Titchmarsh-Sims-Weyl theory for Complex Hamiltonian systems

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1 Introduction

In [3] the Sims extension of the Titchmarsh-Weyl limit-point/limit-circle classification of Sturm-Liouville differential equations to the case of a complex potential, was further extended to the equation

$$M[y] := -(py')' + qy = wy \tag{1. 1}$$

on an interval $[a, b), -\infty < a < b \leq \infty$. The novelty in [3] was that the coefficient p as well as q was allowed to be complex-valued, and q allowed to be less restricted than was the case in [11]; in [11] $p = w = 1$ and the imaginary part of q is assumed to be semibounded. A consequence of investigating the problem in this more general context was that interesting features which were hidden in Sims' case were exposed. Specifically, the three classes of Sims now involve a Sobolev-type norm and the region in the complex plane appropriate to the theory was shown to be the complement of a closed convex set Q which contains the spectrum of the natural m-accretive operator defined by the problem. Birger and Kalyabin ([2]) considered a system of equations of the form 1. 1, with $p = w = 1$ and $\Im[e^{i\eta}q(x)] \leq -k < 0$ and established the existence of the first of Sims' cases involving a Sobolev-type norm when $|\eta| = \pi/2$.

Our objective in this paper is the extension of the theory in [3] to Hamiltonian systems

$$Jy'(x,\lambda) = (\lambda A(x) + B(x))y(x,\lambda) \tag{1. 2}$$

on an interval $[a,b)$, where, for each $x \in [a,b), J, A(x), B(x)$ are complex $2n \times 2n$ matrices with $-J^2 = I_{2n}$,the identity $2n \times 2n$ matrix, $A(x) \geq 0$ and $B(x)$ may be a non-Hermitian matrix. This will include results for scalar equations of arbitrary order with complex coefficients; a special case is therefore the Orr-Sommerfeld equation for fluid flow. Our approach owes much to the work of Atkinson [1], Hinton and Shaw, [5], [6] and Krall [8] on the extension of the Everitt-Kodaira theory for general even-ordered scalar differential equations with real coefficients to Hamiltonian systems with symmetric coefficients. The first major challenge was to determine a natural analogue of the set Q in [3] on whose complement the Titchmarsh-Weyl-type analysis is composed. In this case the complement is the union of sets $\Lambda(k,\mathcal{U}_{2n})$, over a set of admissible pairs $(k,\mathcal{U}_{2n})$ of complex numbers k and $2n \times 2n$ matrices $\mathcal{U}_{2n}$. These sets turn out to be open half-planes in the Sturm Liouville case but in general there is no reason to suppose that they are either open or convex. For each $\lambda \in \Lambda(k,\mathcal{U}_{2n})$, a nested sequence of bounded convex sets of $n \times n$ matrices is exhibited which converges to a limiting set $D_b(\lambda)$, the analogue of the limiting Weyl circle. The nature of $D_b(\lambda)$ is what determines the number of linearly independent solutions of (1. 1) lying in a weighted space $L^2_{C_\lambda}$, where, for each $x \in [a,b), C_\lambda(x)$ is an $n \times n$ matrix bounded below by a constant multiple of $\mathcal{U}_{2n}A(x)\mathcal{U}_{2n}^*$. The space $L^2_{C_\lambda}$ incorporates the Sobolev-type norm in the Sturm-Liouville case(and indeed, in the problem for general scalar equations), and the analogue of the Sims -Weyl classification of (1. 1) is expressed in terms of it. The analogue of the Titchmarsh-Weyl $M(\lambda)$-matrix is an element of the limiting set $D_b(\lambda)$, and, for each fixed $\mu \in \Lambda(k,\mathcal{U}_{2n})$ and $M_0 \in D_b(\mu)$, a matrix-valued function M exists on $\Lambda(k,\mathcal{U}_{2n})$ which is equal to M_0 when $\lambda = \mu$. On making a mild assumption to guarantee that $\Lambda(k,\mathcal{U}_{2n})$ is open, this function M is shown to be analytic on $\Lambda(k,\mathcal{U}_{2n})$. Moreover, it is proved that the number of $L^2_A(a,b)$- solutions of (1. 1) is constant on each connected component of $\mathbb{C}\setminus Q$, and when there are precisely n $L^2_A(a,b)$-solutions in a connected component Λ_c, M is analytic on Λ_c. Of course, these last properties are well-known in the Sturm-Liouville (and general scalar equation) cases and their proofs follow readily from the Weyl analysis. However, here we are forced to follow a circuitous route via the spectral properties of a natural operator defined by our problem. This is proceeding in the reverse direction to that for the Sturm-Liouville case when the analytic properties of M serve to describe the spectral properties of an associated m-accretive operator.

