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11th Problem Sheet

Variational Methods and Applications to PDEs

Problem 18

Let Ω be a bounded, open subset of $\mathbb{R}^n$ with $\partial\Omega \in C^1$ and let the functional $L : H^1(\Omega)^m \longrightarrow \mathbb{R}$ be given by

$$L[u] = \frac{1}{2}\sum_{j=1}^{m}\int_\Omega |\nabla u_j|^2\,dx \qquad \text{for } u = (u_1,\dots,u_m) \in H^1(\Omega)^m.$$

Furthermore, let $g \in L^2(\partial\Omega)^m$ and $M := \{u \in H^1(\Omega)^m : u|_{\partial\Omega} = g, |u| = 1 \text{ a.e. on } \Omega\}$ (note that here, $|\cdot|$ denotes the Euclidean norm both in $\mathbb{R}^n$ and $\mathbb{R}^m$).

a) Show that if $M \neq \emptyset$, then $L|_M$ has a minimizer $u \in M$.

b) Prove that if $u \in M$ is a minimizer of $L|_M$, then u satisfies

$$(*) \qquad \sum_{j=1}^{m}\int_\Omega \nabla u_j \cdot \nabla v_j\,dx = \sum_{j=1}^{m}\int_\Omega |\nabla u_j|^2 u \cdot v\,dx$$

for any $v \in H_0^1(\Omega)^m \cap L^\infty(\Omega)^m$.

c) Find a strong formulation for the boundary value problem $(*)$.

Instructions:

- For part a), use that there exist a compact embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ and a bounded linear operator $T : H^1(\Omega) \longrightarrow L^2(\partial\Omega)$ such that $Tu = u|_{\partial\Omega}$ for any $u \in H^1(\Omega) \cap C(\overline{\Omega})$. In this sense, we write $u|_{\partial\Omega} := Tu$ for all $u \in H^1(\Omega)$. T is the so-called *trace-operator*.
- For part b), let $v \in H_0^1(\Omega)^m \cap L^\infty(\Omega)^m$ and consider $w_\tau = \frac{u+\tau v}{|u+\tau v|}$ and $\ell(\tau) = L[w_\tau]$. Later, use (and prove!) that u satisfies $\sum_{j=1}^{m} u_j \nabla u_j = 0$.

– please turn over! –

Problem 19

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $A \in C(\overline{\Omega}, \mathbb{R}^{n\times n})$, $b \in C(\overline{\Omega}, \mathbb{R}^n)$ and $f \in C(\overline{\Omega} \times \mathbb{R})$. We assume that A ist uniformly positive definite, i.e. $\xi^\top A(x)\xi \geq a|\xi|^2$ for any $x \in \Omega$ and any $\xi \in \mathbb{R}^n$ with some $a > 0$, and that there exists a function $\varphi \in C^1(\overline{\Omega})$ such that $\nabla\varphi(x) = A^{-1}(x)b(x)$ for any $x \in \Omega$.

a) Prove that the boundary value problem

$$\begin{aligned} -\operatorname{div}(A(x)\nabla u) + b(x)\cdot\nabla u + f(x,u) &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

can be written as a variational problem by introducing a new unknown w such that $u = e^{\psi} w$.

b) Suppose now that $f(x,u) = c(x)u - r(x)$, where $c, r \in C(\overline{\Omega})$ and $c \geq 0$ in Ω. Prove that a minimizer for the variational problem in part a) exists.

To be discussed in the problem session on Tuesday, February 2, 2010.