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6th Problem Sheet

Variational Methods and Applications to PDEs

Problem 10

a) Let X, Y, Z be normed spaces and $A : X \longrightarrow Y$, $B : Y \longrightarrow Z$ be linear operators. Show that $B \circ A : X \longrightarrow Z$ is compact if either A is compact and B is continuous or B is compact and A is continuous.

b) Let Ω be a non-empty, bounded and open subset of $\mathbb{R}^n$ an $p \in [1, \infty)$. Prove that if the embedding $W^{1,p}(\Omega) \hookrightarrow L^{q_0}(\Omega)$ is compact for some $q_0 > 1$, then the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for all $q \in [1, q_0]$.

Problem 11

Let $\Omega \subset \mathbb{R}^n$ be a non-empty, bounded and open subset of $\mathbb{R}^n$ and let $p \in [1, 2^*)$ where $2^* = \infty$ if $n \in \{1, 2\}$ and $2^* = \frac{2n}{n-2}$ if $n \geq 3$. Consider the functional $L : H_0^1(\Omega) \longrightarrow \mathbb{R}$ given by

$$L[u] = \int_\Omega |\nabla u(x)|^2 dx \qquad \text{for all } u \in H_0^1(\Omega).$$

Prove that the restriction of L to the set

$$M := \left\{ u \in H_0^1(\Omega) : \int_\Omega |u(x)|^p dx = 1 \right\}$$

has a minimizer $u_0 \in M$.

To be discussed in the problem session on Tuesday, December 8, 2009.