Lectures on the Benjamin–Ono equation as an integrable PDE

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The Benjamin–Ono equation (1967)

Long internal gravity waves in a two–layer fluid with infinite depth

$$\partial_t u = \partial_x(|D_x|u - u^2) \quad , \quad u = u(t,x) \ ,$$
$$u(t, x + 2\pi) = u(t,x) \ (x \in \mathbb{T}) \quad \text{or} \quad u(t,x) \underset{x\to\infty}{\longrightarrow} 0 \ ,$$
$$\widehat{|D_x| f}(\xi) := |\xi| \hat{f}(\xi) \quad , \quad \xi \in \mathbb{Z} \text{ or } \xi \in \mathbb{R} \ .$$

Rigorous derivation from the full Euler model by
Bona–Lannes–Saut (2008)

Why this equation ?

In addition to its physical relevance, the Benjamin–Ono equation has been the center of an important mathematical activity.

- Example of a quasilinear dispersive PDE. Dispersive methods from harmonic analysis (Tao, Kenig–Ionescu, Molinet, Molinet–Pilod, Ifrim–Tataru, ...)
- Example of an integrable Hamiltonian PDE. Lax pair, conservation laws,... (Bock–Kruskal, Fokas–Ablowitz, Dobrokhotov–Krichever, Coifman–Wickerhauser, Kaup–Matsuno, Miller–Xu, Miller–Wetzel, Wu, PG–Kappeler, PG–Kappeler–Topalov, Sun, Gassot, ...)

This minicourse will emphasize the second feature, as an introduction to the corresponding methods. No prerequisite on integrability is assumed.

Outline of the course

- Lecture 1. The initial value problem for smooth solutions. Lax pair and conservation laws.
- Lecture 2. An explicit formula for the solution of the initial value problem.
- Lecture 3. Various applications : traveling waves, multi–solitons, low regularity solutions, zero dispersion limit.

Lecture 1. THE INITIAL VALUE PROBLEM FOR SMOOTH SOLUTIONS. LAX PAIR AND CONSERVATION LAWS.

The main result

Theorem (Saut, 1979)

For every $u_0 \in H^2_{\mathrm{real}}$, *there exists a unique solution* $u \in C(\mathbb{R}, H^2_{\mathrm{real}})$ *of the equation*

$$\partial_t u = \partial_x(|D_x|u - u^2) \qquad (BO)$$

such that $u(0) = u_0$. *Furthermore, for* $k \in \mathbb{Z}_{\geq 0}$, *the norms of* $u(t)$ *in* $H^{k/2}$ *are globally controlled .*

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Strategy.

- Local wellposedness in H^2 and propagation of higher regularity.
- Lax pair structure and conservation laws controlling the $H^{k/2}$ norms.

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Recall that we work on $\mathbb{T}$ (periodic boundary conditions) or on $\mathbb{R}$ (solutions vanishing as $x \to \infty$).

Local wellposedness

Proposition

For every $u_0 \in H^2_{\mathrm{real}}$, *there exists* $T > 0$ *and a unique solution* $u \in C_w([-T, T], H^2_{\mathrm{real}}) \cap C([-T, T], L^2)$ *of the equation*

$$\partial_t u - \partial_x |D_x| u + 2u\partial_x u = 0 \qquad (BO)$$

such that $u(0) = u_0$. *Furthermore,* T *is bounded from below if* $\|u_0\|_{H^2}$ *is bounded, and the flow map*

$$u_0 \in H^2_{\mathrm{real}} \longmapsto u \in C_w([-T, T], H^2_{\mathrm{real}}) \cap C([-T, T], L^2)$$

is continuous. Finally, if $u_0 \in H^s$ *for some integer* $s > 2$, *then* $u \in C_w([-T, T], H^s)$.