There is an analogous theory for the formal adjoint of (1. 2), namely

$$Jz'(x,\lambda) = (\overline{\lambda}A(x) + B^*(x))z(x,\lambda). \tag{1. 3}$$

For $\lambda \in \Lambda(k,\mathcal{U}_{2n}) \cap \hat{\Lambda}(k,\mathcal{U}_{2n}^{-1})$, where $\hat{\Lambda}(k,\mathcal{U}_{2n})$ is an analogue of $\Lambda(k,\mathcal{U}_{2n})$ for (1. 3), the limiting set $\hat{D}_b(\lambda)$ for (1. 3) is the adjoint of $D_b(\lambda)$, and intimate connections exist between quantities associated with the Weyl-Sims-type classifications of (1. 2) and (1. 3) as outlined for (1. 2) in the previous paragraph. The natural operator defined by the problem for (1. 3) is the adjoint of that for (1. 2).

We shall use the following notation throughout. The set of $m \times n$ matrices will be denoted by $\mathbb{C}^{m,n}$ with $\mathbb{C}^{m,1}$ written as $\mathbb{C}^m$. A superscript T will stand for the transpose of a vector or matrix and $*$ the conjugate transform. We write $< \cdot, \cdot >, |\cdot|$ for the standard inner product and norm on $\mathbb{C}^n$, namely,

$$< u, v >= v^* u, \quad |u| =< u, u >^{1/2} .$$

For $M \in \mathbb{C}^{n,n}$ we take for its norm $\|M\|$ the largest eigenvalue of $(M^*M)^{1/2}$, and define its real and imaginary parts as $\mathbf{Re}[M] = \frac{1}{2}(M + M^*), \mathbf{Im}[M] = \frac{1}{2i}(M - M^*)$. Furthermore, recall that $M \geq 0$ if $u^* M u \geq 0$ for all $u \in \mathbb{C}^n$ and $M > 0$ if $M \geq 0$ and $Mu = 0$ implies $u = 0$, and hence M^{-1} exists. The weighted function space $L^2_A(a, b)$ consists of $\mathbb{C}^n$-valued functions (or, rather, equivalence classes - see section 4 below) with inner product

$$(u, v)_A := \int_a^b v(x)^* A(x) u(x) dx$$

and norm $\|u\|_A := (u, u)_A^{1/2}$.

2 Preliminaries

We are concerned with the Hamiltonian system

$$Jy^{'}(x, \lambda) = (\lambda A(x) + B(x))\, y(x, \lambda) \tag{2. 1}$$

where, for each $x \in [a, b)$, $A(x),\ B(x)\ \in \mathbf{C^{2n,2n}}$, the set of complex $2n \times 2n$ matrices, $\lambda \in \mathbf{C}$ and

$$J = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix} \tag{2. 2}$$

where $0_n, I_n$ are the zero and unit matrices respectively in $\mathbf{C^{n,n}}$. We always assume that

(i) $A(x) \geq 0$ for a.e. $x \in [a, b)$

(ii) (2. 1) is *regular* at the end-point a of $[a, b)$, but may be *singular* at b.

We mean by the term *regular* that there exists a unique solution of (2. 1) which satisfies a prescribed initial condition

$$y(a) = (\zeta_1, ..., \zeta_{2n})^T.$$

If $A(\cdot)$ and $B(\cdot)$ are locally integrable on $[a, b)$ then the regularity is guaranteed.

Let $y, z \in \mathrm{AC}^{\mathrm{loc}}_{\mathrm{2n}}[a, b)$, the set of $2n \times 1$ vector functions with absolutely continuous components on $[a, b)$. Then

$$z^* Jy^{'} - (Jz^{'})^* y = (z^* Jy)^{'} \tag{2. 3}$$

and, for $X, Y \in [a, b)$, this yields the Green's formula

$$\int_X^Y \left\{ z^* J y' - (J z')^* y \right\} dx = (z^* J y)(Y) - (z^* J y)(X). \tag{2. 4}$$

If we denote the standard Euclidean inner product on $\mathbf{C^{2n}}$ by $< \cdot, \cdot >$ and set

$$[y, z](x) = z^*(x) J y(x) \tag{2. 5}$$

then (2. 4) can be written as

$$\int_X^Y \left\{ < J y', z > - < y, J z' > \right\} dx = [y, z](Y) - [y, z](X) \tag{2. 6}$$

which will aid transparency when the underlying operator theory is discussed later.

Let $\theta(\cdot, \lambda), \phi(\cdot, \lambda)$ be the solutions of (2. 1) which satisfy

$$\theta(a, \lambda) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \quad \phi(a, \lambda) = \begin{pmatrix} -\beta_2 \\ \beta_1 \end{pmatrix} \tag{2. 7}$$

where $\alpha_i, \beta_i \in \mathbf{C^{n,n}}$, $i = 1, 2$, and suppose that,

(iii)

$$\sum_{i=1}^{2} \alpha_i^* \alpha_i = \sum_{i=1}^{2} \beta_i^* \beta_i = I_n, \tag{2. 8}$$

(iv)

$$\alpha_1^* \beta_2 = \alpha_2^* \beta_1. \tag{2. 9}$$

The $\mathbf{C^{2n,2n}}$-valued function $Y = (\theta \mid \phi)$ is then a fundamental matrix for (2. 1) satisfying

$$Y(a, \lambda) = E = \begin{pmatrix} \alpha_1 & -\beta_2 \\ \alpha_2 & \beta_1 \end{pmatrix}. \tag{2. 10}$$

