ON THE ROLE OF GAUSSIAN CURVATURE IN HARMONIC ANALYSIS

RAINER MANDEL

Abstract. Die Gaußsche Krümmung einer Hyperfläche des Euklidischen Raumes ist als das Produkt ihrer Hauptkrümmungen definiert. Ihre immense Bedeutung in der Mathematik speist sich beispielsweise aus dem Satz von Gauß-Bonnet, welcher besagt, dass das Integral der Gaußkrümmung über die gegebene Fläche ihren topologischen Typ eindeutig bestimmt. Weit weniger intuitiv ist die Tatsache, dass die Gaußsche Krümmung einer gegebenen Hyperfläche $M \subset \mathbb{R}^n$ maßgeblich darüber entscheidet, für welche Funktionen $f \in L^p(\mathbb{R}^n)$ die Restriktion der Fouriertransformierten auf M, also $\hat{f}|_M$, in einem geeigneten Sinne wohldefiniert ist. In diesem Vortrag sollen das Zusammenspiel von Gaußscher Krümmung und Fouriertransformation beleuchtet und Anwendungen auf die Theorie linearer Wellen- und Schrödingergleichungen skizziert werden.

Foreword

In this document I elaborate on a talk entitled "Über die Rolle der Gaußschen Krümmung in der Harmonischen Analysis" that I held on December 4th 2019 at the Karlsruhe Institute of Technology on the occasion of my "Habilitationskolloquium". As the abstract indicates, it deals with the role of Gaussian curvature in Harmonic Analysis – a topic that I came accross in the last years. Since my research is not directly related to it, I do not and cannot present advanced theory, but rather concentrate on a short and (hopefully) insightful introduction to the restriction problem for the Fourier transform. As a permanent source of inspiration in the preparation of this talk I would like to single out Stein's book [47]. For further reading about related topics I recommend the books [36, 37, 42, 48] and the article [49] dealing with several other curvature-related problems in Harmonic Analysis which are not included in these notes. I would like to stress that most of the reasonings presented below are not entirely proved within these notes - so a lot more details should be added.

1. Fourier analysis in $\mathbb{R}^n$

In the following we discuss some properties of the Fourier transform, which is defined by

$$(\mathcal{F}_n f)(\xi) := \hat{f}(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\, dx \qquad \text{where } f \in L^1(\mathbb{R}^n). \tag{1}$$

We recall that a measurable function f belongs to $L^q(\mathbb{R}^n)$ for $1 \le q \le \infty$ if and only if the corresponding norm

$$\|f\|_q := \left(\int_{\mathbb{R}^n} |f(x)|^q\, dx \right)^{1/q} \quad (1 \le q < \infty), \qquad \|f\|_\infty := \operatorname{ess\,sup}_{\mathbb{R}^n} |f|$$

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is finite. A basic and important fact is that the Lebesgue spaces $(L^q(\mathbb{R}^n), \|\cdot\|_q)$ are complete, i.e., they are Banach spaces. In any first course on Harmonic Analysis it is shown how to define the Fourier transform for functions in $L^2(\mathbb{R}^n)$ (actually even for tempered distributions). This is quite remarkable given that the integral appearing in (1) need not be absolutely convergent in this case. It is defined by approximation, namely by

$$\hat{f} := L^2(\mathbb{R}^n) - \lim_{R\to\infty} \widehat{f\chi_{B_R(0)}} \qquad \text{where } f \in L^2(\mathbb{R}^n). \tag{2}$$

Here, $\chi_{B_R(0)}$ is the indicator function of the ball with center 0 and radius R in $\mathbb{R}^n$. It is a basic but nontrivial fact that this limit is well-defined. This follows from the density of Schwartz functions in $L^2(\mathbb{R}^n)$ and Plancherel's theorem, i.e., from the isometry property of the Fourier transform in $L^2(\mathbb{R}^n)$. Notice that such approximations are heavily used in the theory of oscillatory integrals and in Harmonic Analysis in general. In the following we first discuss which integrability and decay properties of $\hat{f}$ can be deduced from f. The fundamental result in this direction is the Hausdorff-Young inequality.

Theorem 1 (Hausdorff-Young). *The inequality $\|\hat{f}\|_q \lesssim \|f\|_p$ holds if and only if $q = p' := \frac{p}{p-1}$ and $1 \le p \le 2$.*

Proof. "$\Leftarrow$" This direction follows from the trivial bound $\|\hat{f}\|_\infty \lesssim \|f\|_1$ and Plancherel's identity $\|\hat{f}\|_2 = \|f\|_2$ via the Riesz-Thorin interpolation theorem.
"$\Rightarrow$" Assuming that the inequality holds for all $f \in L^p(\mathbb{R}^n)$, we deduce for the functions $f_\lambda(x) := f(\lambda x)$ with $\lambda > 0$

$$\lambda^{-n+\frac{n}{q}} \|\hat{f}\|_q = \|\lambda^{-n}\hat{f}(\lambda^{-1}\cdot)\|_q = \|\widehat{f_\lambda}\|_q \lesssim \|f_\lambda\|_p = \lambda^{-\frac{n}{p}}\|f\|_p.$$

Since this holds for all $\lambda > 0$ we deduce $p = q'$. The necessity of $1 \le p \le 2$ is not that easy to see. We refer to [52, p.21] for a sketch[1]. $\square$

Remark 1.

