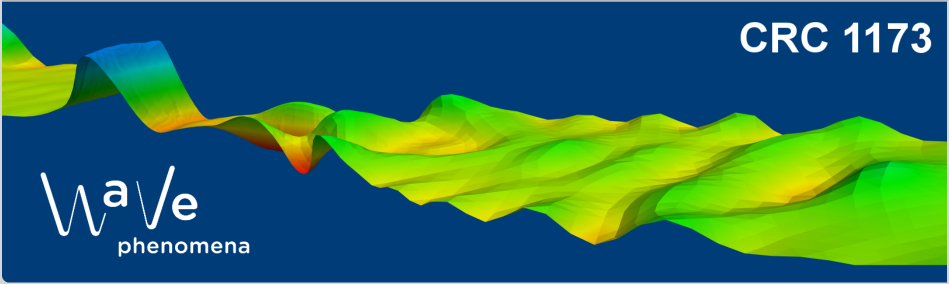
Global bifurcation results for the Lugiato-Lefever equation

joint work with W. Reichel, J. Gärtner (KIT)

Rainer Mandel

Karlsruhe, July 23th 2018

KIT, Institut für Analysis



Outline

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The Lugiato-Lefever equation (1987)

The electric field in a ring resonator $\simeq [0, 2\pi]$

$$-ia_t = da_{xx} + (i - \zeta)a + |a|^2 a - if \qquad \text{on } \mathbb{R}_+ \times [0, 2\pi]$$

- $a =$ complex-valued E–field, 2π-periodic in x
- $d =$ "dispersion"
- $f =$ "forcing" (strength of the laser)
- $\zeta =$ "detuning" (wavelength of the laser)

Aims:

- Soliton-type solutions
- "frequency combs", i.e. Fourier picture = ____|_|_|_|_|_|_|____

Contributions from Analysis

Time-dependent problem:

- 2011 **Miyaji, Ohnishi, Tsutsumi:** Stability results
- 2016 **Jahnke, Mikl, Schnaubelt:** Well-pos. + Strang splitting

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Stationary problem:

- Curve of constant solutions $\mathcal{T} := \{(a_s, \zeta_s) : s \in (-1, 1)\}$
- Spatial dynamics (x "treated as time"):
  2010 **Miyaji, Ohnishi, Tsutsumi**
  2014 **Godey, Balakireva, Coillet, Chembo**
  2017 **Godey**
- Bifurcation of nonconstant solutions with fixed period 2π:
  2016 **Mandel, Reichel**
  2018 **Mandel**

Contributions from Analysis:

For this talk:

- R. Mandel, W. Reichel (SIAM)
  A priori bounds and global bifurcation results for frequency combs modeled by the Lugiato-Lefever equation
- R. Mandel (to appear in TMNA)
  Global secondary bifurcation, symmetry breaking and period-doubling

Extensions:

- J. Gärtner, R. Mandel, W. Reichel (in preparation)

Bifurcation

$$-da'' = (i-\zeta)a + |a|^2 a - if, \quad a \in C^2_{per}([0,2\pi];\mathbb{C})$$

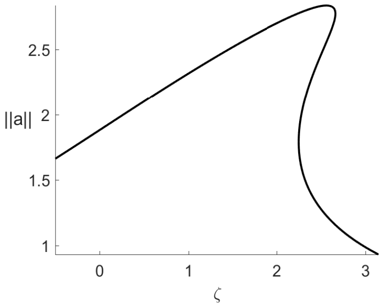

Figure: Constant solutions for $f = 1.6$

Constant solutions:

$$a_s(x) = f(1-s^2) - i \cdot fs\sqrt{1-s^2}$$
$$\zeta_s = f^2(1-s^2) + \frac{s}{\sqrt{1-s^2}}$$

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Bifurcation analysis

Steps:

(1) Find bifurcation points (a_s, ζ_s)
Crandall-Rabinowitz Theorem

(2) Show that the bifurcating branches are "global"
Krasnoselskii-Rabinowitz Theorem

(3) Find global secondary bifurcations

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With the aid of suitable software . . .

