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Calculus of Variations & Applications to PDEs
WS 09/10 – Handout on functional analysis

Definition FA.1 (Banach space) *A pair $(X, \|\cdot\|)$ is called a normed space if X is a vector-space over the field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) and*

(i) $\|x\| \geq 0$ *for all* $x \in X$ *and* $\|x\| = 0 \Leftrightarrow x = 0$,

(ii) $\|\lambda x\| = |\lambda|\|x\|$ *for all* $x \in X$, $\lambda \in \mathbb{K}$,

(iii) $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in \mathbb{K}$.

For a sequence $(x_n)_{n\in\mathbb{N}}$ *in* X *and an element* $x \in X$ *we say*

$$x = \lim_{n\to\infty} x_n$$

if $\lim_{n\to\infty} \|x_n - x\| = 0$. x *is then called the limit of the sequence* $(x_n)_{n\in\mathbb{N}}$.

A sequence $(x_n)_{n\in\mathbb{N}}$ *is called a Cauchy-sequence, if for every* $\epsilon > 0$ *there exist* $N_0 = N_0(\epsilon)$ *such that* $n, m \geq N_0$ *implies* $\|x_n - x_m\| \leq \epsilon$.

The normed space $(X, \|\cdot\|)$ *is called a Banach-space if every Cauchy-sequence in* X *has a limit.*

Definition FA.2 (dual space) *Let* $(X, \|\cdot\|)$ *be a normed space over the field* $\mathbb{K}$. *A linear functional* $\phi : X \to \mathbb{K}$ *is called bounded if*

$$\|\phi\| := \sup_{x\neq 0} \frac{|\phi(x)|}{\|x\|} < \infty.$$

The set $X' = \{\phi : X \to \mathbb{K} \text{ linear, bounded}\}$ *is called the dual space of* X. *Together with the above* $\|\cdot\|$, *the dual space* X' *is a Banach-space.*

Theorem FA.3 (Hahn-Banach) *Let* $(X, \|\cdot\|)$ *be a normed space and let* $V \subset X$ *be a linear subspace. If* $\phi \in V'$ *then there exists* $\psi \in X'$ *sucht that* $\psi|_V = \phi$ *and* $\|\psi\| = \|\phi\|$. *In other words,* ϕ *can be extended to a bounded linear functional* ψ *on* X *with the same norm as* ϕ.

Definition FA.4 (second dual, reflexive spaces) *Let* $(X, \|\cdot\|)$ *be a normed space and* $X'' = (X')'$ *be its second dual. There is an injective map* $I : X \to X''$ *with* $\|I(x)\| = \|x\|$ *called* canonical injection *given by*

$$I : \begin{cases} X & \to & X'' \\ z & \to & I(z) \end{cases} \quad \text{where } I(z) \text{ is defined by } I(z)\phi := \phi(z)$$

The space X *is called* reflexive *if the map* I *is bijective.*

Examples: Hilbert-spaces and $L^p(X)$ for $1 < p < \infty$ are reflexive. $L^1(X)$ and $L^\infty(X)$ are in general not reflexive.

Definition FA.5 (separable spaces) *A normed space $(X, \|\cdot\|)$ is called separable if the exists a countable set M which is dense in X, i.e., $\overline{M} = X$.*

Examples: $C(\overline{\Omega})$ for $\Omega \subset \mathbb{R}^n$ compact and $L^p(\Omega)$ for $1 \le p < \infty$, $\Omega \subset \mathbb{R}^n$ measurable are separable. $L^\infty(\Omega)$ is in general not separable.

Definition FA.6 (compact sets) *Let $(X, \|\cdot\|)$ be a normed space and $A \subset X$. The set $A \subset X$ is called (sequentially-)compact if every sequence in A has a convergent subsequence with limit in A.*

Remark: In metric spaces sequential-compactness is equivalent to the following definition of compacntess: every open covering of A has a finite sub-covering.

Theorem FA.7 (Compactness and finite-dimensionality) *Let $(X, \|\cdot\|)$ be a normed space. Then $\overline{B_1(0)} = \{x \in X : \|x\| \le 1\}$ is compact if and only if X is finite-dimensional.*

Definition FA.8 (weak, weak-$*$ convergence) *Let $(X, \|\cdot\|)$ be a normed space with dual-space X'.*

(i) A sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to converge weakly to $x \in X$ if

$$\phi(x_n) \to \phi(x) \text{ as } n \to \infty \text{ for all } \phi \in X'. \text{ Notation: } x_n \rightharpoonup x \text{ as } n \to \infty.$$

(ii) A sequence $(\phi_n)_{n\in\mathbb{N}}$ in X' is said to converge weak-$*$ to $\phi \in X'$ if

$$\phi_n(x) \to \phi(x) \text{ as } n \to \infty \text{ for all } x \in X. \text{ Notation: } \phi_n \rightharpoonup^* \phi \text{ as } n \to \infty.$$

Definition FA.9 (weak/weak-$*$ sequential compactness) *Let $(X, \|\cdot\|)$ be a normed space with dualspace X'.*

(i) A set $M \subset X$ is called weakly sequentially compact, if every sequence in M has a weakly convergent subsequence with limit in M.

(ii) A set $M \subset X'$ is called weak-$*$ sequentially compact, if every sequence in M has a weak-$*$ convergent subsequence with limit in M.

Theorem FA.10 (Banach-Alaoglu)

(i) Let X be separable. Then $\overline{B_1(0)} \subset X'$ is weak-$*$ sequentially compact.

(ii) Let X be reflexive. Then $\overline{B_1(0)} \subset X$ is weakly sequentially compact.

Corollary FA.11

(i) Let X be separable and let $(\phi_n)_{n\in N}$ be a bounded sequence of functionals in X'. Then $(\phi_n)_{n\in\mathbb{N}}$ has a weak-$*$ convergent subsequence.

(ii) Let X be reflexive and let $(x_n)_{n\in N}$ be a bounded sequence in X. Then $(x_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence.