Lectures on the Benjamin–Ono equation as an integrable PDE

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Lecture 3. Applications of the explicit formulae : low regularity wellposedness, multisolitons, zero–dispersion limit.

Recall from the second lecture

Theorem (PG, 2022)

On the torus, the solution $u \in C(\mathbb{R}, H^2_{\rm real}(\mathbb{T}))$ *of the Benjamin–Ono equation with* $u(0) = u_0$ *is given by*

$$\begin{aligned} u(t) &= \Pi u(t) + \overline{\Pi u(t)} - \langle u_0 | 1\rangle \ , \\ \forall z \in \mathbb{D}\ ,\ \Pi u(t,z) &= \langle (\mathrm{Id} - z\mathrm{e}^{it}\mathrm{e}^{2itL_{u_0}}S^*)^{-1}\Pi u_0 | 1\rangle \end{aligned}$$

On the line, the solution $u \in C(\mathbb{R}, H^2_{\rm real}(\mathbb{R}))$ *of the Benjamin–Ono equation with* $u(0) = u_0$ *is given by*

$$\begin{aligned} u(t) &= \Pi u(t) + \overline{\Pi u(t)} \ , \\ \forall z \in \mathbb{C}_+\ ,\ \Pi u(t,z) &= \frac{1}{2i\pi} I_+[(G - 2tL_{u_0} - z\mathrm{Id})^{-1}\Pi u_0] \ . \end{aligned}$$

The meaning of the explicit formula on the line

$$\forall z \in \mathbb{C}_+ \ , \ \Pi u(t,z) = \frac{1}{2i\pi} I_+[(G - 2tL_{u_0} - z\mathrm{Id})^{-1}\Pi u_0] \ .$$

We claim that the operator

$$A_t := -i(G - 2tL_0)$$

is maximally dissipative, in the sense that, for every λ with $\mathrm{Re}(\lambda) > 0$, $\lambda\mathrm{Id} - A : \mathrm{Dom}(A) \to L^2_+$ is onto and if, for every $f \in \mathrm{Dom}(A)$, $\mathrm{Re}\langle Af|f\rangle \leq 0$. Indeed, the expression of A_t in the Fourier representation is given by

$$\widehat{A_t f}(\xi) = \frac{d}{d\xi}\hat{f}(\xi) + 2it\xi\hat{f}(\xi) \ ,$$

and therefore it is easy to check by explicit calculations that

$$\mathrm{Dom}(A_t) = \{f \in L^2_+(\mathbb{R}) : \xi \mapsto \mathrm{e}^{it\xi^2}\hat{f}(\xi) \in H^1(0,\infty)\}$$

with $\forall f \in \mathrm{Dom}(A_t) \ , \ \mathrm{Re}\langle A_t f|f\rangle \leq 0$, and that $A_t + iz\mathrm{Id}$ is bijective for every $z \in \mathbb{C}_+$.

The meaning of the explicit formula on the line, continued

From standard perturbation theory, we infer that, for every bounded antiselfadjoint operator B on $L^2_+(\mathbb{R})$, $A_t + B$ is maximally dissipative. In particular, if $u_0 \in L^\infty(\mathbb{R}) \cap L^2_{\mathrm{real}}(\mathbb{R})$,

$$-i(G - 2tL_{u_0}) = A_t - 2itT_{u_0}$$

is maximally dissipative, so that the formula

$$\forall z \in \mathbb{C}_+ \ , \ \Pi u(t,z) = \frac{1}{2i\pi} I_+[(G - 2tL_{u_0} - z\mathrm{Id})^{-1}\Pi u_0]$$

still holds, and in fact provides a formula for the extension of the flow map of the Benjamin–Ono equation to $H^s_{\mathrm{real}}(\mathbb{R})$ for every $s > 1/2$, as obtained by Molinet–Pilod (2012) by using Tao's normal form (2004). More to be done in this direction of low regularity wellposedness on the line (wellposedness on $L^2(\mathbb{R})$:Ionescu–Kenig (2007), ..., H^s for $s < 0$?)

Low regularity extension of the flow map on the torus

Lemma (PG-Kappeler-Topalov, 2020)

If $u \in H^s_{\mathrm{real}}(\mathbb{T})$ with $s > -1/2$, then L_u is selfadjoint on $L^2_+(\mathbb{T})$.