- Bifurcation diagrams
- Plots of solutions in space (and frequency domain)
- Time-dependent detuning

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(1) Finding bifurcation points

Crandall-Rabinowitz Theorem:

Bifurcation from (a_s, ζ_s) occurs provided

$$-d\phi'' = (i - \zeta_s)\phi + 2|a_s|^2\phi + a_s^2\overline{\phi},$$
$$\phi(0) = \phi(2\pi), \quad \phi'(0) = \phi'(2\pi).$$

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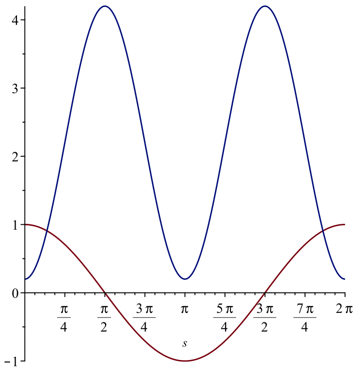
Consequence: Local bifurcation of synchronized solutions from

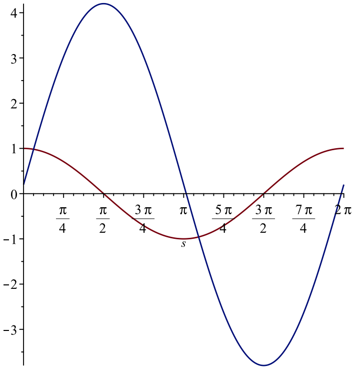
$$(a_{s_1}, \zeta_{s_1}),\ (a_{s_2}, \zeta_{s_2}),\ \ldots,\ (a_{s_N}, \zeta_{s_N}).$$

(1) Finding bifurcation points

Synchronized



Nonsynchronized



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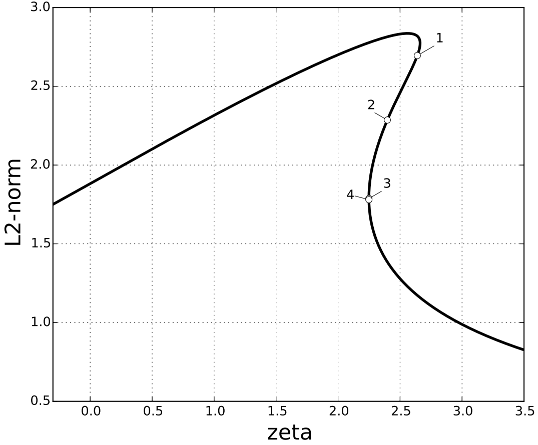

Figure: Bifurcation points at $s_1, \ldots, s_4$ for $d = -0.1$, $f = 1.6$

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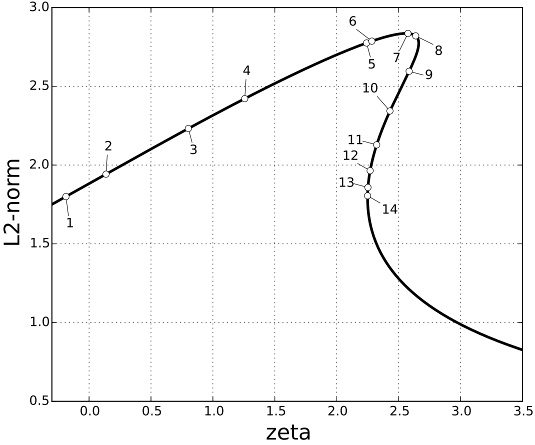

Figure: Bifurcation points at $s_1, \ldots, s_{14}$ for $d = 0.1$, $f = 1.6$

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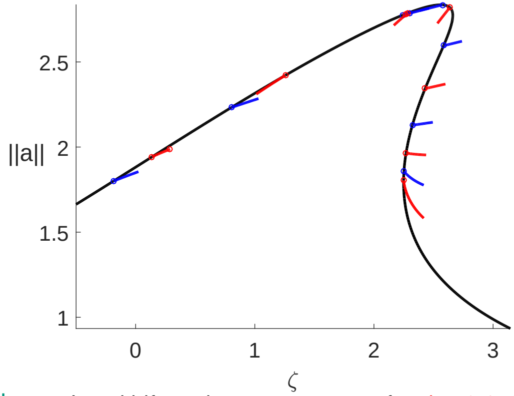

