $\mathcal{C}^1$, $I(u) \geq 0$ for any $u \in X$, and I is $\mathcal{T}$-sequentially lower semicontinuous.
b) If $u_n \xrightarrow{\mathcal{T}} u$ and $I(u_n) \to I(u)$ then $u_n \to u$.
c) There is $r > 0$ such that $0 < \inf_{\substack{v \in \mathcal{V} \\ \|v\|_{\mathcal{V}} = r}} J(v)$.
d) $\|v_n\| + I(v_n + w_n) \to \infty$ as $\|v_n + w_n\| \to \infty$.
e) $I(t_n(v_n + w_n))/t_n^2 \to \infty$ if $t_n \to \infty$ and $v_n \to v_0 \neq 0$ as $n \to \infty$.
f) J satisfies the $(PS)_c^{\mathcal{T}}$ condition on $\mathcal{M}$.

a) Note that $I(v + w) \geq 0$ for any $v \in \mathcal{V}$, $w \in \mathcal{W}$. The convexity condition implies that I is $\mathcal{T}$-sequentially lower semicontinuous, and I is of class $\mathcal{C}^1$. Thus we obtain a).

b) Consider $u_n, u \in X$ such that $u_n \xrightarrow{\mathcal{T}} u$ and $I(u_n) \to I(u)$. Writing $u_n = v_n + w_n$, $u = v + w$ with $v_n, v \in \mathcal{V}$, $w_n, w \in \mathcal{W}$ we have $v_n \to v$ in $\mathcal{V}$, $w_n \rightharpoonup w$ in $\mathcal{W}$. Hence

$$\frac{1}{2}\int_\Omega |v_n^+ + w_n|^2\,dx + \int_\Omega F(x, v_n + w_n)\,dx \\ \to \frac{1}{2}\int_\Omega |v^+ + w|^2\,dx + \int_\Omega F(x, v + w)\,dx.$$

By the weakly sequentially lower semicontinuity

$$|v_n^+ + w_n|_2 \to |v^+ + w|_2. \tag{4.15}$$

Since $v_n^+ + w_n \rightharpoonup v^+ + w$ in $L^p(\Omega, \mathbb{R}^3)$ then, up to a subsequence, $v_n^+ + w_n \rightharpoonup v^+ + w$ in $L^2(\Omega, \mathbb{R}^3)$, and by (4.15) we have $v_n^+ + w_n \to v^+ + w$ in $L^2(\Omega, \mathbb{R}^3)$. Hence

$$u_n = v_n + w_n \to u = v + w \text{ a.e. on } \Omega.$$

Taking $F(x, u) = \frac{1}{p}|u|^p$, we finally observe that

$$\int_\Omega F(x, u_n) - F(x, u_n - u)\,dx \\ = \int_\Omega \int_0^1 \frac{d}{dt} F(x, u_n + (t-1)u)\,dtdx \\ = \int_0^1 \int_\Omega \langle f(x, u_n + (t-1)u), u \rangle\,dxdt.$$

Since $f(x, u_n + (t-1)u) \to f(x, tu)$ a.e. on Ω Vitali's convergence theorem yields

$$\int_\Omega F(x, u_n) - F(x, u_n - u)\, dx$$
$$\to \int_0^1 \int_\Omega \langle f(x, tu), u\rangle\, dx dt = \int_\Omega F(x, u)\, dx$$

as $n \to \infty$. Moreover, since $\int_\Omega F(x, u_n) \to \int_\Omega F(x, u)\, dx$ there holds

$$\int_\Omega F(x, u_n - u)\, dx \to 0, \tag{4.16}$$

hence $|u_n - u|_p \to 0$, and $w_n \to w$ in $L^p(\Omega, \mathbb{R}^3)$. This shows b).

c) In order to prove c) we deduce for $v \in \mathcal{V}$

$$\begin{aligned} J(v) &= \frac{1}{2}Q(v) - \int_\Omega F(x, v)\, dx \geq \frac{\delta}{2}\|v\|^2 - \frac{1}{p}|v|_p^p \\ &\geq \frac{\delta}{2}\|v\|_{\mathcal{V}}^2 - C_1\|v\|_{\mathcal{V}}^p \end{aligned}$$

for some constant $\delta, C_1 > 0$ which proves c).

