CORRIGENDUM TO: ON THE TWO-DIMENSIONAL BOUSSINESQ EQUATIONS WITH TEMPERATURE-DEPENDENT THERMAL AND VISCOSITY DIFFUSIONS IN GENERAL SOBOLEV SPACES

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We make the following corrections:

1. We remove the endpoint results concerning the propagation of the initial regularities $(\theta_0, u_0) \in H^2(\mathbb{R}^2) \times (L^2(\mathbb{R}^2))^2$ resp. $H^1(\mathbb{R}^2) \times (H^2(\mathbb{R}^2))^2$ by the two dimensional viscous Boussinesq flow in [1, Theorem 1.2], which correspond to the two endpoints $(2, 0)$ resp. $(1, 2)$ in [1, Figure 1]. See the grey points in Figure 1 below for the corrected admissible regularity exponent set $\{(s_\theta, s_u) \in [1, \infty) \times [0, \infty) \,|\, s_u - 1 \leqslant s_\theta \leqslant s_u + 2\} \backslash \{(2, 0), (1, 2)\}$.
2. We correct the proof of the uniqueness result in [1, Theorem 1.2] under the initial data assumption $(\theta_0, u_0) \in H^1(\mathbb{R}^2) \times (L^2(\mathbb{R}^2))^2$, which corresponds to the endpoint $(1, 0)$ in Figure 1.

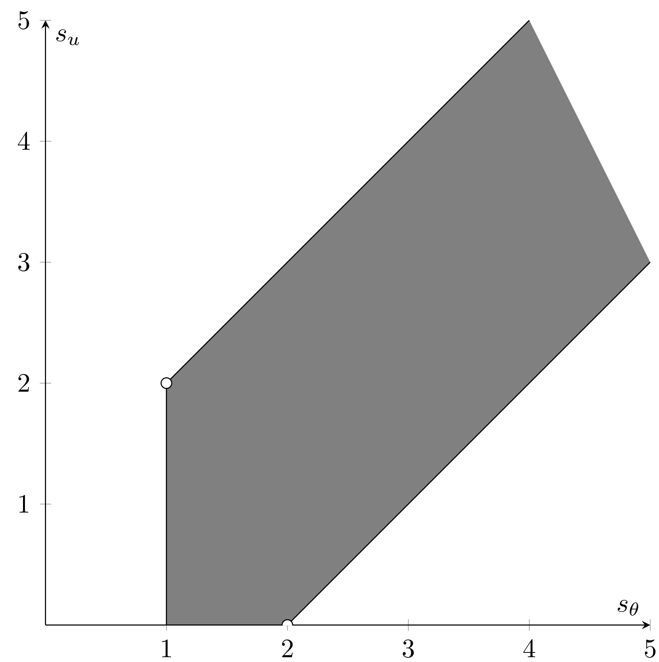

FIGURE 1. Admissible regularity exponents fig:s

In the proofs of the results for the three endpoint regularity cases $(s_\theta, s_u) = (2,0)$, or $(1,2)$, or $(1,0)$ stated in [1, Theorem 1.2], we used the wrong embedding $L^1(\mathbb{R}^2) \hookrightarrow H^{-1}(\mathbb{R}^2)$ in [1]. We sketch the corrected proofs below, using the same notations as well as the numbering of equations as in [1].

1. Since we remove the two endpoints $(2,0)$ and $(1,2)$ in Figure 1, that is, the two (technical) inequalities in [1, (1.18) and (1.20)]:
   - H^2-Estimate for θ, if $u \in L^\infty_{\mathrm{loc}}([0,\infty);(L^2(\mathbb{R}^2))^2) \cap L^2_{\mathrm{loc}}([0,\infty);(H^1(\mathbb{R}^2))^2)$;
   - H^2-Estimate for u, if $\theta \in L^\infty_{\mathrm{loc}}([0,\infty);H^1(\mathbb{R}^2)) \cap L^2_{\mathrm{loc}}([0,\infty);H^2(\mathbb{R}^2))$,

   we have to show
   - H^2-Estimate for θ, if $u \in L^\infty_{\mathrm{loc}}([0,\infty);(H^{0+}(\mathbb{R}^2))^2) \cap L^2_{\mathrm{loc}}([0,\infty);(H^{1+}(\mathbb{R}^2))^2)$;
   - H^2-Estimate for u, if $\theta \in L^\infty_{\mathrm{loc}}([0,\infty);H^{1+}(\mathbb{R}^2)) \cap L^2_{\mathrm{loc}}([0,\infty);H^{2+}(\mathbb{R}^2))$,

   such that the vertical line $\{(2,s_u) \mid s_u \in (0,2]\}$ and the horizontal line $\{(s_\theta,2) \mid s_\theta \in (1,2]\}$ are included in the admissible regularity exponent set.

