Karlsruher Institut für Technologie
Institut für Analysis
Prof. Dr. Michael Plum
Prof. Dr. Wolfgang Reichel
Hans-Jürgen Freisinger

WS 2009/10

9th Problem Sheet

Variational Methods and Applications to PDEs

Problem 15

Let X be a Banach space, U an open subset of X and $L, m \in C^1(X, \mathbb{R})$. Furthermore, let $c \in \mathbb{R}$, $M := \{u \in U : m[u] \leq c\}$ and $u_0 \in M$ be such that

$$L[u_0] = \inf_{u \in M} L[u].$$

Prove that there exist multipliers $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $(\lambda_1, \lambda_2) \neq (0, 0)$ and

$$\lambda_1 DL[u_0] = \lambda_2 Dm[u_0].$$

Problem 16

Let $\Omega \neq \emptyset$ be a bounded and open subset of $\mathbb{R}^n$ and $2 < p < 2^*$ where 2^* is defined as in Problem 11.

a) Prove that the functional $L : H_0^1(\Omega) \longrightarrow \mathbb{R}$ given by

$$L[u] = \int_\Omega |u|^p dx, \qquad u \in H_0^1(\Omega)$$

has a maximizer $u_0 \in M$ on the set

$$M := \left\{ u \in H_0^1(\Omega) : \int_\Omega \left(|\nabla u|^2 + u^2 \right) \, dx = 1 \right\}.$$

b) By means of part a), show that the boundary value problem

$$\begin{aligned} -\Delta u + u &= |u|^{p-2} u \quad \text{in } \Omega, \\ u &= 0 \qquad\qquad \text{on } \partial\Omega \end{aligned}$$

has a nontrivial weak solution $u \in H_0^1(\Omega) \setminus \{0\}$.

To be discussed in the Problem session on Tuesday, January 19, 2010.