Karlsruher Institut für Technologie
Institut für Analysis
Prof. Dr. Michael Plum
Prof. Dr. Wolfgang Reichel
Hans-Jürgen Freisinger

WS 2009/10

3rd Problem Sheet

Variational Methods and Applications to PDEs

Problem 5

Let X be a Banach space and $L : X \longrightarrow \mathbb{R}$ be a Gâteaux differentiable and convex functional.

a) Prove that $L[u] - L[v] \leq dL[u](u - v)$ for all $u, v \in X$.

b) Show that if $dL[u_0] = 0$ for $u_0 \in X$, then $L[u_0] = \inf_X L[u]$.

c) Find the minimizer of the functional

$$L[u] = \int_0^1 \big(u(x)^4 + e^x u(x)\big)\, dx$$

on the space $X = C[0, 1]$.

Problem 6

Let Ω be a bounded, open subset of $\mathbb{R}^n$. As in Problem 1, let $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function satisfying

$$|f(x, s)| \leq C(1 + |s|^{p-1}) \qquad \text{for all } (x, s) \in \Omega \times \mathbb{R}$$

with constants $C > 0$ and $p > 1$ and $F : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$F(x, t) = \int_0^t f(x, s) ds \qquad \text{for all } (x, t) \in \Omega \times \mathbb{R}.$$

Prove that the functional $L : L^p(\Omega) \longrightarrow \mathbb{R}$ given by

$$L[u] = \int_\Omega F(x, u(x)) dx, \qquad u \in L^p(\Omega).$$

is Gâteaux differentiable and determine its Gâteaux derivative.

To be discussed in the problem session on Tuesday, November 24, 2009.