Proof. If $s = 0$, easy because $T_u f$ is well defined for

$$u \in L^2(\mathbb{T}) \ , \ f \in \mathrm{Dom}(L_u) = H^1_+(\mathbb{T}) \subset L^\infty(\mathbb{T}) \ .$$

If $s < 0$, use the Hermitian form

$$Q_u(f,g) := \langle Df|g\rangle - \langle u|g\overline{f}\rangle \ , \ f, g \in H^{1/2}_+(\mathbb{T}) := H^{1/2}(\mathbb{T}) \cap L^2_+(\mathbb{T}) \ .$$

Then

$$\forall \varepsilon > 0 \ , \ g\overline{f} \in H^{1/2-\varepsilon}(\mathbb{T}) \subset H^{-s}(\mathbb{T})$$

and, for some $C_u > 0$, $Q_u(f,g) + C_u\langle f|g\rangle$ is positive definite on $H^{1/2}_+(\mathbb{T})$. Then define $\langle L_u f|g\rangle := Q_u(f,g)$ if f belongs to

$$\mathrm{Dom}(L_u) := \{f \in H^{1/2}_+(\mathbb{T}) : \forall g \in H^{1/2}_+(\mathbb{T}), |Q_u(f,g)| \leq A\|g\|_{L^2}\} \ .$$

Low regularity extension on the torus, continued

If $u_0 \in H^s_{\mathrm{real}}(\mathbb{T})$, $s > -1/2$, the operator $\mathrm{e}^{-2itL^2_{u_0}}$ sends the form domain $H^{1/2}_+(\mathbb{T})$ into itself, and therefore, for every $z \in \mathbb{D}$, the vector

$$f_t(z) := (I - \overline{z} S \mathrm{e}^{-it - 2itL^2_{u_0}})^{-1}(1)$$

belongs to $H^{1/2}_+(\mathbb{T})$. Consequently, the inner product

$$\langle \Pi u_0 | f_t(z) \rangle = \langle (\mathrm{Id} - z \mathrm{e}^{it} \mathrm{e}^{2itL_{u_0}} S^*)^{-1} \Pi u_0 | 1 \rangle$$

is well defined and provides an extension of the flow map to $H^s_{\mathrm{real}}(\mathbb{T})$ for every $s > -1/2$.

One can check [PG–Kappeler–Topalov] that this extension is strongly continuous, and that this extension is optimal : there exists a sequence of smooth data u_0^ε which converge to 0 in $H^{-1/2}_{\mathrm{real}}(\mathbb{T})$ and such that the corresponding solution $u^\varepsilon(t)$ has no limit in $\mathcal{D}'(\mathbb{T})$ on any time interval of positive length.

Solitons

A soliton solution of the Benjamin–Ono equation is a solution of the form $u(t,x) = u_0(x - ct)$. Amick and Toland (1991) characterized them as

- On the line,

$$u_0(x) = \frac{2\lambda}{\lambda^2 + (x - x_0)^2} \ , \ \lambda > 0.$$

- On the torus,

$$u_0(x) = \frac{N(1 - r^2)}{1 - 2r\cos(N(x - x_0)) + r^2} + c \ , \ N \in \mathbb{Z}_{\geq 1}, c \in \mathbb{R} \ .$$

Nice properties w.r.t. L_{u_0} and Hardy, e.g. on the line

$$u_0(x) = \frac{2}{1 + x^2} = \frac{i}{x + i} - \frac{i}{x - i} \ ,$$

we have $L_{u_0}(\Pi u_0) = -\frac{1}{2}\Pi u_0$, $G(\Pi u_0) = -i\Pi u_0$ so that

$$(G - 2tL_{u_0} - z\mathrm{Id})^{-1}\Pi u_0 = \frac{\Pi u_0}{-i + t - z} \ , \ \Pi u(t,z) = \frac{1}{2i\pi} I_+(\Pi u_0) = \frac{i}{z - t + i}$$

and $u(t,x) = u_0(x - t)$.

Multisolitons on the line

Theorem (Ruoci Sun, 2021)

Given $u_0 \in L^2 - \mathrm{real}(\mathbb{R}, (1+x^2)dx)$, the following conditions are equivalent

- $u_0(x) = \sum_{j=1}^N \frac{2\lambda_j}{\lambda_j^2 + (x - x_j)^2}$.
- L_{u_0} *has exactly* N *simple negative eigenvalues and* Πu_0 *is a linear combination of the eigenfunctions.*

Then the eigenfunctions of L_u *span a space of rational functions of the form* $\mathbb{C}_{N-1}[x]/Q_u(x)$ *where* $\deg Q_u = N$.

In this case, the explicit formula

$$\forall z \in \mathbb{C}_+ \ , \ \Pi u(t,z) = \frac{1}{2i\pi} I_+[(G - 2tL_{u_0} - z\mathrm{Id})^{-1}\Pi u_0]$$

lead to a finite dimensional linear system and to an expression of the form

$$u(t,x) = \sum_{j=1}^N \frac{2\lambda_j(t)}{(x - x_j(t))^2 + \lambda_j(t)^2} .$$

The zero–dispersion limit

Problem : study the $\varepsilon \to 0$ limit of u^ε defined by

$$\partial_t u^\varepsilon = \partial_x(\varepsilon|D_x|u^\varepsilon - (u^\varepsilon)^2)\ ,\ u^\varepsilon(0) = u_0\ .$$

Theorem (PG, 2023, improves Miller–Xu (2011), Miller–Wetzel (2016), Gassot (2022))

