Nonlinear Maxwell equations – a variational approach

Course at the Karlsruher Institut für Technologie

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We are interested in the propagation of electromagnetic waves is described by the Maxwell equations for the electric field $\mathcal{E}$, the electric displacement field $\mathcal{D}$, the magnetic field $\mathcal{H}$, and the magnetic induction $\mathcal{B}$. These are time-dependent vector fields in a domain $\Omega \subset \mathbb{R}^3$. Given the current intensity $\mathcal{J}$ and the scalar charge density ρ, the *Maxwell equations* in differential form are as follows:

$$\begin{cases} \partial_t \mathcal{B} + \nabla \times \mathcal{E} = 0 & \text{(Faraday's Law)} \\ \nabla \times \mathcal{H} = \mathcal{J} + \partial_t \mathcal{D} & \text{(Ampere's Law)} \\ \operatorname{div}(\mathcal{D}) = \rho & \text{(Gauss' Electric Law)} \\ \operatorname{div}(\mathcal{B}) = 0 & \text{(Gauss' Magnetic Law).} \end{cases}$$

These fields are related by constitutive equations determined by the material. The relation between the electric displacement field and the electric field is given by $\mathcal{D} = \varepsilon\mathcal{E} + \mathcal{P}_{NL}(x, \mathcal{E})$ where $\varepsilon = \varepsilon(x) \in \mathbb{R}^{3\times 3}$ is the (linear) permittivity tensor of the material, and $\mathcal{P}_{NL}$ is the nonlinear part of the polarization. The relation between magnetic field and magnetic induction is $\mathcal{B} = \mu\mathcal{H} - \mathcal{M}$ where $\mu = \mu(x) \in \mathbb{R}^{3\times 3}$ denotes the magnetic permeability tensor and $\mathcal{M}$ the magnetization of the material. In a linear medium one has $\mathcal{P}_{NL} = 0$ leading to the linear Maxwell equations.

Suppose there are no currents, charges nor magnetization, i.e. $\mathcal{J} = 0$, $\rho = 0$, $\mathcal{M} = 0$. Then multiplying Faraday's law with μ^{-1}, taking the curl and using the constitutive relations and Ampere's law leads to the *nonlinear electromagnetic wave equation* of the form

$$\nabla \times (\mu(x)^{-1} \nabla \times \mathcal{E}) + \varepsilon(x)\partial_t^2 \mathcal{E} + \partial_t^2 \mathcal{P}_{NL}(x, \mathcal{E}) = 0 \tag{1}$$

for the electric field $\mathcal{E}$. Solving this one obtains $\mathcal{D} = \varepsilon\mathcal{E} + \mathcal{P}_{NL}(x, \mathcal{E})$ by the constitutive relation and $\mathcal{B}$ by time integrating Faraday's law. Finally $\mathcal{H} = \mu^{-1}\mathcal{B}$ is also determined by the constitutive relation.

Equation (1) is particularly challenging and in the literature there are several simplifications relying on approximation of the nonlinear electromagnetic wave equation. The most prominent one is the scalar or vector nonlinear Schrödinger equation. In order to justify this approximation one assumes that the term $\nabla(\operatorname{div}(\mathcal{E}))$ in $\nabla \times (\nabla \times \mathcal{E}) = \nabla(\operatorname{div}(\mathcal{E})) - \Delta\mathcal{E}$ is negligible and can be dropped, and that one can use the so-called *slowly varying envelope* *approximation.* However, this approach may produce non-physical solutions and *our goal* is to find *exact* solutions of the Maxwell equations and develop analytical tools which allow to look for *time-harmonic fields* $\mathcal{E}$ of the form

$$\mathcal{E}(x,t) = u(x)\cos(\omega t) \quad \text{for } x \in \Omega \text{ and } t \in \mathbb{R}$$

with frequency $\omega > 0$. Suppose that the nonlinear polarization is of the form

$$\mathcal{P}_{NL}(x,\mathcal{E}) = \chi\big(x, |u(x)|^2\big)\mathcal{E}$$

i.e. the scalar susceptibility χ depends only on the intensity of $\mathcal{E}$. Then (1) reduces to the *curl-curl* equation which is the main subject of our research

$$\nabla \times \big(\mu(x)^{-1}\nabla \times u\big) - V(x)u = f(x,u) \qquad \text{in } \Omega, \tag{2}$$

where $f(x,u) := \chi\big(x,|u|^2\big)u$ and $V(x) = \omega^2\varepsilon(x) \in \mathbb{R}^{3\times 3}$. Probably the most common type of nonlinearity in the physics and engineering literature is the *Kerr nonlinearity* $f(x,u) = \chi^{(3)}(x)|u|^2u$. Other examples for f that appear in applications are nonlinearities with *saturation* like $f(x,u) = \chi^{(3)}(x)\frac{|u|^2}{1+|u|^2}u$. The problem (2) has a variational nature, i.e. weak solutions correspond to critical points of the functional associated with (2). The problem is strongly indefinite and during the lectures we will develop variational tools which allow to find ground state and bound state solutions.

Plan of the lectures

- Nonlinear Dirichlet problem on a bounded domain. Mountain Pass Theorem and Nehari manifold approach.
- General conditions imposed on the nonlinear term. Mountain Pass Theorem vs. Nehari manifold approach.
- Functional and variational setting for the curl-curl equation on a bounded domain.
- The role of the cylindrical symmetry in the curl-curl problems.
- Generalized Nehari manifold approach for strongly indefinite problems. Critical point theory I.
- Generalized Nehari manifold approach for strongly indefinite problems. Critical point theory II.
- Ground state solutions and the multiplicity of bound state solutions.
- Curl-curl equations in $\mathbb{R}^3$ and the lack of compactness.
- Recent results and the list of open problems.