(a) *The inequality goes back to Young [55, p.80] for $p' \in \{4,6,8,\dots\}$ and Hausdorff [18] generalized it to all $p' \in [2,\infty]$, which means $p \in [1,2]$. Both papers actually deal with the corresponding result for Fourier series instead of the Fourier transform.*

(b) *The above proof provides an explicit constant for the Hausdorff-Young inequality, which, however, is not sharp. Thanks to Babenko [1] for $p' \in \{4,6,8,\dots\}$ and Beckner [4] for all $p' \in [2,\infty)$ we know that the best possible inequality reads*

$$\|\hat{f}\|_{p'} \le (2\pi)^{n(\frac{1}{p'}-\frac{1}{2})} \left(p^{\frac{1}{p}}(p')^{-\frac{1}{p'}}\right)^{\frac{n}{2}} \|f\|_p \qquad \text{where } 1 \le p \le 2.$$

Notice that this constant differs from the one in [1,4] because of the slightly different definition of the Fourier transform that we use. Maximizers for this inequality are known to exist. Beckner [4] and Lieb [30] proved that in case $1 < p < 2$ all of them

[1]The idea is to consider a sequence of functions f_k consisting of k bumps of the form $e^{-|x|^2}$ that are distributed over $\mathbb{R}^n$ with sufficiently large (and k-dependent) distance to each other. Then one shows $\|\hat{f}_k\|_{p'}\|f_k\|_p^{-1} \sim k^{1-\frac{2}{p}}$ as $k\to\infty$, which implies $p \le 2$ and thus the result.

are multiples of $x \mapsto e^{-x\cdot Ax + v\cdot x}$ *where the real part of the matrix* $A \in \mathbb{C}^{n\times n}$ *is positive definite and* $v \in \mathbb{C}^n$*.*

(c) *Since we are interested in restrictions of the Fourier transform to compact manifolds, one may first investigate the properties of* $\hat{f}|_\Omega = \hat{f}\chi_\Omega$ *where* $\Omega \subset \mathbb{R}^n$ *is an "n-dimensional" set of positive finite measure with indicator function* χ_Ω*. From the Hausdorff-Young inequality one gets for* $q \geq 2$

$$\|\hat{f}\chi_\Omega\|_q \lesssim \|\hat{f}\|_q \lesssim \|f\|_{q'}, \qquad \|\hat{f}\chi_\Omega\|_q \lesssim \|\hat{f}\|_\infty \|\chi_\Omega\|_q \lesssim \|f\|_1.$$

Interpolating these two estimates we find

$$\|\hat{f}\|_{L^q(\Omega)} \lesssim \|f\|_p \qquad \text{for } 1 \leq p \leq 2,\ 2 \leq q \leq p'. \tag{3}$$

(d) *For* $f \in L^1(\mathbb{R}^n)$ *we actually have more information than* $\hat{f} \in L^\infty(\mathbb{R}^n)$*. Indeed, the dominated convergence theorem and the Riemann-Lebesgue-Lemma imply that* $\hat{f}$ *is uniformly continuous and converges to zero at infinity.*

Part (d) tells us that the assumption $f \in L^1(\mathbb{R}^n)$ provides pointwise information for its Fourier transform $\hat{f}$. On the other hand, the Fourier inversion theorem shows that $f \in L^2(\mathbb{R}^n)$ does not give any more than $\hat{f} \in L^2(\mathbb{R}^n)$, which provides no pointwise information at all. So we ask the question in which sense this pointwise information gets lost. To which sets can we restrict $\hat{f}$ and in which sense? The surprising answer is that for general submanifolds $M \subset \mathbb{R}^n$ the integrability properties of $\hat{f}|_M$ with respect to the induced measure essentially depend on "curvature properties" of M. We will see why!

2. Fourier restriction to curved hypersurfaces in $\mathbb{R}^n$

We consider the question in which sense the Fourier transform may be restricted to smooth m-dimensional submanifolds $M \subset \mathbb{R}^n$ that we equip with the canonical measure μ induced by the Lebesgue measure on $\mathbb{R}^m$. We concentrate on the case of a hypersurface $m = n-1$. The Fourier transform of some function $f \in L^1(\mathbb{R}^n)$ is uniformly continuous and in particular the function $\hat{f}|_M$ is (pointwise) well-defined for every subset $M \subset \mathbb{R}^n$. Is this true if we assume $f \in L^p(\mathbb{R}^n)$ for some $1 < p < 2$? Notice that the delicate point here is that M is a set of Lebesgue measure zero so that along the way from $p = 1$ to $p = 2$ we definitely have to lose pointwise information. We show that a small but very important amount of information about $\hat{f}|_M$ persists provided that the submanifold M is a hypersurface with positive Gaussian curvature. Then $\hat{f}|_M$ is well-defined as an element of $L^2(d\mu)$ whenever $f \in L^p(\mathbb{R}^n)$ with $1 \leq p \leq \frac{2(n+1)}{n+3}$. Recall that

$$g \in L^2(d\mu) \;\Leftrightarrow\; \|g\|_{L^2(d\mu)} := \left(\int_M |g|^2\, d\mu\right)^{1/2} < \infty \qquad \text{where } \mu(A) = \mathcal{H}^{n-1}(A \cap M),$$

where A is a Lebesgue-measurable subset of $\mathbb{R}^n$. This is the famous theorem of Tomas and Stein [54], which we recapitulate below. For a better understanding of this result let us mention that the dual characterization of Lebesgue spaces yields $(1 \le q,p < \infty)$

$$\|\hat{f}\|_{L^q(d\mu)} \lesssim \|f\|_p \quad \forall f \in L^p(\mathbb{R}^n) \quad \Leftrightarrow \quad \|\widehat{g\,d\mu}\|_{p'} \lesssim \|g\|_{L^{q'}(d\mu)} \quad \forall g \in L^{q'}(d\mu)$$
$$\text{where} \quad \widehat{g\,d\mu}(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_M g(x) e^{-ix\cdot\xi}\,d\mu(x). \tag{4}$$

Why is curvature necessary? Assume for the moment that the manifold M is entirely contained in the subspace $\mathbb{R}^k \times \{0\}^{n-k} \subset \mathbb{R}^n$. Then $\widehat{g\,d\mu}$ does not depend on the last k coordinates and in particular no decay to zero is possible unless the function is identically zero. In particular, for generic $g \in C^\infty(M)$ the function $\widehat{g\,d\mu}$ does not lie in any Lebesgue space $L^{p'}(\mathbb{R}^n)$ with $p > 1$, so no estimate of the form (4) can be expected to hold for $p > 1$. This observation remains true if we substitute $\mathbb{R}^k \times \{0\}^{n-k}$ by any proper affine subspace of $\mathbb{R}^n$ and consider a suitable ξ-direction. As a consequence, in order to prove $\hat{f}|_M \in L^q(d\mu)$ we have to require that the manifold bends away from its tangent spaces in a sufficiently strong manner. This is the motivation for imposing a positive lower bound on the Gaussian curvature of M.