Note that (2. 8), (2. 9) are equivalent to E being unitary. Let $\eta(\cdot, \lambda), \chi(\cdot, \lambda)$ be the solutions of the adjoint problem

$$J z'(x, \lambda) = \left(\overline{\lambda} A(x) + B^*(x) \right) z(x, \lambda), \quad x \in [a, b) \tag{2. 11}$$

which satisfy

$$\eta(a, \lambda) = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \quad \chi(a, \lambda) = \begin{pmatrix} -\alpha_2 \\ \alpha_1 \end{pmatrix}. \tag{2. 12}$$

Then $Z = (\eta \mid \chi)$ is a fundamental matrix for (2. 11) satisfying

$$Z(a, \lambda) = \hat{E} := \begin{pmatrix} \beta_1 & -\alpha_2 \\ \beta_2 & \alpha_1 \end{pmatrix}. \tag{2. 13}$$

In fact, $\hat{E} = -J(E^{-1})^* J$ and

$$Z = -J(Y^{-1})^* J. \tag{2. 14}$$

This follows from the uniqueness theorem for initial value problems, since the right-hand side of (2. 14) can be shown to be a solution matrix of (2. 11).

It follows that

$$Z^* JY = -J^* Y^{-1} J^* JY = J. \tag{2. 15}$$

Thus

$$(Z^* JY)(x) = \begin{pmatrix} \eta^* J\theta & \eta^* J\phi \\ \chi^* J\theta & \chi^* J\phi \end{pmatrix}(x) = J \tag{2. 16}$$

or, in the notation (2. 5)

$$\begin{pmatrix} [\theta, \eta](x) & [\phi, \eta](x) \\ [\theta, \chi](x) & [\phi, \chi](x) \end{pmatrix} = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix}. \tag{2. 17}$$

Throughout the paper, we exhibit dependence on λ only when necessary.

3 Weyl-Sims type analysis

The definition of our Weyl-Sims sets will be motivated by that for the Weyl circles in [8, section 4] modified for our non-self-adjoint problem in a way which recovers the description in [3] for the Sturm-Liouville case.

We choose $U, H_1, H_2 \in \mathbf{C^{n,n}}$ such that H_1 and H_2 are Hermitian, and define

$$\mathcal{U}_{2n} := \begin{pmatrix} -U & H_1 \\ -H_2 & U^* \end{pmatrix}, \tag{3. 1}$$

so that

$$\mathcal{U}_{2n} J = (\mathcal{U}_{2n} J)^*. \tag{3. 2}$$

We assume moreover that U, H_1, H_2 are chosen such that

$$\mathcal{U}_{2n} J \text{ has exactly } n \text{ positive and exactly } n \text{ negative eigenvalues} \tag{3. 3}$$

which is true, e.g., if $H_1 = H_2 = 0$ and U is invertible, or if $H_1 < 0, H_2 > 0$ and $U = 0$.

It follows that, with Y a $\mathbf{C^{2n,2n}}$-valued column-wise solution of (2. 1),

$$\begin{aligned} Y^* \mathcal{U}_{2n} JY \,\big|_a^X &= \int_a^X (Y^* \mathcal{U}_{2n} JY)' dx \\ &= \int_a^X \left\{ Y^* \mathcal{U}_{2n} JY' + [Y^* (\mathcal{U}_{2n} J)^* Y']^* \right\} dx \\ &= 2 \int_a^X \mathbf{Re}\, [Y^* \mathcal{U}_{2n} JY'] dx \\ &= 2 \int_a^X \mathbf{Re}\, [Y^* \mathcal{U}_{2n} (\lambda A(x) + B(x)) Y] dx = 2 \int_a^X Y^* C_\lambda Y dx \end{aligned} \tag{3. 4}$$

where

$$C_\lambda(x) := \mathbf{Re}[\mathcal{U}_{2n}(\lambda A(x) + B(x))]. \tag{3. 5}$$

The integral (3. 4) at $X = b$ plays the role of the modified Dirichlet integral and is the analogue of [3, (2.25)]; see Example 3.2 below. We define $(k, \mathcal{U}_{2n}) \in \mathbf{C} \times \mathbf{C}^{\mathbf{2n},\mathbf{2n}}$ to be an *admissible pair* if (3. 1)-(3. 3) hold and, for all $x \in [a, b)$,

$$C_k(x) \equiv \mathbf{Re}[\mathcal{U}_{2n}(kA(x) + B(x))] \geq 0 \tag{3. 6}$$

and for any admissible pair $(k, \mathcal{U}_{2n})$ we define

$$\Lambda(k, \mathcal{U}_{2n}) := \{\lambda \in \mathbf{C} : \text{for some } \delta > \ 0, \ \mathbf{Re}\ [(\lambda - k)\mathcal{U}_{2n}A(x)] \geq \delta \mathcal{U}_{2n}A(x)\mathcal{U}_{2n}^*, \ \text{for all } x \in [a, b)\} \tag{3. 7}$$

