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Calculus of Variations & Applications to PDEs
WS 09/10 – Handout on the Divergence Theorem

Definition G.1 *An open, connected set $\Omega \subset \mathbb{R}^n$ is called a C^1-domain (a Lipschitz-domain) if for every $x_0 \in \partial\Omega$ there exists a radius $r > 0$ and a C^1-function (Lipschitz-function) $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that (up to rotation of the coordinate frame)*

$$\Omega \cap B_r(x_0) = \{x = (x', x_n) \in B_r(x_0) : x_n > \gamma(x')\}.$$

Note that as a consequence

$$\partial\Omega \cap B_r(x_0) = \{x = (x', x_n) \in B_r(x_0) : x_n = \gamma(x')\}.$$

For $x = (x', \gamma(x')) \in \partial\Omega \cap B_r(x_0)$ the vector

$$\nu(x) = \frac{(\nabla\gamma(x'), -1)}{\sqrt{1 + |\nabla\gamma(x')|^2}}$$

is called the exterior unit-normal of $\partial\Omega$ at x. In case of a Lipschitz-domain the exterior unit-normal ν exists a.e. with respect to the surface-measure on $\partial\Omega$.

Definition G.2 (Divergence theorem, Gauss theorem) *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^1-domain and let ν be the exterior unit-normal on $\partial\Omega$. Then*

$$\int_\Omega \frac{\partial f}{\partial x_i}\, dx = \oint_{\partial\Omega} f \nu_i \, d\sigma$$

for every function $f \in C^1(\overline{\Omega})$. The result also holds if Ω is a bounded Lipschitz-domain.

Often the divergence theorem appears in the following form

$$\int_\Omega \operatorname{div} F \, dx = \oint_{\partial\Omega} F \cdot \nu \, d\sigma$$

for a vectorfield $F : \overline{\Omega} \to \mathbb{R}^n$. The components of $F = (F_1, \ldots, F_n)$ have to satisfy $F_i \in C^1(\overline{\Omega})$ for $i = 1, \ldots, n$.

Lemma G.3 (Green's identities) *Let $u, v \in C^2(\overline{\Omega})$. Then*

$$\begin{aligned}
\int_\Omega \Delta u \, dx &= \oint_{\partial\Omega} \nabla u \cdot \nu \, d\sigma, \\
\int_\Omega \nabla u \cdot \nabla v \, dx &= -\int_\Omega u \Delta v \, dx + \oint_{\partial\Omega} u \nabla v \cdot \nu \, d\sigma, \\
\int_\Omega (u\Delta v - v\Delta u) \, dx &= \oint_{\partial\Omega} (u\nabla v - v\nabla u) \cdot \nu \, d\sigma.
\end{aligned}$$