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10th Problem Sheet

Variational Methods and Applications to PDEs

Preliminaries: On this Problem Sheet, we are going to apply the calculus of variation to some „higher order" problems. In the following let Ω be an open and bounded subset of $\mathbb{R}^n$ and $\partial\Omega$ be smooth. Then, the Sobolev space $H_0^2(\Omega)$ is defined as the completion of the space $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{H_0^2(\Omega)}$ given by

$$\|u\|_{H_0^2(\Omega)} := \|\Delta u\|_{L^2(\Omega)} = \left(\int_\Omega (\Delta u)^2 dx\right)^{\frac{1}{2}}.$$

$H_0^2(\Omega)$ is a Hilbert space which has the following embedding properties:

- If $j \in \mathbb{N}$ is such that $2 > j + \frac{n}{2}$, one has a compact embedding $H_0^2(\Omega) \hookrightarrow C^j(\overline{\Omega})$.
- If $p > 1$ is such that $\frac{1}{p} > \frac{1}{2} - \frac{2}{n}$, one has a compact embedding $H_0^2(\Omega) \hookrightarrow L^p(\Omega)$.

Moreover, one has the following version of Poincaré's inequality: for any $u \in H_0^2(\Omega)$, it holds

$$\|u\|_{L^2(\Omega)} \leq C\|\Delta u\|_{L^2(\Omega)}$$

with a constant $C > 0$ depending only on Ω.

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a) Let $f \in L^2(\Omega)$. Prove that the functional $L : H_0^2(\Omega) \longrightarrow \mathbb{R}$ given by

$$L[u] = \int_\Omega \left(\frac{1}{2}(\Delta u)^2 + f(x)u\right) dx \qquad (u \in H_0^2(\Omega))$$

has a minimizer $u_0 \in H_0^2(\Omega)$ and find the boundary value problem that is solved by u_0.

b) Let $p > 1$ be arbitrary if $n \leq 4$ and $p \in \left(1, \frac{2n}{n-4}\right)$ if $n > 4$. On the set

$$M := \left\{u \in H_0^2(\Omega) : \int_\Omega |u|^p dx = 1\right\}$$

we consider the functional $L : M \longrightarrow \mathbb{R}$ given by

$$L[u] = \frac{1}{2}\int_\Omega (\Delta u)^2 dx \qquad (u \in M).$$

Show, that L has a minimizer $u_0 \in M$ and find the boundary value problem that is solved by u_0.

To be discussed in the problem session on Tuesday, February 2, 2010.