and suppose that

$$Q := \mathbf{C} \setminus \bigcup_{\mathcal{S}} \Lambda(k, \mathcal{U}_{2n}) \neq \mathbf{C} \tag{3. 8}$$

where the union is over the set $\mathcal{S}$ of admissible pairs $(k, \mathcal{U}_{2n})$. In particular (3. 6), (3. 7), and $A \geq 0$ imply

$$C_\lambda(x) \geq 0 \text{ for all } x \in [a, b), \lambda \in \Lambda(k, \mathcal{U}_{2n}), \ (k, \mathcal{U}_{2n}) \in \mathcal{S}. \tag{3. 9}$$

Let $Y = (\theta \mid \phi)$ in (3. 4) and set (see (3. 2))

$$Y^* \mathcal{U}_{2n} J Y =: 2 \begin{pmatrix} S & T \\ T^* & P \end{pmatrix}. \tag{3. 10}$$

Then, with $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$, where ϕ_1, ϕ_2 are respectively the $\mathbf{C}^{\mathbf{n},\mathbf{n}}$-valued functions consisting of the first and last $n-$components, and θ_1, θ_2 similarly defined for $\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$, we have

$$\begin{aligned} S(x, \lambda) &\equiv \mathbf{Re}[\theta_1^* U \theta_2](x) + \frac{1}{2}(\theta_1^* H_1 \theta_1 + \theta_2^* H_2 \theta_2)(x), \\ T(x, \lambda) &\equiv \frac{1}{2}\{\theta_1^* U \phi_2 + \theta_2^* U^* \phi_1 + \theta_1^* H_1 \phi_1 + \theta_2^* H_2 \phi_2\}(x), \\ P(x, \lambda) &\equiv \mathbf{Re}[\phi_1^* U \phi_2](x) + \frac{1}{2}(\phi_1^* H_1 \phi_1 + \phi_2^* H_2 \phi_2)(x). \end{aligned} \tag{3. 11}$$

We shall require U, H_1, H_2, and the initial values β_1, β_2 in (2. 7) to be such that

$$P(a) = -\mathbf{Re}\ (\beta_2^* U \beta_1) + \frac{1}{2}(\beta_2^* H_1 \beta_2 + \beta_1^* H_2 \beta_1) \geq 0. \tag{3. 12}$$

This generates the subset

$$\mathcal{S}_a = \{(k, \mathcal{U}_{2n}) \in \mathcal{S} : (3.12) \text{ is satisfied}\} \tag{3. 13}$$

which is the analogue of the set $\mathcal{S}(\alpha)$ in [3, (2.6) and (2.7)]. In addition to (3. 9), (3. 12) we require the following definiteness condition to hold for $(k, \mathcal{U}_{2n}) \in S_a$ and $\lambda \in \Lambda(k, \mathcal{U}_{2n})$: for $\zeta \in \mathbf{C}^n$,

$$[P(a)\zeta = 0 \text{ and } (C_\lambda \phi)(x)\zeta = 0 \text{ for } a.e.\ x \in [a,b)] \Rightarrow \zeta = 0. \tag{3. 14}$$

We will show in Lemma 3.5 below that this implies $P(X) > 0$ for X sufficiently large.

Remark 3.1 *1. Since (3. 6), (3. 7) imply $C_\lambda(x) \geq \delta \mathcal{U}_{2n} A(x) \mathcal{U}_{2n}^* \geq 0$ for all $x \in [a,b)$, $\lambda \in \Lambda(k, \mathcal{U}_{2n}), (k, \mathcal{U}_{2n}) \in \mathcal{S}$, the condition $(C_\lambda \phi)(x)\zeta = 0$ in the premise of (3. 14) may be replaced by*

$$A(x)\mathcal{U}_{2n}^* \phi(x)\zeta = 0. \tag{3. 15}$$

2. It is an interesting observation that, for $\zeta \in \mathbf{C}^{\mathbf{n}}$ satisfying $(C_\lambda \phi)(x)\zeta = 0$, $z(x) := \mathcal{U}_{2n}^* \phi(x)\zeta$ solves the adjoint problem (2. 11), since

$$(\bar{\lambda} A + B^*)\mathcal{U}_{2n}^* \phi \zeta = -\mathcal{U}_{2n}(\lambda A + B)\phi\zeta = -\mathcal{U}_{2n} J \phi' \zeta = -(\mathcal{U}_{2n} J)^* \phi' \zeta = J \mathcal{U}_{2n}^* \phi' \zeta.$$

The *Weyl-Sims sets* for (2. 1) are defined by

$$D_X(\lambda) := \{ l \in \mathbf{C}^{\mathbf{n},\mathbf{n}} : [(\theta + \phi l)^* (\mathcal{U}_{2n} J)(\theta + \phi l)](X) \leq 0 \}. \tag{3. 16}$$

