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WS 2009/10

7^{th} Problem Sheet

Variational Methods and Applications to PDEs

Problem 12

Let $\Omega \subset \mathbb{R}^n$ be a non-empty, bounded and open subset of $\mathbb{R}^n$ and let $p \in [1, 2^*)$ where $2^* = \infty$ if $n \in \{1, 2\}$ and $2^* = \frac{2n}{n-2}$ if $n \geq 3$. Consider the functional $L : H_0^1(\Omega) \longrightarrow \mathbb{R}$ given by

$$L[u] = \int_\Omega |u(x)|^p dx \qquad \text{for all } u \in H_0^1(\Omega)$$

and the set

$$M := \left\{ u \in H_0^1(\Omega) : \int_\Omega |\nabla u(x)|^2 dx = 1 \right\}.$$

Prove:

a) $\inf_M L[u] = 0$ and no minimizer exists,

b) $\sup_M L[u] < \infty$ and a maximizer exists.

To be discussed in the Problem session on Tuesday, December 22, 2009.