Introduction to harmonic analysis with applications to evolution equations
by Hajer Bahouri

These lectures intend to prepare the students how to apply tools from Fourier analysis to directly solve linear and non linear partial differential equations arising in physics and fluid mechanics.

The first aim of these courses is to review briefly the Paley-Littlewood decomposition which is a classical tool of harmonic analysis that has established itself as a very powerful tool in the study of functional inequalities and evolution equations. For a detailed and complete exposition of the Littlewood-Paley theory, we refer for instance to [1, 2, 5].

In the second part, we will illustrate the effectiveness of this decomposition in the study of partial differential equations starting by the issue of dispersion phenomena for the Schrödinger equation on $\mathbb{R}^d$

$$(S) \qquad \begin{cases} i\partial_t u - \Delta u &= 0 \\ u_{|t=0} &= u_0. \end{cases}$$

Dispersion phenomena, which express that waves with different frequencies move at different velocities, correspond to a pointwise inequality in time decay. Then we will see how to pass from these pointwise estimates to Strichartz estimates which are estimates of the space-time norms of the solutions by the norm of the initial datum, and which play a central tool in the study of semilinear and quasilinear equations.

The last part will be devoted to the introduction of harmonic analysis on non commutative Lie groups (see [3, 4, 6]), and specifically on the Heisenberg group which is a Carnot group of step 2 and the Engel group which is rather a Carnot group of step 3. As it will be seen, this is strongly tied to the spectral analysis of the harmonic and quartic oscillators.

## References


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