Note that, by (3. 10),

$$[(\theta + \phi l)^* (\mathcal{U}_{2n} J)(\theta + \phi l)](X) = 2 \left[l^* P(X) l + T(X) l + l^* T^*(X) + S(X) \right]. \tag{3. 17}$$

For X sufficiently large (such that $P(X) > 0$, in view of Lemma 3.5 below), we use the notation

$$\mathcal{C}(X, \lambda) := -(P^{-1} T^*)(X, \lambda), \quad \mathcal{R}(X, \lambda) := (T P^{-1} T^* - S)(X, \lambda), \tag{3. 18}$$

in which case

$$[(\theta + \phi l)^* (\mathcal{U}_{2n} J)(\theta + \phi l)](X) = 2 \left[(l - \mathcal{C})^*(X) P(X) (l - \mathcal{C})(X) - \mathcal{R}(X) \right]. \tag{3. 19}$$

Thus,

$$\begin{aligned} D_X(\lambda) = \quad & \{ l \in \mathbf{C}^{\mathbf{n},\mathbf{n}} : (l - \mathcal{C}(X,\lambda))^* P(X,\lambda)(l - \mathcal{C}(X,\lambda)) \leq \mathcal{R}(X,\lambda) \} && (3.\ 20) \\ = \quad & \{ l \in \mathbf{C}^{\mathbf{n},\mathbf{n}} : l - \mathcal{C}(X,\lambda) = P^{-1/2}(X,\lambda) V \mathcal{R}^{1/2}(X,\lambda) \\ & \text{for some } V \in \mathbf{C}^{\mathbf{n},\mathbf{n}},\ V^* V \leq I_n \} && (3.\ 21) \end{aligned}$$

is some kind of generalised (Weyl) circle; note that $\mathcal{R}(X, \lambda) > 0$ will be shown in Lemma 3.5 below.

Example 3.2 *We now consider the case $n = 1$, and make the choice*

$$\mathcal{U}_2 = \begin{pmatrix} -u & 0 \\ 0 & \bar{u} \end{pmatrix} \tag{3. 22}$$

for some nonzero $u \in \mathbf{C}$, which clearly satisfies the assumption (3. 3). Note that (3. 12) here means

$$\mathbf{Re}(u\beta_1\bar{\beta}_2) \leq 0.$$

Equation (3. 10) gives

$$2\begin{pmatrix} S & T \\ T^* & P \end{pmatrix} = (\theta \mid \phi)^* \begin{pmatrix} 0 & u \\ \bar{u} & 0 \end{pmatrix} (\theta \mid \phi)\,,$$

so that when $P(x) > 0$ by a straightforward calculation,

$$P = \mathbf{Re}(u\bar{\phi}_1\phi_2), \quad \mathcal{R} = \frac{|\, u\,|^2 |\, W\,|^2}{4P}, \;\; \mathcal{C} = -\frac{1}{2P}(u\bar{\phi}_1\theta_2 + \bar{u}\theta_1\bar{\phi}_2),$$

with $W = \theta_1\phi_2 - \theta_2\phi_1$. Thus (3. 20) is

$$D_X(\lambda) = \left\{ l \in \mathrm{C} : |\, l - \mathcal{C}(X)\,| \leq r(X) := \frac{|\, u\,||\, W(X)\,|}{2P(X)} \right\}.$$

As a special case, we consider the second-order scalar Sturm-Liouville problem

$$-(pv')' + qv = \lambda wv \;\text{ on } [a, b), \tag{3. 23}$$

where p, q are complex valued functions, $p \neq 0$ a.e. on $[a, b)$ and $\frac{1}{p}, q \in L^1_{loc}[a, b)$, and $w \in L^1_{loc}[a, b)$ is positive throughout $[a, b)$. The equation (3. 23), interpreted in the quasi-derivative sense, can be written in the form (2. 1) with

$$A = \begin{pmatrix} w & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -q & 0 \\ 0 & 1/p \end{pmatrix}, \quad y = \begin{pmatrix} v \\ pv' \end{pmatrix}.$$

Choosing $u = e^{i\eta}$ (for some $\eta \in \mathbf{R}$) in (3. 22), we easily obtain

$$C_\lambda(x) = \begin{pmatrix} \mathbf{Re}[e^{i\eta}(q - \lambda w)(x)] & 0 \\ 0 & \frac{1}{|p(x)|^2}\mathbf{Re}[e^{i\eta}p(x)] \end{pmatrix} \tag{3. 24}$$

so that (for our special choice of $\mathcal{U}_2$)

$$(k, \mathcal{U}_2) \in \mathcal{S} \Leftrightarrow \{\mathbf{Re}[e^{i\eta}(q - kw)(x)] \geq 0 \text{ and } \mathbf{Re}[e^{i\eta}p(x)] \geq 0 \text{ for all } x \in [a, b)\}$$

a condition which is equivalent to (2. 3) in [3]. Moreover, for (k, η) such that $(k, \mathcal{U}_2) \in \mathcal{S}$,

$$\Lambda(k, \mathcal{U}_2) = \left\{\lambda \in \mathbf{C} : \mathbf{Re}[(\lambda - k)e^{i\eta}] < 0\right\}, \tag{3. 25}$$

which corresponds to (2.4) in [3].