   More precisely, if $u \in L^\infty_{\mathrm{loc}}([0,\infty);(H^\varepsilon(\mathbb{R}^2))^2) \cap L^2_{\mathrm{loc}}([0,\infty);(H^{1+\varepsilon}(\mathbb{R}^2))^2)$ for some $\varepsilon \in (0,1)$, then we have (instead of [1, (1.18)])

   theta:Hs2 $$\begin{aligned} &\|\theta\|^2_{L^\infty_T H^2_x} + \|\nabla\theta\|^2_{L^2_T H^2_x} \leqslant C(\kappa_*, \|a\|_{C^2}, \kappa^*)\|\theta_0\|^2_{H^2}(1+\|\nabla\theta_0\|^2_{L^2}) \\ &\times \exp\Big(C(\kappa_*,\varepsilon,\|a\|_{\mathrm{Lip}})(\|u\|^2_{L^2_T H^{1+\varepsilon}_x} + \|u\|^4_{L^4_T L^4_x} + \|\nabla\theta\|^4_{L^4_T L^4_x})\Big), \end{aligned} \tag{(1.18)'}$$

   which follows from the same argument as in [1, Subsection 2.3.2], but with the following inequality (instead of [1, the inequality at the top of Page 16])

   $$\begin{aligned} \int_{\mathbb{R}^2} |\nabla\Delta\eta\cdot\nabla u\cdot\nabla\eta|\,dx &\leqslant \frac{\kappa^*}{4}\|\nabla\Delta\eta\|^2_{L^2_x} + C(\kappa^*)\|\nabla u\cdot\nabla\eta\|^2_{L^2_x} \\ &\leqslant \frac{\kappa^*}{4}\|\nabla\Delta\eta\|^2_{L^2_x} + C(\kappa^*)\|\nabla u\|^2_{L_x^{\frac{2}{1-\varepsilon}}}\|\nabla\eta\|^2_{L_x^{\frac{2}{\varepsilon}}} \\ &\leqslant \frac{\kappa^*}{4}\|\nabla\Delta\eta\|^2_{L^2_x} + C(\kappa^*,\varepsilon)\|\nabla u\|^2_{H^\varepsilon_x}\|\nabla\eta\|^2_{H^1_x}. \end{aligned}$$

   Similarly, we have (instead of [1, (1.20)])

   u:Hs2 $$\begin{aligned} &\|u\|^2_{L^\infty_T H^2_x} + \|\nabla u\|^2_{L^2_T H^2_x} \leqslant (\|u\|^2_{L^\infty_T H^1_x} + \|\nabla u\|^2_{L^2_T H^1_x}) \\ &+ C\Big(\|\Delta u_0\|^2_{L^2_x} + \|u\|^2_{L^\infty_T H^1_x \cap L^2_T \dot H^2_x}(\|u\|^2_{L^\infty_T H^1_x\cap L^2_T \dot H^2_x} + \|\nabla\theta\|^2_{L^2_T H^{1+\varepsilon}_x}) \\ &+ \|\Delta\theta\|_{L^2_T L^2_x}\|\Delta u\|_{L^2_T L^2_x}\Big) \times \exp\Big(C\big(\|(u,\nabla\theta)\|^4_{L^4_T L^4_x} + \|\nabla^2\theta\|^2_{L^2_T H^\varepsilon_x}\big)\Big). \end{aligned} \tag{(1.20)'}$$

   where the constant C depends on $\mu_*, \varepsilon, \|b\|_{C^2}, \|\theta\|_{L^\infty_T H^{1+\varepsilon}_x}, \|\nabla\theta\|_{L^2_T H^1_x}$.

   We remark here that in general we can not show $\nabla\Delta\eta\cdot\nabla u\cdot\nabla\eta \in L^1_{\mathrm{loc}}L^1_x$ if $(\eta,u) \in L^\infty_{\mathrm{loc}}(H^2_x \times (L^2_x)^2) \cap L^2_{\mathrm{loc}}(H^3_x \times (H^1_x)^2)$ because of the failure of the Sobolev embedding $H^1(\mathbb{R}^2) \not\hookrightarrow L^\infty(\mathbb{R}^2)$.
2. We now correct the proof of the uniqueness result with initial data $(\theta_0,u_0) \in H^1(\mathbb{R}^2)\times(L^2(\mathbb{R}^2))^2$ given in [1, Section 2.2]. We are going to show $H^{1+\delta}\times H^\delta$, $\delta \in (-1,0)$-Estimates (instead of $H^1\times L^2$-Estimates in [1]) for the difference $(\dot\theta,\dot u)$ of two weak solutions (θ_1,u_1) and (θ_2,u_2) satisfying

   $$\begin{aligned} &\theta_1,\theta_2 \in C([0,\infty);H^1(\mathbb{R}^2)) \cap L^2_{\mathrm{loc}}([0,\infty);H^2(\mathbb{R}^2)), \\ &u_1,u_2 \in C([0,\infty);(L^2(\mathbb{R}^2))^2) \cap L^2_{\mathrm{loc}}([0,\infty);(H^1(\mathbb{R}^2))^2), \end{aligned}$$

   following the arguments in [1, Subsection 2.3.1].

