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WS 2009/10

8[th] Problem Sheet

Variational Methods and Applications to PDEs

Problem 13

Let $I := (-1,1)$ and $L : H^1(I) \longrightarrow \mathbb{R}$ be given by

$$L[u] = \int_{-1}^{1} u'(x)^2 dx, \qquad u \in H^1(I).$$

Compute all minimizers of $L|_M$ where

a) $M = \left\{ u \in H_0^1(I) : \int_{-1}^{1} u^2(x)dx = 1 \right\}$,

b) $M = \left\{ u \in H^1(I) : \int_{-1}^{1} u^2(x)dx = 1 \text{ and } \int_{-1}^{1} u(x)dx = 0 \right\}$.

Hint: Without proof, you may use that the minimizers belong to $C^2[-1,1]$.

Problem 14

Let Ω be a bounded and open subset of $\mathbb{R}^n$, $2 < p < 2^*$ where 2^* is defined as in Problem 11 and $H := H_0^1(\Omega) \setminus \{0\}$. Prove that the the functional $L : H \longrightarrow \mathbb{R}$ given by

$$L[u] = \frac{\int_\Omega |\nabla u(x)|^2 dx}{\left(\int_\Omega |u(x)|^p dx\right)^{\frac{2}{p}}}$$

has a minimizer $u_0 \in H$ and find a PDE for u_0.
Hint: Use Problem 11.

To be discussed in the Problem session on Tuesday, January 19, 2010.

We wish you a merry Christmas and a happy new year 2010 !