Figure: Local bifurcations at $s_1, \ldots, s_{14}$ for $d = 0.1$, $f = 1.6$

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Additional information:

- $|f|$ large $\Rightarrow$ Many bifurcation points exist.
- $|f| \leq 1$ $\Rightarrow$ No bifurcation points exist.
- 1D solution space $=$ span$\left\{z_k \cos(kx)\right\}$ for $z_k \in \mathbb{C} \setminus \{0\}$
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Krasnoselskii-Rabinowitz:

The branch C_{s_i} emanating from (a_{s_i}, ζ_{s_i})

(A) is unbounded OR

(B) returns to another constant solution $(a_{s_j}, \zeta_{s_j}), j \neq i$.

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Theorem (Mandel, Reichel (2016))

Every nonconstant synchronized solution (a, ζ) satisfies

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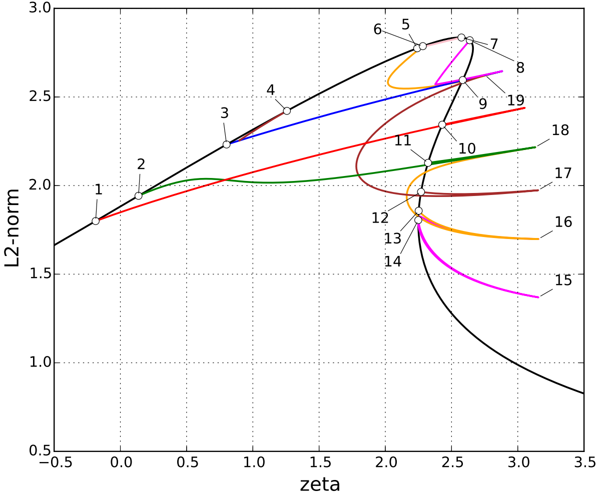

Figure: Bifurcation diagram for $f = 1.6, d = 0.1$

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Solitons:

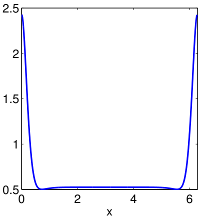

Figure: 1-soliton (15)

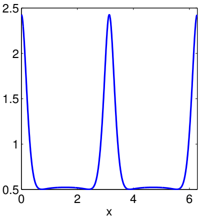

Figure: 2-soliton (16)

(3) Secondary bifurcations

- Secondary bifurcation = "Bifurcation from bifurcating branches"
- $X_p := H^1_{per}([0, \frac{2\pi}{p}]; \mathbb{C})$ where $p \in \mathbb{N}$

Satz (Krasnoselskii-Rabinowitz-Dancer)

If alternative (B) holds then the bifurcating branch $\mathcal{C}_{s_i}$

$$\sum_{(a_{s_j},\zeta_{s_j})\in\mathcal{C}_{s_i}} \underbrace{\Big(\text{"LS-Index change" at } (a_{s_j},\zeta_{s_j})\Big)}_{=:\delta_j} = 0.$$

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If alternative $(B)_p$ holds then the bifurcating branch $\mathcal{C}^p_{s_i} \subset X_p$