d) Consider a sequence $(v_n + w_n)$ in X such that $\|v_n + w_n\| \to \infty$ as $n \to \infty$ and $(\|v_n\| + I(v_n + w_n))$ is bounded. Then $|w_n|_p \to \infty$ and taking into account

$$I(v_n + w_n) \geq \frac{1}{p}|v_n + w_n|_p^p,$$

we get the boundedness of $(v_n + w_n)$ in $L^p(\Omega, \mathbb{R}^3)$, hence $|w_n|_p$ is bounded and we get a contradiction.

e) Consider sequences $t_n \to \infty$ and $v_n \in \mathcal{V}$, $w_n \in \mathcal{W}$ such that $v_n \to v_0 \neq 0$ as $n \to \infty$. Note that

$$I(t_n(v_n + w_n))/t_n^2 \geq \frac{1}{p}t_n^{p-2}|v_n + w_n|_p^p,$$

and if $\|v_n + w_n\| \to \infty$ as $n \to \infty$ then $|v_n + w_n|_p \to \infty$, hence

$$I(t_n(v_n + w_n))/t_n^2 \to \infty \tag{4.17}$$

and we are done. Now suppose $(\|v_n + w_n\|)$ is bounded, hence $(|v_n + w_n|_p)$ is bounded. If $|v_n + w_n|_p \to 0$ then $|v_n + w_n|_2 \to 0$ which implies $v_n \to 0$ in $L^2(\Omega, \mathbb{R}^3)$ contradicting $v_0 \neq 0$. Therefore $t_n^{p-2}|v_n + w_n|_p \to \infty$ as $n \to \infty$ and again (4.17) holds.

f) Let $u_n = v_n + w_n \in \mathcal{M}$ be a $(PS)_c$-sequence for J. We need to show that $u_n \xrightarrow{\mathcal{T}} u_0$ in X for some $u_0 \in X$ along a subsequence. Observe that

$$J(u_n) - \frac{1}{2}J'(u_n)(u_n) = \left(\frac{1}{2} - \frac{1}{p}\right)\int_\Omega |v_n + w_n|^p\, dx$$

and

$$\begin{aligned} & J(u_n) - \frac{1}{p} J'(u_n)(u_n) \\ & \geq \left(\frac{1}{2} - \frac{1}{p}\right) \left(\int_\Omega |\nabla \times v_n|^2 + \lambda |v_n + w_n|^2 \, dx\right) \end{aligned}$$

The above inequalities and the Hölder inequality imply that

$$2J(u_n) - \left(\frac{1}{2} + \frac{1}{p}\right) J'(u_n)(u_n) \geq \left(\frac{1}{2} - \frac{1}{p}\right) Q(v_n) + \left(\frac{1}{2} - \frac{1}{p}\right) |v_n + w_n|_p^p$$

Therefore $\|v_n\|$ and $|v_n + w_n|_p$ must be bounded. Since $u_n = v_n + w_n$ is bounded in X and we may assume, up to a subsequence,

$$v_n \rightharpoonup v_0 \text{ in } \mathcal{V}, \; v_n \to v_0 \text{ in } L^p(\Omega, \mathbb{R}^3) \text{ and } w_n \rightharpoonup w_0 \text{ in } \mathcal{W}$$

for some $(v_0, w_0) \in \mathcal{V} \times \mathcal{W}$. Note that

$$\begin{aligned} J'(v_n, w_n)[v_n - v_0, 0] &= \|v_n - v_0\|^2 + \int_\Omega \langle \nabla \times v_0, \nabla \times (v_n - v_0) \rangle \, dx \\ &\quad + \lambda \int_\Omega \langle v_n + w_n, v_n - v_0 \rangle \, dx - \int_\Omega \langle |v_n + w_n|^{p-2}(v_n + w_n), v_n - v_0 \rangle \, dx. \end{aligned}$$

Since (v_n) is bounded in $\mathcal{V}$, $v_n \to v_0$ in $L^2(\Omega, \mathbb{R}^3)$ and $(|v_n + w_n|^{p-2}(v_n + w_n))$ is bounded in $L^{\frac{p}{p-1}}(\Omega, \mathbb{R}^3)$ we deduce $\|v_n - v_0\| \to 0$. □

4.6 Critical point theory – Nehari manifold approach

We use the same notations as in Section 4.4. Now we consider the set

$$\mathcal{N} := \{u \in X \setminus \widetilde{X} : J'(u)|_{\mathbb{R}u \oplus \widetilde{X}} = 0\} = \{u \in \mathcal{M} \setminus \widetilde{X} : J'(u)[u] = 0\} \subset \mathcal{M}. \tag{4.18}$$

and we require the following condition on I:

(I8) $\frac{t^2-1}{2} I'(u)[u] + tI'(u)[v] + I(u) - I(tu + v) < 0$ for every $u \in \mathcal{N}$, $t \geq 0$, $v \in \widetilde{X}$ such that $u \neq tu + v$.

