

# Lectures on the Benjamin–Ono equation as an integrable PDE

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The Benjamin–Ono equation was introduced in 1967 as a model of long, one-way internal gravity waves in a two-layer fluid. The unknown  $u$  is a real variable function of two real variables  $t, x$ , and the equation reads

$$\partial_t u = \partial_x (|D|u - u^2) ,$$

where  $|D|$  denotes the Fourier multiplier associated to the symbol  $|\xi|$ . In addition to the contributions in the Fluid Mechanics literature, this equation has been the center of many mathematical works, mainly for the following two reasons.

- It is an example of a quasilinear dispersive equation, on which techniques issued from harmonic analysis and normal form theory has been successfully developed.
- It is also an example of an integrable PDE, in the strong sense that it admits a Lax pair structure leading to explicit descriptions of the solutions.

Here I will try to explain why this equation is one of the best introductory examples to the theory of integrable PDEs. **No prerequisite on integrability is necessary** to follow these lectures.

## Plan of the lectures.

- (1) The initial value problem for sufficiently smooth data on the line and on the torus. The Lax pair. Global wellposedness.
- (2) The explicit formula on the torus and on the line.
- (3) Applications to low regularity wellposedness and to (multi)–solitons.

## REFERENCES

- [1] T. BENJAMIN, *Internal waves of permanent form in fluids of great depth*, J. Fluid Mech., 29(1967), 559–592.
- [2] P. GÉRARD, T. KAPPELER, *On the integrability of the Benjamin–Ono equation on the torus*, Comm. Pure Appl. Math. 74 (2021), 1685–1747.
- [3] P. GÉRARD, *An explicit formula for the Benjamin–Ono equation*, arXiv:2212.03139
- [4] H. ONO, *Algebraic solitary waves in stratified fluids*, J. Physical Soc. Japan 39(1975), 1082–1091.