Lectures on the Benjamin–Ono equation as an integrable PDE

Patrick Gérard

Univ. Paris-Saclay, Laboratoire de Mathématiques d'Orsay

Minicourse, Workshop on Analysis and PDEs in Karlsruhe, March 27–28, 2023

Lecture 2. An explicit formula for the solution of the initial value problem.

Recall from the first lecture

The initial value problem for the Benjamin–Ono equation

$$\partial_t u = \partial_x(|D_x|u - u^2) \quad , \quad u = u(t,x) \ ,$$
$$u(t, x+2\pi) = u(t,x) \ (x \in \mathbb{T}) \quad \text{or} \quad u(t,x) \underset{x\to\infty}{\longrightarrow} 0 \ ,$$
$$\widehat{|D_x| f}(\xi) := |\xi| \hat{f}(\xi) \quad , \quad \xi \in \mathbb{Z} \text{ or } \xi \in \mathbb{R} \ .$$

is globally wellposed on the Sobolev space $H^2_{\rm real}$, and satisfies a Lax pair identity with the following operators on the Hardy space L^2_+,

$$L_u := D_x - T_u \ , \ B_u = i(T_{|D_x|u} - T_u^2) \ .$$

This leads to the conservation laws

$$\langle L_u^k \Pi u, \Pi u \rangle \ , \ k \geq 0 \ ,$$

which control the norms $H^{k/2}$ of u.

Another consequence of the Lax pair identity

Theorem (Fokas–Ablowitz (1983),... Wu (2016), PG–Kappeler (2021))

If $u \in C(\mathbb{R}, H^2_{\mathrm{real}})$ *solves the Benjamin–Ono equation, then*

$$\frac{dL_{u(t)}}{dt} = [B_{u(t)}, L_{u(t)}] \ .$$

Corollary

Define the family of unitary operators $\{U(t)\}_{t\in\mathbb{R}}$ *by*

$$U'(t) = B_{u(t)}U(t) \ , \ U(0) = \mathrm{Id} \ .$$

Then $L_{u(t)} = U(t)L_{u(0)}U(t)^* \ .$

Proof. Compute the time derivative of $U(t)^* L_{u(t)} U(t)$.

Inverse Spectral Theory and explicit formulae

The spectrum of L_u is a conservation law of the Benjamin–Ono equation.

→ Strategy : solve the initial value problem by inverse spectral theory.

- On the line. Fokas–Ablowitz (1983), Coifman–Wickerhauser (1991) $u \in \mathscr{S}(\mathbb{R})$ and small, ...
- On the torus. Recent complete resolution by PG–Kappeler–Topalov (2021) through some nonlinear Fourier Transform. Sharp wellposedness in $H^s(\mathbb{T}), s > -1/2$.

Inverse Spectral Theory and explicit formulae

The spectrum of L_u is a conservation law of the Benjamin–Ono equation.

$\to$ Strategy : solve the initial value problem by inverse spectral theory.

- On the line. Fokas–Ablowitz (1983), Coifman–Wickerhauser (1991) $u \in \mathscr{S}(\mathbb{R})$ and small, ...
- On the torus. Recent complete resolution by PG–Kappeler–Topalov (2021) through some nonlinear Fourier Transform. Sharp wellposedness in $H^s(\mathbb{T}), s > -1/2$.

In this lecture, we shall bypass the inverse spectral step and establish directly explicit formulae for the solution thanks to commuting properties of the Lax operators with the structure of the Hardy space.

The explicit formula on the torus

$L^2_+(\mathbb{T})$ is equipped with the shift operator and its adjoint

$$S := T_{e^{ix}} \ , \ S^* = T_{e^{-ix}}$$

and with the inner product $\langle f|g\rangle = \int_0^{2\pi} f(x)\overline{g}(x)\,\frac{dx}{2\pi}$.