It implies that for any $u^+ \in X^+ \setminus \{0\}$ the functional J has a unique critical point $n(u^+)$ on the half space $\mathbb{R}^+ u^+ + \widetilde{X}$. Moreover $n(u^+)$ is the global maximum of J on the half space $\mathbb{R}^+ u^+ + \widetilde{X}$. Then the map

$$n : SX^+ = \{u^+ \in X^+ : \|u^+\| = 1\} \to \mathcal{N}$$

is a homeomorphism and the set $\mathcal{N}$ is a topological manifold, the Nehari-Pankov manifold, and it is enough to look for critical point of $J \circ n$.

We also consider a slightly stronger condition

(I8)$_{\mathcal{M}}$ $\frac{t^2-1}{2} I'(u)[u] + tI'(u)[v] + I(u) - I(tu+v) < 0$ for every $u \in \mathcal{M}$, $t \geq 0$, $v \in \widetilde{X}$ such that $u \neq tu + v$.

Lemma 23. *(I8)$_{\mathcal{M}}$ implies (I5).*

Proof. Take any $u \in \mathcal{M}$ and take $t = 1$ in (I8). Observe that

$$I(u) - I(u+v) = I'(u)[v] + I(u) - I(u+v) < 0$$

for all $v \neq 0$, hence (I5) is satisfied.

□

Theorem 24 ([8])**.** *Let $J \in \mathcal{C}^1(X, \mathbb{R})$ satisfy (I1)-(I4), (I6)-(I8), set $c_{\mathcal{N}} = \inf_{\mathcal{N}} J$ and let J be coercive on $\mathcal{N}$, i.e. $J(u) \to \infty$ as $\|u\| \to \infty$ and $u \in \mathcal{N}$. Then the following holds:*

a) $c_{\mathcal{N}} \geq a > 0$ and J has a $(PS)_{c_{\mathcal{N}}}$-sequence in $\mathcal{N}$.
b) If J satisfies the $(PS)^{\mathcal{T}}_{c_{\mathcal{N}}}$-condition in $\mathcal{N}$ then $c_{\mathcal{N}}$ is achieved by a critical point of J.
c) If J satisfies the $(PS)^{\mathcal{T}}_{c}$-condition in $\mathcal{N}$ for every c and if J is even then it has an unbounded sequence of critical values.
d) If in addition $(I5)$ holds and $c_{\mathcal{M}}$ is attained by a critical point, then $c_{\mathcal{M}} = c_{\mathcal{N}}$.

Proof. As in Proof of Theorem 17, cf. [29, Section 4], one proves that

(i) $n := \widehat{n}|_{S^+} : S^+ \to \mathcal{N}$ is a homeomorphism with inverse $\mathcal{N} \to S^+$, $u \mapsto u^+/\|u^+\|$.
(ii) $J \circ n : SX^+ \to \mathbb{R}$ is $\mathcal{C}^1$.
(iii) $(J \circ n)'(u) = \|n(u)^+\| \cdot J'(u)|_{T_u SX^+} : T_u SX^+ \to \mathbb{R}$ for every $u \in S^+$.
(iv) $(u_n)_n \subset S^+$ is a Palais-Smale sequence for $J \circ n$ if, and only if, $(m(u_n))_n$ is a Palais-Smale sequence for J in $\mathcal{N}$.
(v) $u \in S^+$ is a critical point of $J \circ n$ if, and only if, $n(u)$ is a critical point of J.
(vi) If J is even, then so is $J \circ n$.

The existence of a $(PS)_{c_{\mathcal{N}}}$-sequence (u_n) for J in $\mathcal{N}$ follows from (ii) and (iv) because $c_{\mathcal{N}} = \inf_{\mathcal{N}} J \circ n$. We claim that

(vii) If J satisfies the $(PS)_c^{\mathcal{T}}$-condition in $\mathcal{N}$ for some $c > 0$ then $J \circ n$ satisfies the $(PS)_c$-condition.