Proof

Kato's iterative scheme

$$\partial_t u^{n+1} - \partial_x |D_x| u^{n+1} + 2u^n \partial_x u^{n+1} = 0 \ , \ \ u^{n+1}(0) = u_0 \ .$$

Proof

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Lemma

Given $v_0 \in L^2$, $f \in L^1([-T, T], L^2)$, $u \in L^1([-T, T], \mathrm{Lip}_{\mathrm{real}})$, *there exists a unique* $v \in C([-T;, T], L^2)$ *such that*

$$\partial_t v - \partial_x |D_x| v + 2u\partial_x v = f \ , \ \ v(0) = v_0 \ .$$

Furthermore, for $t \in [-T, T]$, *the following estimate holds,*

$$\|v(t)\|_{L^2} \leq \|v_0\|_{L^2} + C \left| \int_0^t (\|\partial_x u(\tau)\|_{L^\infty} \|v(\tau)\|_{L^2} + \|f(\tau)\|_{L^2})\, d\tau \right| \ .$$

Proof, continued

Start with some $u^0 \in C([-T, T], H^2_{\text{real}})$ such that $u^0(0) = u_0$. At each step n, the lemma provides $u^{n+1} \in C([-T, T], H^2_{\text{real}})$ with, $\forall t \in [0, T]$,

$$\partial_t u^{n+1} - \partial_x |D_x| u^{n+1} + 2u^n \partial_x u^{n+1} = 0 \,, \qquad u^{n+1}(0) = u_0$$

$$\|u^{n+1}(t) - u^n(t)\|_{L^2} \leq C \int_0^t \|\partial_x u^n(\tau)\|_{L^\infty} \, [\|u^{n+1}(\tau) - u^n(\tau)\|_{L^2} + \|u^n(\tau) - u^{n-1}(\tau)\|_{L^2}] \, d\tau$$

$$\|u^{n+1}(t)\|_{H^2} \leq \|u_0\|_{H^2} + C \int_0^t [\|\partial_x u^n(\tau)\|_{L^\infty} \|u^{n+1}(\tau)\|_{H^2} + \|u^n(\tau)\|_{H^2} \|\partial_x u^{n+1}(\tau)\|_{L^\infty}] \, d\tau \,.$$

Proof, continued

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$$\|u^{n+1}(t) - u^n(t)\|_{L^2} \le C \int_0^t \|\partial_x u^n(\tau)\|_{L^\infty} \quad [\|u^{n+1}(\tau) - u^n(\tau)\|_{L^2} + \|u^n(\tau) - u^{n-1}(\tau)\|_{L^2}]\, d\tau$$

$$\|u^{n+1}(t)\|_{H^2} \le \|u_0\|_{H^2} + C \int_0^t \quad [\|\partial_x u^n(\tau)\|_{L^\infty} \|u^{n+1}(\tau)\|_{H^2} + \|u^n(\tau)\|_{H^2} \|\partial_x u^{n+1}(\tau)\|_{L^\infty}]\, d\tau \ .$$

If $\|u_0\|_{H^2} \le R$, choose $T > 0$ so small that $R\,\mathrm{e}^{\tilde C TR} \le 2R$ and $\sup_{|t|\le T} \|u^0(t)\|_{H^2} \le 2R$. Then, by Grönwall's inequality,

$$\forall n \ge 0 \ , \ \sup_{|t|\le T} \|u^n(t)\|_{H^2} \le 2R \ , \ \sum_{n=0}^{\infty} \sup_{|t|\le T} \|u^{n+1}(t) - u^n(t)\|_{L^2} < +\infty \ .$$

Proof, conclusion

The L^2 contraction argument also leads to uniqueness of the solution in $C_w([-T,T], H^2_{\mathrm{real}}) \cap C([-T,T], L^2)$ and to continuity of the flow map.

The H^2 bound can be extended to H^s bound for $s > 2$ on the same time interval $[-T,T]$. □

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Remark. General considerations (Bona–Smith, Tao,...) lead to strong continuity $u \in C([-T, T], H^2_{\rm real})$. We shall obtain it alternatively, thanks to the conservation laws.
For this, we need to introduce the Lax pair structure.

The Hardy space

Introduce, on $\mathbb{T}$ or on $\mathbb{R}$,

$$L^2_+ := \{f \in L^2 : \forall \xi < 0, \hat{f}(\xi) = 0\} \ .$$

Space of holomorphic functions.

$$L^2_+(\mathbb{T}) \simeq \{f \text{ holomorphic on } \mathbb{D} : \sup_{r<1} \int_0^{2\pi} |f(re^{ix})|^2 \, dx < +\infty\}$$

$$L^2_+(\mathbb{R}) \simeq \{f \text{ holomorphic on } \mathbb{C}_+ : \sup_{y>0} \int_{\mathbb{R}} |f(x+iy)|^2 \, dx < +\infty\}$$

The Riesz–Szegő projector is

$$\Pi : L^2 \to L^2_+ \ , \ \widehat{\Pi f}(\xi) = \mathbf{1}_{\xi \geq 0} \hat{f}(\xi) \ .$$