Theorem (PG, 2022)

The solution $u \in C(\mathbb{R}, H^2_{\mathrm{real}}(\mathbb{T}))$ of the Benjamin–Ono equation with $u(0) = u_0$ is given by

$$\begin{aligned} u(t) &= \Pi u(t) + \overline{\Pi u(t)} - \langle u_0|1\rangle \ , \\ \forall z \in \mathbb{D} \ , \ \Pi u(t,z) &= \langle (\mathrm{Id} - z e^{it} e^{2itL_{u_0}} S^*)^{-1} \Pi u_0 | 1\rangle \end{aligned}$$

Proof (torus)

Because of the equation $\partial_t u = \partial_x(|D_x|u - u^2)$, we have $\langle u(t)|1\rangle = \langle u_0|1\rangle$, and therefore, since $u(t)$ is real valued,

$$u(t) = \Pi u(t) + \overline{\Pi u(t)} - \langle u(t)|1\rangle = \Pi u(t) + \overline{\Pi u(t)} - \langle u_0|1\rangle \ .$$

Fourier expansion of $\Pi u(t)$:

$$\forall z \in \mathbb{D} \ , \ \Pi u(t,z) = \sum_{n=0}^{\infty} z^n \langle \Pi u(t)|S^n 1\rangle = \langle (\mathrm{Id} - zS^*)^{-1} \Pi u(t)|1\rangle \ .$$

Apply the unitary operator $U(t)^*$ to both sides of this inner product,

$$\forall z \in \mathbb{D} \ , \ \Pi u(t,z) = \langle (\mathrm{Id} - zU(t)^* S^* U(t))^{-1} U(t)^* \Pi u(t)|U(t)^* 1\rangle \ .$$

Proof (torus)

Because of the equation $\partial_t u = \partial_x(|D_x|u - u^2)$, we have $\langle u(t)|1\rangle = \langle u_0|1\rangle$, and therefore, since $u(t)$ is real valued,

$$u(t) = \Pi u(t) + \overline{\Pi u(t)} - \langle u(t)|1\rangle = \Pi u(t) + \overline{\Pi u(t)} - \langle u_0|1\rangle \ .$$

Fourier expansion of $\Pi u(t)$:

$$\forall z \in \mathbb{D} \ , \ \Pi u(t,z) = \sum_{n=0}^{\infty} z^n \langle \Pi u(t)|S^n 1\rangle = \langle (\mathrm{Id} - zS^*)^{-1}\Pi u(t)|1\rangle \ .$$

Apply the unitary operator $U(t)^*$ to both sides of this inner product,

$$\forall z \in \mathbb{D} \ , \ \Pi u(t,z) = \langle (\mathrm{Id} - zU(t)^* S^* U(t))^{-1} U(t)^* \Pi u(t)|U(t)^* 1\rangle \ .$$

We are going to calculate explicitly $U(t)^*1, U(t)^*\Pi u(t), U(t)^* S^* U(t)$ using $U'(t) = B_{u(t)} U(t)$ and some commutator identities.

Proof (torus), continued

Lemma

$[S^*, B_u] = i((L_u + \mathrm{Id})^2 S^* - S^* L_u^2)$

Assume this lemma. Then we have

$$\begin{aligned}
\frac{d}{dt} U(t)^* S^* U(t) &= U(t)^* [S^*, B_{u(t)}] U(t) = i U(t)^* ((L_{u(t)} + \mathrm{Id})^2 S^* - S^* L_{u(t)}^2) \\
&= i(L_{u_0} + \mathrm{Id})^2 U(t)^* S^* U(t) - i U(t)^* S^* U(t) L_{u_0}^2 \ .
\end{aligned}$$

and $U(t)^* S^* U(t) = \mathrm{e}^{it(L_{u_0} + \mathrm{Id})^2} S^* \mathrm{e}^{-itL_{u_0}^2}$. Recall that $L_u(1) = -\Pi u$, $B_u(1) = -iL_u^2(1)$, so that

$$\begin{aligned}
\frac{d}{dt} U(t)^* 1 &= -U(t)^* B_{u(t)}(1) = i U(t)^* L_{u(t)}^2(1) = i L_{u_0}^2 U(t)^* 1 \\
U(t)^* 1 &= \mathrm{e}^{itL_{u_0}^2}(1) \ , \\
U(t)^* \Pi u(t) &= -U(t)^* L_{u(t)} 1 = -L_{u_0} U(t)^* 1 = \mathrm{e}^{itL_{u_0}^2} \Pi u_0 \ .
\end{aligned}$$

Proof (torus), conclusion

Plug the obtained expressions

$$\begin{aligned} U(t)^*1 &= \mathrm{e}^{itL_{u_0}^2}(1)\ ,\ U(t)^*\Pi u(t) = \mathrm{e}^{itL_{u_0}^2}\Pi u_0\ , \\ U(t)^*S^*U(t) &= \mathrm{e}^{it(L_{u_0}+\mathrm{Id})^2}S^*\mathrm{e}^{-itL_{u_0}^2} \end{aligned}$$

into the formula

$$\forall z \in \mathbb{D}\ ,\ \Pi u(t,z) = \langle (\mathrm{Id} - zU(t)^*S^*U(t))^{-1}U(t)^*\Pi u(t) | U(t)^*1\rangle\ .$$

