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5^{th} Problem Sheet

Variational Methods and Applications to PDEs

Problem 9

Let $\Omega \neq \emptyset$ be a bounded and open subset of $\mathbb{R}^N$, $\lambda > 0$ and $L : H_0^1(\Omega) \longrightarrow \mathbb{R}$ be given by

$$L[u] = \int_\Omega \big(|\nabla u(x)|^2 + \lambda \cos(u(x)) \big)\, dx \qquad \text{for all } u \in H_0^1(\Omega).$$

Prove:

a) L is weakly lower semicontinuous and coercive.

b) L has a minimizer $u_0 \in H_0^1(\Omega)$.

c) There exists $\lambda_* > 0$ such that $u_0 = 0$ for all $0 < \lambda < \lambda_*$.

d) There exists $\lambda^* > 0$ such that $u_0 \neq 0$ for all $\lambda > \lambda^*$.

e) L is Fréchet-differentiable. Moreover, find the Euler-Lagrange equation for u_0.

Hint: For e), use Problem 7.

To be discussed in the problem session on Tuesday, December 8, 2009.