The Lax operators

Given $b \in L^\infty$, define the Toeplitz operator of symbol b,

$$T_b : L^2_+ \to L^2_+ \ , \ f \mapsto T_b f := \Pi(bf) \ .$$

Notice that $T_b^* = T_{\overline{b}}$.
For $u \in L^\infty$, real valued, define $L_u : H^1_+ := H^1 \cap L^2_+ \to L^2_+$ by

$$L_u(f) = D_x f - T_u f = \frac{1}{i}\frac{df}{dx} - T_u f \ .$$

L_u is unbounded selfadjoint on L^2_+ with $\mathrm{Dom}(L_u) := H^1_+$.
Also define, for $u \in H^2_{\mathrm{real}}$,

$$B_u := i(T_{|D_x|u} - T_u^2) \ .$$

Notice that $B_u : L^2_+ \to L^2_+$ and $B_u : H^1_+ \to H^1_+$ and $B_u^* = -B_u$.

The Lax pair

Theorem (Fokas–Ablowitz (1983),... Wu (2016), PG–Kappeler (2021))

If $u \in C(\mathbb{R}, H^2_{\rm real})$ solves the Benjamin–Ono equation, then

$$\frac{dL_{u(t)}}{dt} = [B_{u(t)}, L_{u(t)}] \ .$$

Proof. Recall that $B_u^* = -B_u$ and that $T_u^* = T_u$. We have

$$\frac{d}{dt} L_{u(t)} = -T_{\partial_t u(t)} = -T_{\partial_x |D_x| u(t)} + 2T_{u(t)\partial_x u(t)} := (1)$$

Since $[\partial_x, T_b] = T_{\partial_x b}$ and $D_x = \frac{1}{i}\partial_x$,

$$(1) = i\,[T_{|D_x|u}, D_x] + 2T_{u\partial_x u} = i\,[T_{|D_x|u}, D_x - T_u] + 2T_{u\partial_x u} + i[T_{|D_x|u}, T_u].$$

Proof of the Lax pair identity, continued

Consequently, $\frac{d}{dt}L_{u(t)} = i\,[T_{|D_x|u}, L_u] + 2T_{u\partial_x u} + i[T_{|D_x|u}, T_u]$.

Lemma

For $a, b \in L^\infty, f \in L^2_+$,

$(T_{ab} - T_aT_b)f = \Pi\Big(\Pi(a)(\mathrm{Id} - \Pi)\{(\mathrm{Id} - \Pi)(b)f\}\Big)$

Assume this lemma. Then

$(2) := i[T_{|D_x|u}, T_u]f = i\Big(T_{|D_x|u}T_u - T_{u|D_x|u}\Big)f + i\Big(T_{u|D_x|u} - T_uT_{|D_x|u}f\Big)$.

Apply the lemma with $a = |D_x|u, b = u$, then $a = u, b = |D_x|u$.

$$(2) = -i\Pi\Big(\Pi(|D_x|u)(\mathrm{Id} - \Pi)\{(\mathrm{Id} - \Pi)(u)f\}\Big) + i\Pi\Big(\Pi(u)(\mathrm{Id} - \Pi)\{(\mathrm{Id} - \Pi)(|D_x|u)f\}\Big).$$

Proof of the Lax pair identity, conclusion

Since
$\Pi(|D_x|v)(x) = \frac{1}{i}(\Pi\partial_x v)(x)$, $(\mathrm{Id}-\Pi)(|D_x|v)(x) = i(\mathrm{Id}-\Pi)\partial_x v(x)$,

$(2) = -\Pi\Big(\Pi(\partial_x u)(\mathrm{Id}-\Pi)\{(\mathrm{Id}-\Pi)(u)f\}\Big) - \Pi\Big(\Pi(u)(\mathrm{Id}-\Pi)\{(\mathrm{Id}-\Pi)(\partial_x u)f\}\Big)$.