To show that the definiteness condition (3. 14) is always satisfied in this example, let $\zeta \in \mathbf{C}^n$, $y(x) := \phi(x)\zeta$, *and* $C_\lambda(x)y(x) = 0$ *for a.e.* $x \in [a,b)$. *This implies* $y_1 \equiv 0$ *by (3. 24), since* $\mathbf{Re}\ [e^{i\eta}(q-\lambda w)(x)] > 0$ *(for all* $x \in [a,b)$*) if* $\lambda \in \Lambda(k,\mathcal{U}_2), (k,\mathcal{U}_2) \in \mathcal{S}$. *Since* y *is of the form* $(v, pv')^T$, $y_1 \equiv 0$ *implies* $y \equiv 0$, *and thus,* $\zeta = 0$.

Example 3.3 *Here, we consider the fourth order scalar problem*

$$(p_2 v'')'' - (p_1 v')' + p_0 v = \lambda w v \tag{3. 26}$$

where p_0, p_1, p_2 *are complex-valued functions,* $p_2 \neq 0$ *a.e. on* (a,b), *and* $w \in L^2_{loc}[a,b)$ *is positive throughout* $[a,b)$. *We introduce the vector* $y = (v, v^{[1]}, v^{[3]}, v^{[2]})^T$ *of quasi-derivatives defined by*

$$v^{[1]} = v',\, v^{[2]} = p_2 v'',\, v^{[3]} = -(v^{[2]})' + p_1 v'.$$

In this notation, (3. 26), interpreted in the quasi-derivative sense, can be written in the form (2. 1) where

$$A = \begin{pmatrix} w & & & \\ & & & \\ & & & \\ & & & \end{pmatrix} \quad B = \begin{pmatrix} -p_0 & & & \\ & -p_1 & 1 & \\ & 1 & 0 & \\ & & & \frac{1}{p_2} \end{pmatrix},$$

with all remaining entries being zero. Choosing, for some $\eta \in \mathbf{R}$,

$$\mathcal{U}_4 := \begin{pmatrix} -e^{i\eta} I_2 & 0_2 \\ 0_2 & e^{-i\eta} I_2 \end{pmatrix},$$

we obtain, by straightforward calculations,

$$C_\lambda(x) = \mathrm{diag}\left(\mathbf{Re}[e^{i\eta}(p_0 - \lambda w)(x)], \mathbf{Re}[e^{i\eta} p_1(x)], 0, \frac{1}{|\ p_2(x)\ |^2}\mathbf{Re}[e^{i\eta} p_2(x)]\right).$$

Consequently, for this choice of $\mathcal{U}_4$,

$$(k,\mathcal{U}_4) \in \mathcal{S} \Leftrightarrow \mathbf{Re}[e^{i\eta}(p_0 - kw)(x)] \geq 0, \mathbf{Re}[e^{i\eta} p_1(x)] \geq 0, \text{ and } \mathbf{Re}[e^{i\eta} p_2(x)] \geq 0 \text{ for all } x \in (a,b),$$

and, for $(k,\mathcal{U}_4) \in \mathcal{S}$,

$$\Lambda(k,\mathcal{U}_4) = \left\{\lambda \in \mathbf{C} : \mathbf{Re}[(\lambda - k)e^{i\eta}] < 0\right\}.$$

By a similar argument as in Example 3.2, we see that the definiteness condition (3. 14) holds.

Example 3.4 *The Orr-Sommerfeld equation*

$$(-D^2 + a^2)^2 u + iaR[V(-D^2 + a^2)u + V'' u] = \lambda(-D^2 + a^2)u, \quad D \equiv \frac{d}{dx}, \tag{3. 27}$$

posed on an interval $I \subseteq \mathbf{R}$, is one of the governing equations of (linearised) hydrodynamic stability. It is strongly related to the stable or unstable reaction of a flow, perpendicular to I, with given real-valued flow profile $V \in \mathbf{C}^2(I)$ and Reynolds number $R > 0$, to a single mode perturbation with wave number $a > 0$, (see, e.g., [9, 10]). Introducing the variables

$$y_1 = -u'' + a^2 u, \quad y_2 = u, \quad y_3 = (-u'' + a^2 u)', \quad y_4 = u',$$

we find that (3. 27) is equivalent to the Hamiltonian system (2. 1) with

$$A = \begin{pmatrix} 1 & & & \\ & & & \\ & & & \\ & & & \end{pmatrix}, \quad B = \begin{pmatrix} -a^2 - iaRV & -iaRV'' & \multicolumn{2}{c}{0_2} \\ 1 & -a^2 & & \\ \multicolumn{2}{c}{0_2} & \multicolumn{2}{c}{I_2} \end{pmatrix},$$

all remaining entries of A being zero. Choosing the same matrix $\mathcal{U}_4$ (depending on a real parameter η) as in the previous example, we calculate