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- Prove $C^4_{s_i} \cap \mathcal{T} = \{(a_{s_i}, \zeta_{s_i}), (a_{s_j}, \zeta_{s_j})\}$ (so that $\delta^4_i + \delta^4_j = 0, \ i \neq j$).
- If $\delta^2_i + \delta^2_j \neq 0$, then $C^2_{s_i} \supsetneq C^4_{s_i}$.
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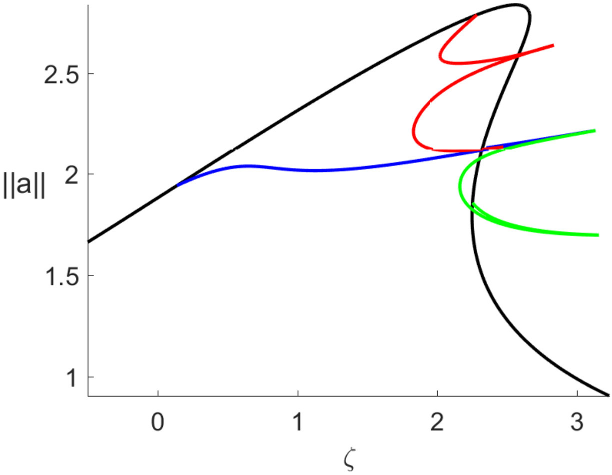

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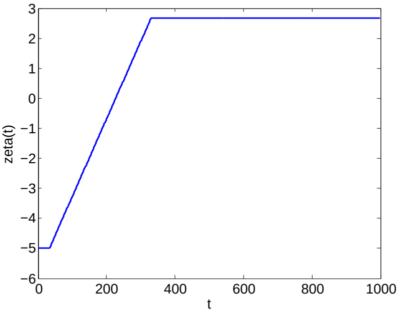
$$-ia_t = da_{xx} + (i - \zeta(t))a + |a|^2 a - if$$
$$a_x(0,t) = a_x(\pi,t) = 0,$$
$$a(x,0) = \text{constant solution for } \zeta \approx -5 + \text{ small perturbation}$$



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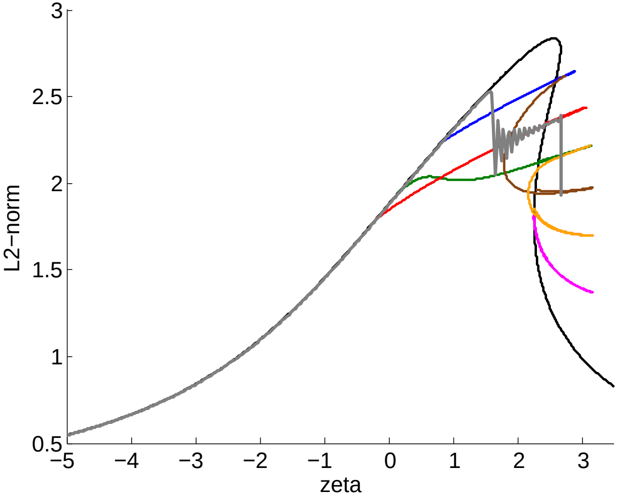

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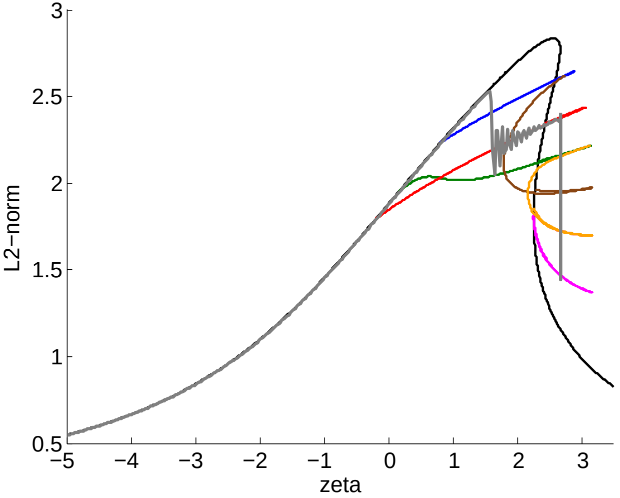

Open problems

- Existence of nonsynchronized solutions
- Analytical prediction of solitons
- Stability of solutions (→ Talk of J. Gärtner)
- Rigorous time-dependent detuning

Extensions:

- Multi-mode excitation
- Nonlinear damping (→ Talk of J. Gärtner)
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