We finally infer

$$\forall z \in \mathbb{D}\ ,\ \Pi u(t,z) = \langle (\mathrm{Id} - z\mathrm{e}^{it+2itL_{u_0}}S^*)^{-1}\Pi u_0 | 1\rangle\ .$$

□

Proof of the lemma

Lemma

$L_u(1) = -\Pi u \ , \ B_u(1) = -iL_u^2(1) \ , \ [S^*, B_u] = i((L_u + \mathrm{Id})^2 S^* - S^* L_u^2)$

Proof. Commutation identity with Toeplitz operators,

$$\forall b \in L^\infty(\mathbb{T}) \ , \ [S^*, T_b] = \langle \ . \ |1\rangle S^* \Pi b \ .$$

Adjoint Leibniz formula : $S^* D = DS^* + S^*$. Combining these two identities, we infer $S^* L_u = (L_u + \mathrm{Id}) S^* - \langle \ . \ |1\rangle S^* \Pi u$ and finally

$$\begin{aligned}
[S^*, B_u] &= i([S^*, T_{|D|u}] - T_u [S^*, T_u] - [S^*, T_u] T_u) \\
&= i(\langle \ . \ |1\rangle S^* D \Pi u - T_u \langle \ . \ |1\rangle S^* \Pi u - (\langle \ . \ |1\rangle S^* \Pi u) T_u) \\
&= i(\langle \ . \ |1\rangle (D S^* \Pi u - T_u S^* \Pi u + S^* \Pi u) - \langle \ . \ | T_u 1\rangle S^* \Pi u) \\
&= i(\langle \ . \ |1\rangle (L_u S^* \Pi u + S^* \Pi u) + \langle \ . \ | L_u 1\rangle S^* \Pi u) \\
&= i((L_u + \mathrm{Id}) \langle \ . \ |1\rangle S^* \Pi u + (\langle \ . \ |1\rangle S^* \Pi u) L_u) \\
&= i((L_u + \mathrm{Id})((L_u + \mathrm{Id}) S^* - S^* L_u) + ((L_u + \mathrm{Id}) S^* - S^* L_u) L_u) \\
&= i((L_u + \mathrm{Id})^2 S^* - S^* L_u^2) \ . \quad \square
\end{aligned}$$

The explicit formula on the line

The shift operator has to be replaced by the Lax–Beurling semigroup

$$S(\eta) := T_{\mathrm{e}^{i\eta x}} \ , \ \eta \geq 0 \ , \ S(\eta)f(x) = \mathrm{e}^{i\eta x} f(x) \ .$$

Infinitesimal generator : multiplication by x. We define $G = x^*$, so that

$$S(\eta)^* = T_{\mathrm{e}^{-i\eta x}} \quad = \quad \mathrm{e}^{-i\eta G} \ , \ \eta \geq 0 \ , \ \widehat{Gf}(\xi) = i \frac{d\hat{f}}{d\xi} \mathbf{1}_{\xi > 0} \ ,$$

$$\mathrm{Dom}(G) \quad = \quad \{ f \in L^2_+(\mathbb{R}) : \hat{f}_{|]0,+\infty[} \in H^1(]0,+\infty[) \} \ .$$

Define $I_+(f) := \hat{f}(0^+)$ if $\hat{f}_{|]0,\delta[} \in H^1(]0,\delta[)$ for some $\delta > 0$.

Theorem (PG, 2022)

The solution $u \in C(\mathbb{R}, H^2_{\mathrm{real}}(\mathbb{R}))$ of the Benjamin–Ono equation with $u(0) = u_0$ is given by $u(t) = \Pi u(t) + \overline{\Pi u(t)}$ with

$$\forall z \in \mathbb{C}_+ \ , \ \Pi u(t,z) = \frac{1}{2i\pi} I_+[(G - 2tL_{u_0} - z\mathrm{Id})^{-1} \Pi u_0] \ .$$

Proof (line) : inverse Fourier transform

Start with the inverse Fourier transform for every $f \in L^2_+(\mathbb{R})$,

$$\forall z \in \mathbb{C}_+ \ , \ f(z) = \frac{1}{2\pi} \int_0^\infty \mathrm{e}^{iz\xi} \hat{f}(\xi)\, d\xi \ .$$