Applying again the lemma, we obtain

$$\begin{aligned} i[T_{|D_x|u}, T_u]f &= -\Big(T_{u\partial_x u} - T_{\partial_x u}T_u\Big)f - \Big(T_{u\partial_x u} - T_u T_{\partial_x u}\Big)f, \\ &= -2T_{u\partial_x u}f + T_{\partial_x u}T_u f + T_u T_{\partial_x u}f. \end{aligned}$$

Then observe that

$$\begin{aligned} [T_u^2, D_x] = T_u[T_u, D_x] + [T_u, D_x]T_u &= -\frac{1}{i}T_u T_{\partial_x u} - \frac{1}{i}T_{\partial_x u}T_u, \\ &= i(T_u T_{\partial_x u} + T_{\partial_x u}T_u), \end{aligned}$$

so that $i[T_{|D_x|u}, T_u] = -2T_{u\partial_x u} - i[T_u^2, D_x] = -2T_{u\partial_x u} - i[T_u^2, L_u]$.
Finally,

$$\frac{d}{dt}L_u = i\,[T_{|D_x|u}, L_u] - i[T_u^2, L_u] = [B_u, L_u]\ .$$

Proof of the lemma

Lemma

For $a, b \in L^\infty, f \in L^2_+,$
$$(T_{ab} - T_a T_b)f = \Pi\Big(\Pi(a)(\mathrm{Id} - \Pi)\{(\mathrm{Id} - \Pi)(b)f\}\Big)$$

Proof. Main observation : if f, g have positive (resp. negative) frequencies, then fg has positive (resp.negative) frequencies.

$$T_{ab}f - T_a T_b f = \Pi(abf) - \Pi(a\Pi(bf)) = \Pi(aU), \quad U := (\mathrm{Id} - \Pi)(bf)$$
$$\Pi(aU) = \Pi(\Pi(a)U) + \Pi((\mathrm{Id} - \Pi)(a)U) = \Pi(\Pi(a)U).$$

Finally, write $bf = \Pi(b)f + (\mathrm{Id} - \Pi)(b)f$, and $\Pi(b)f \in L^2_+$, so that

$$U = (\mathrm{Id} - \Pi)(bf) = (\mathrm{Id} - \Pi)\{(\mathrm{Id} - \Pi)(b)f\}\ .$$

□

Conservation laws

Proposition

If u is a $H^{k/2}$ solution of (BO) with an integer $k \geq 4$, then

$$\langle L_u^k(\Pi u), \Pi u\rangle$$

is conserved by the evolution.

Proof. First assume that $u \in H^s$ with $s >> k/2$. We observe that

$$\partial_t \Pi u = B_u(\Pi u) + iL_u^2(\Pi u) \ .$$

Indeed, if $x \in \mathbb{T}$, $\Pi u = -L_u(1)$, $B_u(1) = i(T_{|D_x|u}(1) - T_u^2(1)) = iL_u(\Pi u)$ and

$$\partial_t(\Pi u) = -[B_u, L_u]1 = B_u(-L_u(1)) + L_u(B_u(1)) \ .$$

If $x \in \mathbb{R}$, replace 1 by $(1 - i\varepsilon x)^{-1}, \varepsilon \to 0^+$.

Proof of the conservation law, continued

Using the Lax pair identity and the previous one,

$$\begin{aligned}\frac{d}{dt}\langle L_u^k \Pi u, \Pi u\rangle &= \langle [B_u, L_u^k]\Pi u, \Pi u\rangle + \langle L_u^k(\Pi u), B_u(\Pi u) + iL_u^2(\Pi u)\rangle \\ &+ \langle L_u^k(B_u(\Pi u) + iL_u^2(\Pi u)), \Pi u\rangle \\ &= 0\ .\end{aligned}$$

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By continuity of the flow, we get the conservation law if $s > k/2$, while, for $s = k/2$, we only have

$$\langle L_u^k(\Pi u), \Pi u\rangle \leq \langle L_{u_0}^k(\Pi u_0), \Pi u_0\rangle \ .$$

Proof of the conservation law, continued

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Reversing the time and using the uniqueness of the initial value problem, we finally get the equality for $s = k/2$.

Proof of the conservation law, continued

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Reversing the time and using the uniqueness of the initial value problem, we finally get the equality for $s = k/2$. Furthermore,

$$\langle L_u^k \Pi u, \Pi u\rangle = \frac{1}{2}\|u\|_{H^{k/2}}^2 + r_k(u)$$

where r_k is continuous on $H^{(k-1)/2}$ if $k \ge 2$, and $r_1 = O(\|u\|_{H^{1/2}}\|u\|_{L^2}^2)$. This provides the control of all the $H^{k/2}$ norms, as well as the strong continuity $u \in C([-T, T], H^2)$, the strong continuity of the flow map, and the global wellposedness.