Theorem 2 (Herz (1962), Hlawka (1950))**.** *Let $M \subset \mathbb{R}^n$ be a smooth, compact and closed hypersurface with positive Gaussian curvature. Then its canonical surface measure μ satisfies*

$$|\widehat{d\mu}(\xi)| \lesssim (1+|\xi|)^{\frac{1-n}{2}}.$$

If additionally $M = \partial\Omega$ for a bounded set $\Omega \subset \mathbb{R}^n$ then

$$|\hat{\chi}_\Omega(\xi)| \lesssim (1+|\xi|)^{-\frac{n+1}{2}}.$$

Proof. (sketch!) Assume $M = \{(x', \phi(x')) : x' \in \mathbb{R}^{n-1}\}$ with $\phi \in C^\infty(\mathbb{R}^{n-1}, \mathbb{R})$. Then we have for all $\psi \in C_0^\infty(\mathbb{R}^n)$ with small support near the point $(0, \phi(0)) \in M$

$$\begin{aligned}
\widehat{\psi\,d\mu}(\xi) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \psi(x) e^{-ix\cdot\xi}\,d\mu(x) \\
&= \int_{\mathbb{R}^{n-1}} (2\pi)^{-n/2} \psi(x', \phi(x')) \sqrt{1+|\phi(x')|^2} e^{-i(x'\cdot\xi' + \phi(x')\xi_n)}\,dx' \\
&= \int_{\mathbb{R}^{n-1}} \tilde{\psi}(x') e^{-i(x'\cdot\xi' + \phi(x')\xi_n)}\,dx' \\
&\approx e^{-i\phi(0)\xi_n} \int_{\mathbb{R}^{n-1}} \tilde{\psi}(x') e^{-i(\xi' + \xi_n \nabla\phi(0))\cdot x'} e^{-i\frac{1}{2}\xi_n \langle x', D^2\phi(0) x'\rangle}\,dx'
\end{aligned}$$

where $\tilde{\psi} \in C_0^\infty(\mathbb{R}^{n-1}, \mathbb{R})$. The method of nonstationary phase yields

$$\xi' + \xi_n \nabla\phi(0) \neq 0 \quad \Rightarrow \quad |\widehat{\psi\,d\mu}(\xi)| \lesssim_N (1+|\xi|)^{-N} \quad (N \in \mathbb{N}).$$

Since the Gaussian curvature of M does not vanish at the point $(0, \phi(0)) \in M$ by assumption, the Hessian matrix $D^2\phi(0) \in \mathbb{R}^{(n-1)\times(n-1)}$ is invertible and thus

$$\xi' + \xi_n \nabla\phi(0) = 0 \quad \Rightarrow \quad |\widehat{\psi\,d\mu}(\xi)| \lesssim (1+|\xi_n|)^{\frac{1-n}{2}} \lesssim (1+|\xi|)^{\frac{1-n}{2}}. \tag{5}$$

Indeed, for large $|\xi_n|$ this follows from[2]

$$\begin{aligned}|\widehat{\psi\, d\mu}(\xi)| &\approx \left|\int_{\mathbb{R}^{n-1}} \tilde{\psi}(x')e^{-i\frac{1}{2}\xi_n\langle x', D^2\phi(0)x'\rangle}\,dx'\right| \\ &= \left|\int_{\mathbb{R}^{n-1}} \mathcal{F}_n(\tilde{\psi})(\eta)\overline{\mathcal{F}_n(e^{i\xi_n\langle x', D^2\phi(0)x'\rangle})(\eta)}\,d\eta\right| \\ &\lesssim \int_{\mathbb{R}^{n-1}} |\mathcal{F}_n(\tilde{\psi})(\eta)||\xi_n|^{\frac{1-n}{2}}\,d\eta \\ &\lesssim |\xi_n|^{\frac{1-n}{2}}.\end{aligned}$$

Making these arguments rigorous with the aid of suitable partitions of unity one finds the first inequality. The proof of the second result may be found in [19] or [48, p.336]. □

We mention that negatively curved parts of manifolds satisfy the same estimates. Regions with small absolute value of the Gaussian curvature, however, are responsible for weaker decay of the Fourier transform. As indicated in (5), the critical directions are parallel to the normal vector field of the manifold at the point where the Gaussian curvature vanishes.

Remark 2.

(a) *The result does not hold for arbitrary convex sets. For instance, explicit computations reveal that the Fourier transform of* $\chi_{[0,1]^n}$ *does not decay better than* $(1+\xi_1)\cdot\ldots\cdot(1+\xi_n)$. *A similar statement holds for polytopes (i.e. finite unions of simplices). The explicit formula for the ball (involving Bessel functions) can be found in [20, p.1].*

(b) *For general bounded sets* $\Omega \subset \mathbb{R}^n$, *the decay properties of* $\hat{\chi}_\Omega$ *depend only on the regularity of* Ω *near its boundary. Indeed, given any compact subset* $K \subset \Omega$ *one may choose a smooth function* $\rho \in C_0^\infty(\Omega)$ *with* $\rho|_K = 1$ *and obtain* $\hat{\chi}_\Omega = \mathcal{F}(\chi_\Omega - \rho) + \hat{\rho}$ *where* $\hat{\rho}$, *being a Schwartz function, decays faster than any polynomial. Hence, the pointwise decay and integrability properties of* $\hat{\chi}_\Omega$ *only depend on* $\chi_\Omega - \rho$, *which is supported in* Ω *but as close to the boundary* $\partial\Omega$ *as we wish.*

(c) *By the Hausdorff-Young inequality we have* $\hat{\chi}_\Omega \in L^s(\mathbb{R}^n)$ *for all* $s \in [2,\infty]$ *and any bounded set* Ω. *For* $C^{1,1}$*-domains one has* $\hat{\chi}_\Omega \in L^s(\mathbb{R}^n)$ *if and only if* $s \in (\frac{2n}{n+1}, \infty]$, *see [29, Corollary 2]. Parts of these results carry over to Lipschitz domains where one can even show* $\hat{\chi}_\Omega \in L^{\frac{2n}{n+1},\infty}(\mathbb{R}^n)$, *see [25, Corollary 1.4].*