Plancherel theorem : we have, in L^2,

$$\hat{f}(\xi) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}} \mathrm{e}^{-ix\xi} \frac{f(x)}{1 + i\varepsilon x}\, dx = \lim_{\varepsilon \to 0} \langle S(\xi)^* f | \chi_\varepsilon \rangle \ ,$$

where $\chi_\varepsilon(x) := (1 - i\varepsilon x)^{-1}$. Plugging the second formula into the first one, we infer

$$\begin{aligned} f(z) &= \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_0^\infty \mathrm{e}^{iz\xi} \langle S(\xi)^* f | \chi_\varepsilon \rangle d\xi \\ &= \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_0^\infty \mathrm{e}^{iz\xi} \langle \mathrm{e}^{-i\xi G} f | \chi_\varepsilon \rangle d\xi \\ &= \lim_{\varepsilon \to 0} \frac{1}{2i\pi} \langle (G - z\mathrm{Id})^{-1} f | \chi_\varepsilon \rangle \\ &= \frac{1}{2i\pi} I_+[(G - z\mathrm{Id})^{-1} f] \ . \end{aligned}$$

Proof (line), continued

Since $u(t)$ is real valued, $u(t) = \Pi u(t) + \overline{\Pi u(t)}$.
The previous inverse Fourier transform formula reads

$$\forall z \in \mathbb{C}_+ \ , \ \Pi u(t,z) = \frac{1}{2i\pi} \lim_{\varepsilon \to 0^+} \langle (G - z\mathrm{Id})^{-1} \Pi u(t) | (1 - i\varepsilon x)^{-1} \rangle \ .$$

Apply the unitary operator $U(t)^*$ to both sides of this inner product,

$$\forall z \in \mathbb{C}_+ \ , \ \Pi u(t,z) = \frac{1}{2i\pi} \langle (U(t)^* G U(t) - z\mathrm{Id})^{-1} U(t)^* \Pi u(t) | U(t)^* (1 - i\varepsilon x)^{-1} \rangle \ .$$

Proof (line), continued

Since $u(t)$ is real valued, $u(t) = \Pi u(t) + \overline{\Pi u(t)}$.
The previous inverse Fourier transform formula reads

$$\forall z \in \mathbb{C}_+ \ , \ \Pi u(t,z) = \frac{1}{2i\pi} \lim_{\varepsilon \to 0^+} \langle (G - z\mathrm{Id})^{-1} \Pi u(t) | (1 - i\varepsilon x)^{-1} \rangle \ .$$

Apply the unitary operator $U(t)^*$ to both sides of this inner product,

$$\forall z \in \mathbb{C}_+ \ , \ \Pi u(t,z) = \frac{1}{2i\pi} \langle (U(t)^* G U(t) - z\mathrm{Id})^{-1} U(t)^* \Pi u(t) | U(t)^* (1 - i\varepsilon x)^{-1} \rangle \ .$$

Again, we are going to calculate explicitly

$$\lim_{\varepsilon \to 0^+} U(t)^* (1 - i\varepsilon x)^{-1}, U(t)^* \Pi u(t), U(t)^* G U(t)$$

using $U'(t) = B_{u(t)} U(t)$ and some commutator identity.

Proof(line), continued

Lemma

$$[G, B_u] = -2L_u + i[L_u^2, G] \ .$$

Assume this lemma and calculate

$$\begin{aligned} \frac{d}{dt} U(t)^* G U(t) &= U(t)^* [G, B_{u(t)}] U(t) \\ &= U(t)^* (-2L_{u(t)} + i[L_{u(t)}^2, G]) U(t) \\ &= -2L_{u_0} + i[L_{u_0}^2, U(t)^* G U(t)] \ . \end{aligned}$$

Integrating this ODE, we get

$$U(t)^* G U(t) = -2tL_{u_0} + \mathrm{e}^{itL_{u_0}^2} G \mathrm{e}^{-itL_{u_0}^2} \ .$$