If $u_0 \in L^\infty \cap L^2_{\mathrm{real}}$, then $u^\varepsilon(t) \rightharpoonup u(t)$ for every t, with

1. *On the torus,* $u(t) = \Pi u(t) + \overline{\Pi u(t)} - \langle u_0|1\rangle$,

$$\forall z \in \mathbb{D}\ ,\ \Pi u(t,z) = \langle (\mathrm{Id} - z\mathrm{e}^{-2itT_{u_0}}S^*)^{-1}\Pi u_0|1\rangle\ .$$

2. *On the line,* $u(t) = \Pi u(t) + \overline{\Pi u(t)}$,

$$\forall z \in \mathbb{C}_+\ ,\ \Pi u(t,z) = \frac{1}{2i\pi}I_+[(G + 2tT_{u_0} - z\mathrm{Id})^{-1}\Pi u_0)]\ .$$

Proof

Denote by $\Phi(t)$ the Benjamin–Ono flow map. Then one checks that

$$u^\varepsilon(t) = \varepsilon\Phi(\varepsilon t)\left[\frac{u_0}{\varepsilon}\right] \ .$$

Plug this identity in the explicit formulae with $L_{u_0} = -i\partial_x - T_{u_0}$, On the torus,

$$\begin{aligned} u^\varepsilon(t) &= \Pi u^\varepsilon(t) + \overline{\Pi u^\varepsilon(t)} - \langle u_0|1\rangle \ , \\ \forall z \in \mathbb{D} \ , \ \Pi u^\varepsilon(t,z) &= \langle (\mathrm{Id} - z\mathrm{e}^{i\varepsilon t}\mathrm{e}^{2i\varepsilon tD - 2itT_{u_0}}S^*)^{-1}\Pi u_0|1\rangle \end{aligned}$$

On the line,

$$\begin{aligned} u^\varepsilon(t) &= \Pi u^\varepsilon(t) + \overline{\Pi u^\varepsilon(t)} \ , \\ \forall z \in \mathbb{C}_+ \ , \ \Pi u^\varepsilon(t,z) &= \frac{1}{2i\pi} I_+[(G - 2t\varepsilon D + 2tT_{u_0} - z\mathrm{Id})^{-1}\Pi u_0] \ . \end{aligned}$$

Then use the strong convergence of the resolvents.

Special cases

Some special cases can be calculated explicitly. For instance, on the line, if u_0 is a rational function. Indeed, G and T_{u_0} act on rational functions,

$$Gf(x) = xf(x) - \lim_{x\to\infty} xf(x) \ , \ T_{u_0}f(x) = u_0(x)f(x) - \sum_{j=1}^{N}\sum_{k=1}^{m_j} \frac{c_{j,k}(f)}{(x-\overline{p}_j)^k} \ ,$$

which reduce the calculation of the resolvent of $G + 2tT_{u_0}$ acting on Πu_0 to a finite linear system.

Let us consider the special case

$$u_0(x) = \frac{2}{1+x^2} \ , \ \Pi u_0(x) = \frac{i}{x+i} \ .$$

The special case $u_0(x) = \frac{2}{1+x^2}$

In this case, one has

$$(G + 2tT_{u_0} - z\mathrm{Id})^{-1}\Pi u_0(y) = \frac{\lambda(t,z) + \frac{\mu(t,z)}{y-i} + \frac{i}{y+i}}{y + \frac{4t}{1+y^2} - z} \ ,$$

where the constants $\lambda(t,z)$, $\mu(t,z)$ are chosen so that the rational function in the right hand side has no pole in the upper half–plane. Two different cases may occur for $(t,x) \in \mathbb{R} \times \mathbb{R}$.

- The equation $y + \frac{4t}{1+y^2} = x$ has only one real solution $y(t,x)$. In this case,

$$u(t,x) = u_0(y(t,x))$$

  is the smooth solution of the Burgers equation obtained from the method of characteristics.
- The equation $y + \frac{4t}{1+y^2} = x$ has three real solutions $y_1(t,x) < y_2(t,x) < y_3(t,x)$, and

$$u(t,x) = \sum_{j=1}^{3} (-1)^{j-1} u_0(y_j(t,x)) \ .$$

Perspectives

- Describe the oscillating part in the zero–dispersion limit (Dubrovin,s conjecture, see Claeys–Grava (2009) in the KdV case).
- Soliton resolution : describe the long time Benjamin–Ono dynamics on the line. Expected : every (generic ?) solution u reads, as $t \to \infty$,

$$u(t,x) = \sum_{j=1}^{N} \frac{2\lambda_j(t)}{(x - x_j(t))^2 + \lambda_j(t)^2} + r(t,x)\ ,\ \ r(t) \simeq \mathrm{e}^{t\partial_x |D_x|} r_\infty\ .$$

generalizing the formula for multi–solitons ($r(t,x) = 0$).