(d) *Sogge and Stein [45, Theorem 1] proved that for any smooth, closed and compact hypersurface* $M \subset \mathbb{R}^n$ *and* $\psi \in C_0^\infty(M)$ *we have the estimate*

$$\left|\mathcal{F}_n\big(|K|^{2n-2}\psi\, d\mu\big)(\xi)\right| \lesssim (1+|\xi|)^{\frac{1-n}{2}}$$

where K *denotes the Gaussian curvature. This estimate generalizes Theorem 2 to manifolds with possibly vanishing Gaussian curvature. For convex hypersurfaces the*

[2]The underlying fact is the following: Let $\sigma > 0$ and $A \in \mathbb{R}^{(n-1)\times(n-1)}$ symmetric and invertible. Then we have the formula $\mathcal{F}(e^{i\sigma\langle x,Ax\rangle})(\xi) = (2\sigma)^{(1-n)/2}|\det(A)|^{-1/2}\exp(i\frac{\pi}{4}\operatorname{sgn}(A))\exp\left(-i\frac{\langle\xi,A^{-1}\xi\rangle}{4\sigma}\right)$ where $\operatorname{sgn}(A)$ denotes the signature of A, i.e., the number of its positive eigenvalues minus the number of its negative eigenvalues, cf. [32, Proposition 9].

same conclusion holds with a weaker mitigating factor K^α for $\alpha = \lfloor \frac{n+3}{2} \rfloor$, see [7, Theorem 5].

(e) *In [31] one can find the estimate $|\widehat{d\mu}(\xi)| \lesssim (1+|\xi|)^{-j/2}$ provided that j principal curvatures of the hypersurface under consideration are bounded away from zero.*

(f) *In Stein's book [47, pp.350-351] one can find the definition of a smooth m-dimensional manifold of type k. For such manifolds one has $|\widehat{\psi\, d\mu}(\xi)| \lesssim (1+|\xi|)^{-1/k}$ for $\phi \in C_0^\infty(M)$. Roughly speaking, in case $k \geq 2$ it means that in each point of M at least one of the $(k-2)$-th order partial derivatives of at least one principal curvature is nonzero.*

Next we use the information from Theorem 2 to prove the Stein-Tomas Theorem.

Theorem 3 (Stein-Tomas). *Let $M \subset \mathbb{R}^n$ be a smooth, closed and compact hypersurface with positive Gaussian curvature. Then we have*

$$\|\hat{f}\|_{L^2(d\mu)} \lesssim \|f\|_p \qquad \text{whenever } 1 \leq p \leq \frac{2(n+1)}{n+3}.$$

Proof. By density, it suffices to prove this estimate for $f \in C_0^\infty(\mathbb{R}^n)$. For simplicity we only prove the claim for $p < \frac{2(n+1)}{n+3}$, cf. Remark 3(a). Instead of proving that $Tf := \hat{f}|_{S^{n-1}}$ extends to a continuous map $L^p(\mathbb{R}^n) \to L^2(d\mu)$ we show the equivalent statement that T^*T extends to a continuous map $L^p(\mathbb{R}^n) \to L^{p'}(\mathbb{R}^n)$. Notice that

$$T^* f = \widehat{f\, d\mu} \qquad \text{and} \qquad T^*Tf = \widehat{d\mu} * f.$$

The strategy is to estimate the operators $f \mapsto (\phi_j \widehat{d\mu}) * f$ where the ϕ_j form a kind of Littlewood-Paley decomposition. More precisely we choose $\phi_0 \in C_0^\infty(\mathbb{R}^n)$ with $\phi_0(0) = 1$ and define $\phi_j(x) := \phi_0(2^{-j}x) - \phi_0(2^{1-j}x)$ for $j \geq 1$. Then we have

$$\sum_{j=0}^\infty \phi_j = 1 \quad \text{on } \mathbb{R}^n \setminus \{0\}, \qquad |\hat{\phi}_j(\xi)| = 2^{nj}|\hat{\phi}_1(2^j\xi)| \lesssim 2^{nj}(1+2^j|\xi|)^{-n} \quad (j \geq 1).$$

From Theorem 2 we get $|\widehat{d\mu}(\xi)| \lesssim (1+|\xi|)^{(1-n)/2}$ and thus obtain for all $j \in \mathbb{N}_0$

$$\|(\phi_j \widehat{d\mu}) * f\|_\infty \leq \|\phi_j \widehat{d\mu}\|_\infty \|f\|_1 \lesssim 2^{j\frac{1-n}{2}} \|f\|_1.$$

This is the $L^1 - L^\infty$ estimate. Moreover, we have for all $j \in \mathbb{N}_0$

$$\begin{aligned}
&|\mathcal{F}_n(\phi_j \widehat{d\mu})(\xi)| \\
&= |(\hat{\phi}_j * d\mu(-\cdot))(\xi)| \\
&\lesssim \int_{\mathbb{R}^n} |\hat{\phi}_j(\xi+\eta)| d\mu(\eta) \\
&\lesssim 2^{nj} \left(\int_{B_{2^{-j}}(-\xi)} (1+2^j|\xi+\eta|)^{-n} d\mu(\eta) + \sum_{k=-j+1}^\infty \int_{B_{2^k}(-\xi) \setminus B_{2^{k-1}}(-\xi)} (1+2^j|\xi+\eta|)^{-n} d\mu(\eta) \right) \\
&\lesssim 2^{nj} \left(\mu(B_{2^{-j}}(-\xi)) + \sum_{k=-j+1}^\infty 2^{-n(j+k)} \mu\big(B_{2^k}(-\xi) \setminus B_{2^{k-1}}(-\xi)\big) \right)
\end{aligned}$$

$$\begin{aligned}
&\lesssim 2^{nj}\left(2^{-j(n-1)} + \sum_{k=-j+1}^{\infty} 2^{-n(j+k)}2^{k(n-1)}\right) \\
&\lesssim 2^{j} + \sum_{k=-j+1}^{\infty} 2^{-k} \\
&\lesssim 2^{j}.
\end{aligned}$$