In order to see this consider a $(PS)_c$-sequence (u_n) for $J \circ n$. Then $(n(u_n))_n$ is a Palais-Smale sequence for J in $\mathcal{N}$ by (iv), hence $n(u_n) \xrightarrow{\mathcal{T}} v$ after passing to a subsequence. This implies $n(u_n)^+ \to v^+$ and moreover, using (I2),

$$0 < c = \lim_{n\to\infty} J(n(u_n)) \le J(v).$$

Now (I1) implies $v^+ \ne 0$, hence $n(u_n)^+ \ne 0$ for n large. From the continuity of n we deduce

$$n(u_n) = n(n(u_n)^+/\|n(u_n)^+\|) \to n(v^+/\|v^+\|),$$

and therefore $v = n(v^+/\|v^+\|) \in \mathcal{N}$ and $n(u_n) \to v$. It follows that

$$u_n = n(u_n)^+/\|n(u_n)^+\| \to v^+/\|v^+\|.$$

This proves (vii).
Next observe that if J satisfies the $(PS)_{c_0}^{\mathcal{T}}$-condition in $\mathcal{N}$ then $c_{\mathcal{N}}$ is achieved by a critical point $u \in S^+$ of $J \circ n$, hence $n(u) \in \mathcal{N}$ is a critical point of J with $J(m(u)) = c_{\mathcal{N}}$. This proves b).
Finally c) follows from standard Ljusternik-Schnirelman theory. Under the conditions of c) the functional $J \circ n$ is even, bounded below, and satisfies the Palais-Smale condition. Hence it has an unbounded sequence of critical values, and so does J by (v).
It remains to prove d), so we assume that (I5) holds. Given $u \in \mathcal{N}$ by (4.14) there exists $t_0 > 0$ such that $J(m(t_0u^+)) < 0$. Therefore the path $\gamma(t) = m(tt_0u^+)$, $t \in [0,1]$, lies in Γ. Since u is the unique maximum of J on $\mathbb{R}^+u + \widetilde{X}$ there holds $J(\gamma(t)) \le J(u)$, and therefore $c_{\mathcal{M}} \le c_{\mathcal{N}}$. Since $c_{\mathcal{M}}$ is attained by a critical point, we get the reverse inequality. □

4.7 Ground state solution

Theorem 25. *Problem ($\mathcal{MP}2$) has a ground state solution for $-\lambda_1 < \lambda < 0$, which is a least energy solution.*

Proof. $(I8)_{\mathcal{M}}$. Let $u \in \mathcal{V}$, $t \ge 0$, $\psi \in \mathcal{W}$ satisfy $u \ne tu + \psi$. We need to show that

$$\begin{aligned} &I'(u)\left[\frac{t^2-1}{2}u + t\psi\right] + I(u) - I(tu+\psi) \\ &= \frac{\lambda}{2}\int_\Omega |\psi|^2\,dx + \int_\Omega \varphi(t)\,dx < 0 \end{aligned} \tag{4.19}$$

where

$$\varphi(t) = \langle |u|^{p-2}u, \frac{t^2-1}{2}u + t\psi\rangle + \frac{1}{p}|u|^p - \frac{1}{p}|tu+\psi|^p.$$

Assume that $u(x) \neq tu(x)+\psi(x)$ and we easy show that $\varphi(0) < 0$ and $\varphi(t) \to -\infty$ as $n \to \infty$. Hence φ attains a global maximum at some $t_0 \geq 0$. We may assume that $t_0 > 0$ and we get $\varphi'(t_0) = 0$. Hence

$$|u|^{p-2}\langle t_0 u + \psi, u\rangle = |t_0 u + \psi|^{p-2}\langle t_0 u + \psi, u\rangle.$$

If $|u|^{p-2}\langle t_0 u + \psi, u\rangle = 0$, then

$$\begin{aligned}\varphi(t_0) &= \varphi(t_0) - t_0|u|^{p-2}\langle t_0 u + \psi, u\rangle \\ &= \langle |u|^{p-2}u, \frac{-t_0^2-1}{2}u\rangle + \frac{1}{p}|u|^p - \frac{1}{p}|t_0 u + \psi|^p = -\Big(\frac{1}{2}-\frac{1}{p}\Big)|u|^p - \frac{1}{p}|t_0 u+\psi|^p < 0.\end{aligned}$$