$$C_\lambda = \begin{pmatrix} a^2\cos\eta - aRV\sin\eta - \mathbf{Re}(\lambda e^{i\eta}) & \frac{1}{2}(aRV'' i e^{i\eta} - e^{-i\eta}) & 0_2 \\ -\frac{1}{2}(aRV'' i e^{-i\eta} + e^{i\eta}) & a^2\cos\eta & \\ \multicolumn{2}{c}{0_2} & (\cos\eta) I_2 \end{pmatrix}$$

for the matrix defined in (3. 5). Consequently,

$$C_\lambda(x) \geq 0 \iff \begin{cases} \cos\eta > 0 \quad \text{and} \\ \mathbf{Re}(\lambda e^{i\eta}) \leq a^2\cos\eta - aRV(x)\sin\eta - \frac{1 + (aRV''(x))^2 + 2aRV''(x)\sin(2\eta)}{4a^2\cos\eta}, \end{cases} \tag{3. 28}$$

and $C_\lambda(x) > 0$ if the last inequality in (3. 28) is strict. Therefore, $(k, \mathcal{U}_4) \in \mathcal{S}$ if and only if the right-hand side of (3. 28) holds with k in place of λ, for all $x \in I$. Moreover, for such k and η, $\Lambda(k, \mathcal{U}_4)$ is again the same as in the previous example. In particular, by (3. 28) and the remark thereafter, $C_\lambda(x) > 0$ ($x \in I$) for $\lambda \in \Lambda(k, \mathcal{U}_4)$, $(k, \mathcal{U}_4) \in \mathcal{S}$, so that the definiteness condition (3. 14) is satisfied.

Lemma 3.5 *Let $\lambda \in \Lambda(k, \mathcal{U}_{2n})$ for some $(k, \mathcal{U}_{2n}) \in \mathcal{S}_a$. Then, some $X_0 = X_0(\lambda) \in [a, b)$ exists such that*

(i) *$P(X, \lambda)$ is non-decreasing in X, $P(X, \lambda) \geq 0$, and, for $X \geq X_0$, $P(X, \lambda) > 0$,*

(ii) *$D_X(\lambda) \neq \emptyset$ for $X \geq X_0$,*

(iii) *for $X \geq X_0$, $\mathcal{R}(X, \lambda)$ is non-increasing in X, and $\mathcal{R}(X, \lambda) > 0$.*

□

Proof

(i) From (3. 4) and (3. 10)

$$\begin{pmatrix} S & T \\ T^* & P \end{pmatrix} |_a^X = \int_a^X \begin{pmatrix} \theta^* C_\lambda \theta & \theta^* C_\lambda \phi \\ \phi^* C_\lambda \theta & \phi^* C_\lambda \phi \end{pmatrix} dx \tag{3. 29}$$

and hence, by (3. 12) and (3. 9),

$$P(X) = P(a) + \int_a^X \phi^* C_\lambda \phi dx \tag{3. 30}$$

is positive semi-definite and non-decreasing. To prove that $P(X) > 0$ for X sufficiently large, assume, on the contrary, that there exist sequences $(X_m) \to b$ and (ζ_m) in $\mathbf{C}^n$, $\zeta_m^* \zeta_m = 1$, such that

$$\zeta_m^* P(X_m) \zeta_m = 0 \ \text{ for all } m \in \mathbf{N}. \tag{3. 31}$$

We can extract a subsequence (ζ_{m_j}) such that

$$\zeta_{m_j} \to \zeta, \ \text{as } j \to \infty, \tag{3. 32}$$

for some $\zeta \in \mathbf{C}^n$, $\zeta^* \zeta = 1$. Since P is positive semi-definite and non-decreasing, (3. 31) implies

$$\zeta_{m_j}^* P(X) \zeta_{m_j} = 0 \text{ for all } j \in \mathbf{N} \text{ and } X \in [a, X_{m_j}],$$

whence (3. 32) yields

$$\zeta^* P(X) \zeta = 0 \text{ for all } X \in [a, b).$$

Thus, using (3. 30) and (3. 9), (3. 12), we obtain

$$\zeta^* P(a) \zeta = 0, \ \ \zeta^* (\phi^* C_\lambda \phi)(x) \zeta = 0 \text{ for } a.e.\ x \in [a, b).$$

Therefore, on using (3. 9), (3. 12), and (3. 14) we arrive at the contradiction $\zeta = 0$.

(ii) Let $X \geq X_0$. In view of (3. 3) and (3. 10), n eigenvalues of $\begin{pmatrix} S & T \\ T^* & P \end{pmatrix}(X)$ are negative . Let the columns of $\begin{pmatrix} \Xi_1(X) \\ \Xi_2(X) \end{pmatrix} \in \mathbf{C}^{2n,n}$ be corresponding orthonormal eigenvectors. Then,

$$(\Xi_1(X)^* \mid \Xi_2(X)^*) \begin{pmatrix} S & T \\ T^* & P \end{pmatrix} (X) \begin{pmatrix} \Xi_1(X) \\ \Xi_2(X) \end{pmatrix} < 0. \tag{3. 33}$$

Now, assume that some nontrivial $\zeta \in \mathbf{C^n}$ exists such that $\Xi_1(X)\zeta = 0$. Then, $\Xi_2(X)\zeta \neq 0$ since $\begin{pmatrix} \Xi_1(X) \\ \Xi_2(X) \end{pmatrix}$ has rank n, whence **(i)** yields

$$\zeta^* \Xi_2(X)^* P(X) \Xi_2(X) \zeta > 0.$$

On the other hand, the opposite inequality is obtained by multiplication of (3. 33) by ζ^* and ζ from the left and the right, respectively. This contradiction implies that $\Xi_1(X)$ is invertible.