This implies the L^2-L^2-estimate

$$\|(\phi_j\widehat{d\mu}) * f\|_2 = \|\mathcal{F}_n(\phi_j\widehat{d\mu})\hat{f}\|_2 \le \|\mathcal{F}_n(\phi_j\widehat{d\mu})\|_\infty \|\hat{f}\|_2 \lesssim 2^j\|f\|_2.$$

So the Riesz-Thorin interpolation theorem yields for all $j \in \mathbb{N}_0$

$$\|(\phi_j\widehat{d\mu}) * f\|_{p'} \lesssim 2^{j(\frac{n+3}{2}-\frac{n+1}{p})}\|f\|_p \qquad \text{for } 1 \le p \le 2.$$

Hence, the operator

$$T^*Tf = \widehat{d\mu} * f = \sum_{j\in\mathbb{N}_0}\left((\phi_j\widehat{d\mu}) * f\right) : L^{p'}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$$

extends to a bounded linear operator provided $1 \le p < \frac{2(n+1)}{n+3}$. □

Remark 3.

(a) *The endpoint case $p = \frac{2(n+1)}{n+3}$ in the Stein-Tomas Theorem may be proved using Stein's complex interpolation theorem [46] or real interpolation in Lorentz spaces following [3, Remark 2.2].*

(b) *The dual version of the Stein-Tomas inequality* (4) *shows that so-called Herglotz waves T^*g lie in $L^{p'}(\mathbb{R}^n)$ precisely for $1 \le p \le \frac{2(n+1)}{n+3}$. These functions are of interest because all of them satisfy the linear Helmholtz equation in $\mathbb{R}^n$.*

(c) *The above proof indicates that a Stein-Tomas-type inequality holds whenever the underlying finite measure satisfies $|\widehat{d\mu}(\xi)| \lesssim (1+|\xi|)^{-b/2}$ and $|\mu(B_r(x))| \lesssim r^a$ where $0 < b \le a < n$. Indeed, it was proved by Mockenhaupt [34], Mitsis [33], Bak-Seeger [3] (endpoint case) and Hambrook-Łaba [17] (optimality) that in this case the Stein-Tomas inequality holds precisely for $1 \le p \le 1 + \frac{b}{4(n-a)+b}$. The original Stein-Tomas Theorem follows for $a = b = n-1$, and* (3) *formally appears in the limit $a \to n$. As a consequence, instead of studying Fourier restrictions to positively curved compact hypersurfaces one can discuss Fourier restrictions to quite general measure spaces lying in $\mathbb{R}^n$. This may be used to investigate fractals or suitable lower-dimensional surfaces or hypersurfaces with only j non-vanishing curvatures, cf. [47, p.365] or [14].*

(d) *There are interesting converse results. For instance it is proved in [23, Theorem 2] that a smooth compact hypersurface satisfying the Stein-Tomas inequality must have positive Gaussian curvature and therefore must satisfy $|\widehat{d\mu}(\xi)| \lesssim (1+|\xi|)^{\frac{1-n}{2}}$. A quantitative version of this was proved by Nicola [38, Theorem 1.1]. In the aforementioned papers one may as well find corresponding statements for surfaces with at least j non-vanishing curvatures.*

(e) *In the past years the search for extremizers and best constants in the Stein-Tomas inequality became a very active field of research. We refer to the papers [6, 12, 13, 41] for results in this direction.*

By interpolation with the trivial $L^1(\mathbb{R}^n) \to L^\infty(d\mu)$-estimate one obtains the estimate (4) for all exponents p, q satisfying $1 \le p \le \frac{2(n+1)}{n+3}$ and $1 \le q \le \frac{(n-1)p'}{n+1}$. Clearly one may ask whether there are better results than that. A famous counterexample attributed to Knapp (see [51]) shows that $q > \frac{(n-1)p'}{n+1}$ is impossible. Furthermore, $p \ge \frac{2n}{n+1}$ can be excluded by explicit computations for the sphere $M = S^{n-1}$. So the maximal range of validity is given by

$$1 \le p < \frac{2n}{n+1}, \qquad 1 \le q \le \frac{(n-1)p'}{n+1}. \tag{6}$$

The claim that the inequality (4) holds for all these exponents is known as the "Restriction Conjecture" that has been attracting many excellent mathematicians over more than 40 years. For the state of the art in this branch of Harmonic Analysis we refer to Tao's survey article [53]. Let us only mention that the conjecture is proved only in the case $n = 2$. We refer to the Appendix for the corresponding references.

3. Applications

3.1. Number theory.

Let $\Omega \subset \mathbb{R}^n$ be a nonempty, open and bounded set. We indicate how estimates for the Fourier transform of the indicator function χ_Ω may be used in order to study the asymptotics of the quantity

$$N_\Omega(r) := \#\{m \in \mathbb{Z}^n : m \in r\Omega\} \quad \text{as } r \to \infty.$$

Here, $N_\Omega(r)$ counts the number of lattice points in $r\Omega$ and its number theoretical relevance comes from the fact

$$N_{B_1(0)}(r) = \sum_{k=0}^{\lceil r^2 \rceil - 1} \#\{m = (m_1, \ldots, m_n) \in \mathbb{Z}^n : m_1^2 + \ldots + m_n^2 = k\}.$$

In other words, estimating $N_{B_1(0)}(r)$ revals some information about the number of ways integers can be written as sums of squares. An observation due to Gauß yields $N_{B_1(0)}(r) - \pi r^n = O(r^{n-1})$, see [24, pp.1-2] and [28, p.128] for an extension to more general domains. Determining the asymptotics of the remainder, called "lattice point discrepancy", is known as the "Gauß circle problem". Following [48, pp.376-384] we will see that Fourier analysis allows to improve this bound from $O(r^{n-1})$ to $O(r^{\frac{n(n-1)}{n+1}})$ for strongly convex $\Omega \subset \mathbb{R}^n$.