More precisely, we first observe that by virtue of the estimates in [1, (1.13) and (1.14)],

$$B(t) := 1 + \|(\nabla u_1, \nabla u_2)\|_{L^2_x}^2 + \|(u_1, u_2, \nabla\eta_1, \nabla\eta_2)\|_{H_x^{\frac12}}^4 + \|(\nabla\eta_1, \nabla\eta_2)\|_{H^1_x}^2 \\ \in L^1_{\mathrm{loc}}([0,\infty)),$$

where $\eta = A(\theta) := \int_0^\theta a(\alpha)d\alpha$ is the function introduced in [1, (2.10)]. Following the arguments in the proof of [1, Lemma 2.2, Subsection 2.3.1], we derive the $H^{\delta+1}\times H^\delta$-Estimates for $(\dot\eta, \dot u)$ which satisfies the equations in [1, (2.19)], in the following three steps:

Step 1. We use the commutator estimate in [1, (2.28)] for $\delta \in (-1,0)$ and $j \geqslant -1$

$$\|[u_1, \Delta_j]\nabla\dot\eta\|_{L^2_x} \leqslant C l_j 2^{-j\delta}\|\nabla u_1\|_{L^2_x}\|\nabla\dot\eta\|_{H^\delta_x},$$
$$\|[\kappa_1, \Delta_j]\Delta\dot\eta\|_{L^2_x} \leqslant C l_j 2^{-j\delta}\|\nabla\kappa_1\|_{H_x^{\frac12}}\|\Delta\dot\eta\|_{H_x^{\delta-\frac12}},$$

with $(l_j)_{j\geqslant -1}\in \ell^1$, and the product estimates

$$\|\dot\kappa\Delta\eta_2\|^2_{L^2_TH^\delta_x} \lesssim \int_0^T \|\dot\kappa\|^2_{H^{\delta+1}_x}\|\Delta\eta_2\|^2_{L^2_x}\mathrm{dt},$$
$$\|\dot u\cdot\nabla\eta_2\|^2_{L^2_TH^\delta_x} \lesssim \int_0^T \Big(\|\dot u\|^2_{H^\delta_x}\|\nabla\eta_2\|^2_{H^1_x} + \|\nabla\dot u\|^2_{H_x^{\delta-\frac12}}\|\nabla\eta_2\|^2_{H_x^{\frac12}}\Big)\mathrm{dt}.$$

By use of interpolation inequalities, Gagliardo-Nirenberg's inequalities, Young's inequalities and Hölder's inequalities, we derive

$$\|\dot\eta\|^2_{L^\infty_TH^{\delta+1}_x} + \|\nabla\dot\eta\|^2_{L^2_TH^{\delta+1}_x} \leqslant C(\delta, \kappa_*, \|a\|_{C^2}, \|(\theta_1,\theta_2)\|_{L^\infty_TH^1_x}) \\ \times\int_0^T (\|\dot u\|^2_{H^\delta_x} + \|\dot\eta\|^2_{H^{\delta+1}_x})B(t)\mathrm{dt} + \frac12\int_0^T\|\nabla\dot u\|^2_{H^\delta_x}\mathrm{dt}.$$

Step 2. Similarly as Step 1, we derive the following estimate

$$\|\dot u\|^2_{L^\infty_TH^\delta_x} + \|\nabla\dot u\|^2_{L^2_TH^\delta_x} \\ \leqslant C(\delta, \mu_*, \|b\|_{C^2}, \|(\theta_1,\theta_2)\|_{L^\infty_TH^1_x})\int_0^T(\|\dot\eta\|^2_{H^{\delta+1}_x} + \|\dot u\|^2_{H^\delta_x})B(t)\mathrm{dt}.$$

Step 3. We sum the above two estimates up, to derive

$$\|\dot\eta\|^2_{L^\infty_TH^{\delta+1}_x} + \|\nabla\dot\eta\|^2_{L^2_TH^{\delta+1}_x} + \|\dot u\|^2_{L^\infty_TH^\delta_x} + \|\nabla\dot u\|^2_{L^2_TH^\delta_x} \\ \leqslant C(\delta, \kappa_*, \mu_*, \|(a,b)\|_{C^2}, \|(\theta_1,\theta_2)\|_{L^\infty_TH^1_x})\int_0^T(\|\dot\eta\|^2_{H^{\delta+1}_x} + \|\dot u\|^2_{H^\delta_x})B(t)\mathrm{dt}.$$

Finally, the Gronwall's inequality implies $\dot\eta = 0$ and $\dot u = 0$. The uniqueness result follows.

We remark here that we do not expect the uniqueness result below the regularity assumption $(\theta_0, u_0) \in H^1(\mathbb{R}^2)\times(L^2(\mathbb{R}^2))^2$, which is critical by view of the Navier-Stokes-type equation for u and the temperature-dependent diffusion coefficients (we used the wrong duality $(L^1(\mathbb{R}^2))' = BMO(\mathbb{R}^2)$ in [1, Remark 1.3] which should be $(L^1)' = L^\infty$).

References

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[1] Z. He and X. Liao, On the two-dimensional Boussinesq equations with temperature-dependent thermal and viscosity diffusions in general Sobolev spaces, *Z. Angew. Math. Phys.*, **73** (2022), Paper No. 16, 25.

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