Proof(line), continued

Recall — see the first lecture— $\partial_t \Pi u = iL_u^2(\Pi u) + B_u(\Pi u)$. We infer

$$\frac{d}{dt}U(t)^*\Pi u(t) = U(t)^*(\partial_t \Pi u(t) - B_{u(t)}\Pi u(t)) = iU(t)^* L_{u(t)}^2 \Pi u(t)$$
$$= iL_{u_0}^2 U(t)^* \Pi u(t) \ ,$$

from which we conclude $U(t)^*\Pi u(t) = \mathrm{e}^{itL_{u_0}^2}\Pi u_0$.
Finally, we have

$$\frac{d}{dt}U(t)^*\chi_\varepsilon = -U(t)^* B_{u(t)}\chi_\varepsilon = -iU(t)^*(T_{|D|u(t)}\chi_\varepsilon - T_{u(t)}^2\chi_\varepsilon)$$

and the right hand side converges in L^2_+ to

$$-iU(t)^*(D\Pi u(t) - T_{u(t)}\Pi u(t)) = -iU(t)^* L_{u(t)}\Pi u(t)$$
$$= -iL_{u_0}U(t)^*\Pi u(t) = -iL_{u_0}\mathrm{e}^{itL_{u_0}^2}\Pi u_0 = \lim_{\varepsilon\to 0} iL_{u_0}^2 \mathrm{e}^{itL_{u_0}^2}\chi_\varepsilon \ .$$

By integrating in time, we infer $U(t)^*\chi_\varepsilon - \mathrm{e}^{itL_{u_0}^2}\chi_\varepsilon \to 0$ in L^2_+.

Proof(line), conclusion

Plugging the obtained identities

$$\begin{aligned} U(t)^*\Pi u(t) &= \mathrm{e}^{itL_{u_0}^2}\Pi u_0 \ , \ U(t)^*\chi_\varepsilon - \mathrm{e}^{itL_{u_0}^2}\chi_\varepsilon \underset{\varepsilon\to 0^+}{\longrightarrow} 0 \ , \\ U(t)^* G U(t) &= -2tL_{u_0} + \mathrm{e}^{itL_{u_0}^2} G \mathrm{e}^{-itL_{u_0}^2} \ , \end{aligned}$$

into the formula

$$\forall z \in \mathbb{C}_+ \ , \ \Pi u(t,z) = \frac{1}{2i\pi}\langle (U(t)^* G U(t) - z\mathrm{Id})^{-1} U(t)^*\Pi u(t) | U(t)^*\chi_\varepsilon\rangle \ ,$$

we infer

$$\begin{aligned} \Pi u(t,z) &= \lim_{\varepsilon\to 0} \frac{1}{2i\pi}\langle \left(\mathrm{e}^{itL_{u_0}^2} G \mathrm{e}^{-itL_{u_0}^2} - 2tL_{u_0} - z\mathrm{Id}\right)^{-1} \mathrm{e}^{itL_{u_0}^2}\Pi u_0 | \mathrm{e}^{itL_{u_0}^2}\chi_\varepsilon\rangle \\ &= \lim_{\varepsilon\to 0} \frac{1}{2i\pi}\langle (G - 2tL_{u_0} - z\mathrm{Id})^{-1}\Pi u_0 | \chi_\varepsilon\rangle \\ &= \frac{1}{2i\pi} I_+[(G - 2tL_{u_0} - z\mathrm{Id})^{-1}\Pi u_0] \ . \quad \square \end{aligned}$$

Proof of the lemma

Lemma

$$[G, B_u] = -2L_u + i[L_u^2, G] \ .$$

Proof. For every $f \in \mathrm{Dom}(G)$, $b \in H^1(\mathbb{R})$, then $T_b f \in \mathrm{Dom}(G)$ and $[G, T_b]f = \frac{i}{2\pi} I_+(f)\Pi b$. Using this identity and $[G, D] = i\mathrm{Id}$, we obtain

$$\forall f \in \mathrm{Dom}(G) \cap \mathrm{Dom}(L_u) \ , \ [G, L_u]f = if - \frac{i}{2\pi} I_+(f)\Pi u \ .$$

We infer, for $f \in \mathrm{Dom}(G) \cap \mathrm{Dom}(L_u^2)$,

$$\begin{aligned} [G, B_u]f &= i([G, T_{|D|u}]f - T_u[G, T_u]f - [G, T_u]T_u f) \\ &= \frac{i}{2\pi}(iI_+(f)(D\Pi u - T_u\Pi u) - iI_+(T_u f)\Pi u) \\ &= \frac{i}{2\pi}(iI_+(f)L_u\Pi u + iI_+(L_u f)\Pi u) \\ &= i(L_u(if - [G, L_u]f) + iL_u f - [G, L_u]L_u f) \\ &= -2L_u f + i[L_u^2, G]f \ . \quad \square \end{aligned}$$