The idea is to use Poisson's summation formula as follows

$$\begin{aligned} N_\Omega(r) &= \sum_{m \in \mathbb{Z}^n} 1_{r\Omega}(m) \\ &= (2\pi)^{n/2} \sum_{m \in \mathbb{Z}^n} \widehat{1_{r\Omega}}(2\pi m) \\ &= (2\pi)^{n/2} \sum_{m \in \mathbb{Z}^n} r^n \hat{1}_\Omega(2\pi r m) \end{aligned}$$

$$= (2\pi)^{n/2} r^n \left(\hat{1}_\Omega(0) + \sum_{m\in\mathbb{Z}^n\setminus\{0\}} \hat{1}_\Omega(2\pi r m) \right)$$
$$= r^n|\Omega| + (2\pi)^{n/2} r^n \sum_{m\in\mathbb{Z}^n\setminus\{0\}} \hat{1}_\Omega(2\pi r m).$$

This is problematic because the roughness of the indicator function prevents its Fourier transform from decaying fast enough to make the series absolutely convergent. In other words, the above chain of equalities is not rigorous because $1_{r\Omega}$ does not satisfy the assumptions of Poisson's summation formula. For this reason, following Müller [35], we mollify this indicator function with the aid of a kernel function $\rho_\varepsilon := \varepsilon^{-n} * \rho(\varepsilon^{-1}\cdot)$ for a suitably chosen $\varepsilon > 0$. For instance we may take $\rho(z) = c\,e^{-|z|^2}$ where $c > 0$ is chosen such that $\int_{\mathbb{R}^n} \rho = 1$ holds.

Lemma 1 (Lemma 3 in [35]). *Let $\Omega \subset \mathbb{R}^n$ be a nonempty, open, bounded and convex set with smooth boundary. Then there is a $c > 0$ such that for all $\varepsilon \in (0,1)$ we have*

$$\chi_{(1-c\varepsilon)\Omega} * \rho_\varepsilon \le \chi_\Omega \le \chi_{(1+c\varepsilon)\Omega} * \rho_\varepsilon.$$

This allows us to prove a result due to Hlawka [20, Satz 9, p.25].

Theorem 4 (Hlawka). *Let $\Omega \subset \mathbb{R}^n$ be a nonempty, open, bounded and convex set with smooth boundary such that $\partial\Omega$ has positive Gaussian curvature. Then*

$$N_\Omega(r) - r^n|\Omega| = O(r^{\frac{n(n-1)}{n+1}}).$$

Proof. From Lemma 1 we infer for all $\varepsilon \in (0,1)$

$$\begin{aligned}
N_\Omega(r) &= \sum_{m\in\mathbb{Z}^n} \chi_{r\Omega}(m) \\
&\le \sum_{m\in\mathbb{Z}^n} (\chi_{r(1+c\varepsilon)\Omega} * \rho_{r\varepsilon})(m) \\
&= \sum_{m\in\mathbb{Z}^n} \mathcal{F}(\chi_{r(1+c\varepsilon)\Omega} * \rho_{r\varepsilon})(2\pi m) \\
&= \sum_{m\in\mathbb{Z}^n} \mathcal{F}(\chi_{r(1+c\varepsilon)\Omega})(2\pi m)\hat{\rho}_{r\varepsilon}(2\pi m) \\
&= r^n(1+c\varepsilon)^n \sum_{m\in\mathbb{Z}^n} \hat{\chi}_\Omega(2\pi r(1+c\varepsilon)m)\hat{\rho}(2\pi r\varepsilon m) \\
&= r^n(1+c\varepsilon)^n \left(\hat{\chi}_\Omega(0)\hat{\rho}(0) + \sum_{m\in\mathbb{Z}^n\setminus\{0\}} \hat{\chi}_\Omega(2\pi r(1+c\varepsilon)m)\hat{\rho}(2\pi r\varepsilon m) \right) \\
&= r^n(1+c\varepsilon)^n \left(\left(\int_{\mathbb{R}^n} \chi_\Omega\right)\left(\int_{\mathbb{R}^n} \rho\right) + \sum_{m\in\mathbb{Z}^n\setminus\{0\}} \hat{\chi}_\Omega(2\pi r(1+c\varepsilon)m)\hat{\rho}(2\pi r\varepsilon m) \right) \\
&= r^n(1+c\varepsilon)^n \left(|\Omega|\cdot 1 + \sum_{m\in\mathbb{Z}^n\setminus\{0\}} \hat{\chi}_\Omega(2\pi r(1+c\varepsilon)m)\hat{\rho}(2\pi r\varepsilon m) \right).
\end{aligned}$$

Now we use that $\hat{\rho}$ is a Schwartz function so that we may use the estimate $|\hat{\rho}(z)| \lesssim_N (1+z)^{-N}$ for any fixed $N \in \mathbb{N}$. Moreover, in view of the curvature assumption on $\partial\Omega$, we have $|\hat{\chi}_\Omega(z)| \lesssim$

$(1+|z|)^{-\frac{n+1}{2}}$ for all $\xi \in \mathbb{R}^n$, see Theorem 2. Using these facts we conclude for all $\varepsilon \le \frac{1}{r} \le 1$