If $|u|^{p-2}\langle t_0 u + \psi, u\rangle \neq 0$, then $|u| = |t_0 u + \psi|$ and

$$\begin{aligned}\varphi(t_0) &= \langle |u|^{p-2}u, \frac{t_0^2-1}{2}u + t_0\psi\rangle \\ &\leq \langle |u|^{p-2}u, \frac{t_0^2-1}{2}ut_0\psi\rangle + \frac{(|u|^p)^2 - \big(|u|^{p-2}\langle u, t_0 u + \psi\rangle\big)^2}{2|u|^p} \\ &< -\frac{\big(|u|^{p-2}\langle u, \psi\rangle\big)^2}{2|u|^p} \leq 0.\end{aligned}$$

Hence $(\mathrm{I8})_{\mathcal{M}}$ is satisfied.
Coercivity. Suppose that $v_n + w_n \in \mathcal{N}$ and $J(v_n + w_n) \to c$. Similarly as in proof of Theorem 22 f) we estimate

$$c + o(1) = 2J(u_n) - \left(\frac{1}{2}+\frac{1}{p}\right)J'(u_n)(u_n) \geq \left(\frac{1}{2}-\frac{1}{p}\right)Q(v_n) + \left(\frac{1}{2}-\frac{1}{p}\right)|v_n + w_n|_p^p$$

and infer that $v_n + w_n$ is bounded in X.
Now the proof follows from Theorem 21. □

4.8 General curl-curl problem – remarks

In [8–10] the following nonautonomous problem has been considered.

$$(\mathcal{MP}) \qquad \nabla \times \big(\mu(x)^{-1}\nabla \times u\big) - V(x)u = f(x,u) \qquad \text{in } \Omega$$

together with the metallic boundary condition $(\mathcal{BC})$ and Theorem 21 as well as Theorem 24 have been applied. For instance we are able to deal with all $\lambda \leq 0$ in $(\mathcal{MP}2)$.

Question 26. *Does* $(\mathcal{MP}2)$ *has any nontrivial solution for* $\lambda > 0$*?*

4.9 Cylindrical symmetry

We set $G = \mathcal{O}(2) \times \{1\} \subset \mathcal{O}(3)$ and require:

(S) Ω is invariant with respect to G.

The symmetry condition (S) allows to adapt the approach from [11] to our setting. Since Ω is invariant under G we can define an action of $g \in G$ on $v \in \mathcal{V}$ and on $w \in \mathcal{W}$ as follows:

$$(g * v)(x) := g \cdot v(g^{-1}x) \qquad \text{and} \qquad (g * w)(x) := g \cdot w(g^{-1}x).$$

It is not difficult to check that this defines an isometric linear (left) action of G on $X = \mathcal{V} \oplus \mathcal{W}$. In particular there holds $gv \in \mathcal{V}$ and $\int_\Omega |\nabla \times (g*v)|^2 dx = \int_\Omega |\nabla \times v|^2 dx$. Similarly, $g * w \in \mathcal{W}$, and $|\nabla(g*w)|_p = |\nabla w|_p$. Moreover, as a consequence of (S), J is invariant with respect to this action: $J(g*v + g*w) = J(v+w)$. Let $X^G = \mathcal{V}^G \oplus \mathcal{W}^G$ be the fixed point set of this action, so $\mathcal{V}^G$ consists of all G-equivariant vector fields $v \in \mathcal{V}$, and $\mathcal{W}^G$ consists of all G-invariant functions $w \in \mathcal{W}$. By the principle of symmetric criticality, a critical point of the constrained functional $J|_{X^G}$ is a critical point of J.

Next we decompose any $v \in \mathcal{V}^G$ as $v = v_\rho + v_\tau + v_\zeta$ with:

$$v_\rho(x) = \beta(r, x_3) \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}, \quad v_\tau(x) = \alpha(r, x_3) \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}, \quad v_\zeta(x) = \gamma(r, x_3) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

where $r = \sqrt{x_1^2 + x_2^2}$. That the coefficient functions α, β, γ depend only on (r, x_3) is an immediate consequence of the G-equivariance of v, i. e. $v(gx) = g \cdot v(x)$. As in [11, Lemma 1] one sees that $\nabla v_\rho, \nabla v_\tau, \nabla v_\zeta \in L^2(\Omega; \mathbb{R}^3)$. Clearly div $v_\tau = 0$, hence $v_\tau, v_\rho + v_\zeta \in \mathcal{V}$. Therefore the map

$$S : \mathcal{V}^G \to \mathcal{V}^G, \quad S(v_\rho + v_\tau + v_\zeta) := -v_\rho + v_\tau - v_\zeta$$

is well defined. A direct computation shows that

$$\langle \nabla \times v_\tau(x), \nabla \times v_\rho(x) \rangle = 0 = \langle \nabla \times v_\tau(x), \nabla \times v_\zeta(x) \rangle \tag{4.20}$$

and

$$\langle \nabla v_\tau(x), \nabla v_\rho(x) \rangle = 0 = \langle \nabla v_\tau(x), \nabla v_\zeta(x) \rangle \tag{4.21}$$