$$\begin{aligned}
N_\Omega(r) - r^n|\Omega| &\lesssim r^n \left(((1+c\varepsilon)^n - 1)|\Omega| + \sum_{m\in\mathbb{Z}^n\setminus\{0\}} |\hat{\chi}_\Omega(2\pi(1+c\varepsilon)rm)||\hat{\rho}(2\pi\varepsilon rm)| \right) \\
&\lesssim r^n \left(\varepsilon + \sum_{m\in\mathbb{Z}^n\setminus\{0\}} (1+|rm|)^{-\frac{n+1}{2}}(1+|\varepsilon rm|)^{-N} \right) \\
&\lesssim r^n \left(\varepsilon + \int_1^\infty (1+|rz|)^{-\frac{n+1}{2}}(1+\varepsilon|rz|)^{-N} z^{n-1}\,dz \right) \\
&\lesssim r^n \left(\varepsilon + \int_1^{\frac{1}{\varepsilon r}} (1+rz)^{-\frac{n+1}{2}} z^{n-1}\,dz + (\varepsilon r)^{-N} \int_{\frac{1}{\varepsilon r}}^\infty (1+rz)^{-\frac{n+1}{2}} z^{n-1-N}\,dz \right) \\
&\lesssim r^n \left(\varepsilon + r^{-n} \int_1^{\frac{1}{\varepsilon}} (1+y)^{-\frac{n+1}{2}} y^{n-1}\,dy + (\varepsilon r)^{-N} r^{-n+N} \int_{\frac{1}{\varepsilon}}^\infty (1+y)^{-\frac{n+1}{2}} y^{n-1-N}\,dy \right) \\
&\lesssim r^n \left(\varepsilon + r^{-n}\varepsilon^{\frac{1-n}{2}} + r^{-n}\varepsilon^{\frac{1-n}{2}} \right) \\
&\lesssim \varepsilon r^n + \varepsilon^{\frac{1-n}{2}}.
\end{aligned}$$

So we choose $\varepsilon = r^{-\frac{2n}{n+1}}$. Then the previous computation is justified because of $\varepsilon r \le 1$ and we get

$$N_\Omega(r) - r^n|\Omega| \lesssim r^{\frac{n(n-1)}{n+1}}.$$

Similarly, one proves the lower bound and the proof is finished. □

Clearly, lattice point problems have been investigated in far more generality and detail, which we cannot reproduce here. They appear not only in number theoretical considerations, but also in Weyl's law on the asymptotics of eigenvalues of the Laplacian or quantum mechanics when it comes to counting the number of states below a given energy level. The interested reader may find more on the lattice point problem in [48, pp.376-384] or in the papers [5, 15, 22, 24, 26, 27, 35].

3.2. **Strichartz estimates.** We present the original Strichartz estimate [51] for the Schrödinger equation

$$i\partial_t\psi - \Delta\psi = 0 \quad \text{in } \mathbb{R}^d, \qquad \psi(0) = \psi_0, \tag{7}$$

where the initial datum satisfies $\psi_0 \in L^2(\mathbb{R}^d)$. We follow [36, pp.301-303] for our proof of the basic Strichartz inequality.

Theorem 5 (Strichartz (1977))**.** *The Schrödinger equation* (7) *has a unique solution* ψ *satisfying*

$$\|\psi\|_{L^{2+\frac{4}{d}}(\mathbb{R}^{d+1})} \lesssim \|\psi_0\|_{L^2(\mathbb{R}^d)}.$$

Proof. We verify this first for a Schwartz function ψ_0 satisfying $\operatorname{supp}(\hat{\psi}_0) \subset B_1(0)$. The solution formula for (7) tells us

$$\psi(x,t) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \hat{\psi}_0(\xi) e^{i(x,t)\cdot(\xi,|\xi|^2)}\, d\xi = \mathcal{F}^{-1}_{d+1}(\hat{\psi}_0\, d\mu)(x,t)$$

where[3] the measure is given by $\mu(A) := \sqrt{2\pi}\int_{\mathbb{R}^d} (\rho\chi_A)(\xi,|\xi|^2)\, d\xi$ for some $\rho \in C_0^\infty(\mathbb{R}^d)$ such that $\rho \equiv 1$ on the support of $\hat{\psi}_0$. Then one has $|\mu(B_r)| \lesssim r^d$ for each ball $B_r \subset \mathbb{R}^{d+1}$ of radius r as well as $|\widehat{d\mu}(\xi,\eta)| \lesssim (1+|(\xi,\eta)|)^{-d/2}$, which can be proved[4] using the method of stationary phase as in (2). Hence, applying first the generalized Stein-Tomas Theorem from Remark 3(c) in its dual version from (4) (for $n = d+1, p = \frac{2(n+1)}{n+3} = \frac{2(d+2)}{d+4}$) yields

$$\|\psi\|_{L^{2+\frac{4}{d}}(\mathbb{R}^{d+1})} = \|\mathcal{F}^{-1}_{d+1}(\hat{\psi}_0\, d\mu)\|_{L^{2+\frac{4}{d}}(\mathbb{R}^{d+1})} \lesssim \|\hat{\psi}_0\|_{L^2(d\mu)} = \|\psi_0\|_{L^2(\mathbb{R}^d)}.$$

In order to remove the condition $\operatorname{supp}(\hat{\psi}_0) \subset B_1(0)$ one notices that ψ, ψ_0 satisfy (7) if and only if $\psi(\lambda\cdot, \lambda^2\cdot), \psi_0(\lambda\cdot)$ satisfy (7) for any given $\lambda > 0$. Using this scaling property and the density of Schwartz functions, one finally arrives at the conclusion. ◻

As for the Stein-Tomas inequality one may ask for best constants and extremizers in Strichartz inequalities. This is an interesting research direction with promising first results in the case $2 + \frac{4}{d} \in \mathbb{N}$, i.e., for $d \in \{1,2,4\}$, see [11, 21, 40]. Similar results hold for the wave equation

$$\partial_{tt}\psi - \Delta\psi = 0 \quad \text{in } \mathbb{R}^d, \qquad \psi(0) = \psi_0, \quad \partial_t\psi(0) = 0, \tag{8}$$

where the initial datum satisfies $\psi_0 \in \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)$. We follow [36, pp.308-310].

Theorem 6 (Strichartz (1977))**.** *The wave equation* (8) *has a unique solution* ψ *satisfying*

$$\|\psi\|_{L^{2+\frac{4}{d-1}}(\mathbb{R}^{d+1})} \lesssim \|\hat{\psi}_0|\cdot|^{-\frac{1}{2}}\|_{L^2(\mathbb{R}^n)} = \|\psi_0\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)}. \tag{9}$$

Proof. We first prove the estimate for a Schwartz function ψ_0 such that $\operatorname{supp}(\hat{\psi}_0) \subset B_1(0) \setminus B_{1/2}(0)$. The solution formula for (8) yields

$$\begin{aligned}\psi(x,t) &= (\cos(t|\nabla|)\psi_0)(x)\\ &= \mathcal{F}_d^{-1}(\cos(t|\cdot|)\hat{\psi}_0)(x)\\ &= \frac{1}{2}\sum_{\pm}\int_{\mathbb{R}^d} e^{i(x,t)\cdot(\xi,\pm|\xi|)}\hat{\psi}_0(\xi)\, d\xi\\ &= \sum_{\pm}\mathcal{F}^{-1}_{d+1}(\hat{\psi}_0|\cdot|^{-\frac{1}{2}}\, d\mu_\pm)(x,t)\end{aligned}$$

where $\mu_\pm(A) := \frac{1}{2}\sqrt{2\pi}\int_{\mathbb{R}^d} |\xi|^{1/2}(\rho\chi_A)(\xi,\pm|\xi|)\, d\xi$ for some $\rho \in C_0^\infty(\mathbb{R}^{d+1}\setminus\{0\})$ such that $\rho \equiv 1$ on the support of $\hat{\psi}_0$. Then one can check $|\mu_\pm(B_r)| \lesssim r^d$ for each ball $B_r \subset \mathbb{R}^{d+1}$ of radius

[3] In the latter formula $\hat{\psi}_0$ stands for the function $(\xi,\eta) \mapsto \hat{\psi}_0(\xi)$.

[4] In order to keep the computations as simple as possible, one may choose $\rho(z_1,\ldots,z_{d+1}) = \rho^*(z_1)\cdot\ldots\cdot\rho^*(z_{d+1})$ for a suitable test function $\rho^* \in C_0^\infty(\mathbb{R})$, which leads to the estimate $|\widehat{d\mu}(\xi,\eta)| \lesssim (1+|\xi_1|+|\eta|)^{-1/2}\cdot\ldots\cdot(1+|\xi_d|+|\eta|)^{-1/2} \lesssim (1+|(\xi,\eta)|)^{-d/2}$.

r as well as $|\widehat{d\mu_\pm}(\xi,\eta)| \lesssim (1+|(\xi,\eta)|)^{(1-d)/2}$ so that the generalized Stein-Tomas Theorem from Remark 3(c) and (4) yield the result for such functions. (These computations are actually independent of the artificially inserted factor $|\xi|^{1/2}$.) In order to remove the condition $\mathrm{supp}(\hat{\psi}_0) \subset B_1(0) \setminus B_{1/2}(0)$ one notices that ψ, ψ_0 satisfy (8) if and only if $\psi(\lambda\cdot), \psi_0(\lambda\cdot)$ satisfy (8) for any given $\lambda > 0$. Since the Strichartz estimate (9) is invariant under this scaling (that's the reason for the factor $|\xi|^{1/2}$) one obtains the result by Littlewood-Paley decomposition. □

Appendix

I would like to mention the milestones of two-dimensional Fourier restriction theory where the Fourier transform along curves is investigated. The fundamental result in this context is the following.

Theorem 7 (Fefferman, Zygmund). *Let $\Gamma \subset \mathbb{R}^2$ be a closed and compact regular curve with positive curvature. Then*

$$\|\hat{f}\|_{L^q(\Gamma,ds)} \lesssim \|f\|_p$$

if and only if $1 \le q \le \frac{p}{3(p-1)}$ and $1 \le p < \frac{4}{3}$.

Remark 4.

(a) *In Theorem 7 the case $1 \le q < \frac{p}{3(p-1)}$ is covered by Zygmund's analysis in [56, Theorem 3], while the endpoint case $q = \frac{p}{3(p-1)}$ is due to Fefferman [10, pp.33-34] (who acknowledges the help of Stein).*

(b) *Given that the Restriction Conjecture is known to be the best possible result one can hope for, no improvements can be made in the scale of Lebesgue spaces.*

(c) *For results concerning the existence or nonexistence of extremizers in the corresponding Stein-Tomas inequality for $q = 2, p = \frac{6}{5}$ see [39].*

As in the higher-dimensional case (see Remark 2(d)) there is a version without positivity assumption on the curvature. Here, a suitable power of the curvature has to be used as a mitigating factor.

Theorem 8. *(Sjölin [43, Corollary 2]) Let $\Gamma \subset \mathbb{R}^2$ be a closed and compact plane curve with curvature κ. Then, for any $\delta > 0$,*

$$\|\hat{f}|\kappa|^{\frac{1}{p'}+\delta}\|_{L^q(\Gamma,ds)} \lesssim \|f\|_p$$

if and only if $1 \le q \le \frac{p}{3(p-1)}$ and $1 \le p < \frac{4}{3}$. If $\kappa \ge 0$ then this holds for $\delta = 0$ as well.

Again, from [38, Theorem 1.1] we get that such a mitigating factor is in some sense necessary: the validity of the Stein-Tomas estimate $\|\hat{f}w\|_{L^2(\Gamma,ds)} \lesssim \|f\|_{6/5}$ implies $w \lesssim |\kappa|^{1/6}$. So we see that curvature is somewhat necessary for having good Fourier restriction mapping properties in the scale of Lebesgue spaces. It is noteworthy that the estimates get worse the flatter the curve is, which coincides with our previous observations. One early result in this direction is [43, Corollary 3]. We mention that there is also a restriction theory for space curves $\gamma : I \to \mathbb{R}^n$ with $I \subset \mathbb{R}$, where typically a nondegeneracy condition of the kind $|\det(\gamma', \dots, \gamma^{(n)})| \ge c > 0$ is assumed. Notice that in the special case $n = 2$, assuming that

γ is parametrized to arclength, this is equivalent to a lower bound for $|\gamma''|$, which is the absolute value of the curvature of γ. The interested reader may find more on this topic in the papers [2, 8, 9, 16, 44, 50].

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R. Mandel
Karlsruhe Institute of Technology
Institute for Analysis
Englerstrasse 2
D-76131 Karlsruhe, Germany
Email address: `Rainer.Mandel@kit.edu`