(4.20) implies that S is a linear isometry, and so is

$$T : X^G = \mathcal{V}^G \oplus \mathcal{W}^G \to X^G, \quad T(v + w) := Sv - w.$$

Clearly we have $T^2 = \mathrm{id}$, and

$$(X^G)^T := \{v \in \mathcal{V}^G : Sv = v\} = \{v \in \mathcal{V}^G : v = v_\tau\}.$$

As a consequence of (4.20), (4.21), and hypothesis (S), J is invariant under this action:

$$J(Tu) = J(u) \quad \text{for all } u = v + w \in X^G.$$

Applying the principle of symmetric criticality once more we see that it suffices to find critical points of $J|_{(X^G)^T}$.

The above discussion shows that we only need to find critical points of the functional

$$J_Y : Y := \{v \in \mathcal{V} : Sv = v\} \to \mathbb{R}$$

defined by

$$\begin{aligned} J_Y(v) = J(v) &= \frac{1}{2}\int_\Omega |\nabla \times v|^2\, dx + \frac{\lambda}{2}\int_\Omega |v|^2\, dx - \int_\Omega |v|^p\, dx \\ &= \frac{1}{2}\|v\|^2 + \frac{\lambda}{2}\int_\Omega |v|^2\, dx - \frac{1}{p}\int_\Omega |v|^p\, dx. \end{aligned}$$

Here we can apply standard critical point theory. Since F is even as a consequence of (S), the existence of an unbounded sequence of solutions follows from the fountain theorem in [5], see also [30].

The following result is proved in [8, 9].

Theorem 27. *There exist infinitely many solutions u_n of $(\mathcal{MP}2)$ of the form*

$$u(x) = \alpha(r, x_3)\begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}, \qquad r = \sqrt{x_1^2 + x_2^2}, \tag{4.22}$$

and such that $J(u_n) \to \infty$. Moreover $\operatorname{div}(u_n) = 0$ and $(u_n) \subset H_0^1(\Omega, \mathbb{R}^3)$.

Let us recall that in [20] problem $(\mathcal{MP}2)$ with the critical Sobolev exponent $p = 6$ has been considered and solutions of the form (5.1) have been obtained for some $\lambda \le 0$ such that $-\lambda$ is close to eigenvalues of the linear problem Lemma 20. Still it is an open problem to find solutions for $p = 6$ but without condition (S), see [20].

5 Maxwell equations in $\mathbb{R}^3$

In this section we only recall some results and open problems concerning the following problem

$$(\mathcal{MP}3) \qquad \nabla \times (\nabla \times u) + V(x)u = f(x,u) \qquad \text{in } \mathbb{R}^3,$$

Similarly as in section 4.9 we can prove the following result, see [7, 10, 11].

Theorem 28. *If $V \in L^\infty(\mathbb{R}^3)$ is $\mathbb{Z}^3$-periodic and $\operatorname{ess\,inf} V > 0$, there exist a solution $u \in H^1(\mathbb{R}^3, \mathbb{R}^3)$ of $(\mathcal{MP}3)$ with $f(x,u) = |u|^{p-2}u$, $2 < p < 6$, which is of the form*

$$u(x) = \alpha(r, x_3)\begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}, \qquad r = \sqrt{x_1^2 + x_2^2}. \tag{5.1}$$

Periodic and sign-changing potentials V have been considered in [7]. Problem $(\mathcal{MP}3)$ without cylindrical symmetry have been investigated in [18] for negative V and vanishing. From the physical point of view the most important case is when V is negative and bounded away from 0 and the nonlinearity is of the form $f(x,u) = |u|^{p-2}u$ with $p = 4$ modelling the Kerr effect. This problem has not been also studied so far, see also [10][Section 7] for the open problem presented in the manuscript and new ones.

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