On multiplying (3. 33) by $(\Xi_1(X)^{-1})^*$ and $\Xi_1(X)^{-1}$ from the left and the right, respectively, we obtain

$$(I_n \mid l^*)\begin{pmatrix} S & T \\ T^* & P \end{pmatrix}(X)\begin{pmatrix} I_n \\ l \end{pmatrix} \leq 0$$

for $l := (\Xi_2\Xi_1^{-1})(X)$. Thus, $l \in D_X(\lambda)$ by (3. 10) and (3. 16).

(iii) On using (3. 18) and (3. 29), we obtain, for $X \geq X_0$,

$$\begin{aligned} \mathcal{R}' &= T'P^{-1}T^* + TP^{-1}(T^*)' - TP^{-1}P'P^{-1}T^* - S' \\ &= \theta^*C_\lambda\phi P^{-1}T^* + TP^{-1}\phi^*C_\lambda\theta - TP^{-1}\phi^*C_\lambda\phi P^{-1}T^* - \theta^*C_\lambda\theta \\ &= -(\theta + \phi\mathcal{C})^*C_\lambda(\theta + \phi\mathcal{C}). \end{aligned}$$

It follows from (3. 9) that $\mathcal{R}'(X) \leq 0$, and consequently $\mathcal{R}(X)$ is non-increasing in X. Moreover, $\mathcal{R}(X) \geq 0$ by (3. 20) and parts (i) and(ii) of the lemma. To prove that $\mathcal{R}(X) > 0$, observe that $\begin{pmatrix} S & T \\ T^* & P \end{pmatrix}(X)$ is invertible by (3. 3) and (3. 10). Then, with $\begin{pmatrix} e \\ f \end{pmatrix} \in \mathbf{C}^{\mathbf{2n,n}}$ formed by the first n columns of its inverse,

$$\begin{pmatrix} S & T \\ T^* & P \end{pmatrix}\begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} I_n \\ 0_n \end{pmatrix}. \tag{3. 34}$$

This implies $f = -P^{-1}T^*e$ and, by (3. 18),

$$\mathcal{R}e = -I_n, \tag{3. 35}$$

so that $\mathcal{R}$ is invertible, whence $\mathcal{R} > 0$ (since $\mathcal{R} \geq 0$).

□

Theorem 3.6 *Let $\lambda \in \Lambda(k, \mathcal{U}_{2n})$ for some $(k, \mathcal{U}_{2n}) \in \mathcal{S}_a$. Then, with X_0 from Lemma 3.5,*

(i) $X > Y \Rightarrow D_X(\lambda) \subseteq D_Y(\lambda)$,

(ii) $X > Y \Rightarrow D_X(\lambda) - \mathcal{C}(X,\lambda) \subseteq D_Y(\lambda) - \mathcal{C}(Y,\lambda)$,

(iii) $D_X(\lambda)$ *is compact and convex for all* $X \in [X_0, b)$,

(iv) $\mathcal{C}_b(\lambda) := \lim_{X\to b}\mathcal{C}(X,\lambda)$ *exists,*

(v)

$$\bigcap_{X\in[X_0,b)} [D_X(\lambda) - \mathcal{C}(X,\lambda)] = D_b(\lambda) - \mathcal{C}_b(\lambda), \tag{3. 36}$$

where $D_b(\lambda) := \bigcap_{X\in[X_0,b)} D_X(\lambda)$,

(vi) $\mathcal{C}_b(\lambda) \in D_b(\lambda)$.

Proof

(i) We have for every $l \in \mathbf{C^{n,n}}$ by (3. 4), after multiplying by $(I_n \mid l^*)$ on the left and $\begin{pmatrix} I_n \\ l \end{pmatrix}$ on the right,

$$[(\theta+\phi l)^*(\mathcal{U}_{2n}J)(\theta+\phi l)](X) = [(\theta+\phi l)^*(\mathcal{U}_{2n}J)(\theta+\phi l)](a) + 2\int_a^X (\theta+\phi l)^* C_\lambda (\theta+\phi l)dx.$$

This gives

$$l \in D_X(\lambda) \Leftrightarrow \int_a^X (\theta+\phi l)^* C_\lambda(\theta+\phi l)dx \le \mathcal{A}(l) := -\frac{1}{2}[(\theta+\phi l)^*(\mathcal{U}_{2n}J)(\theta+\phi l)](a).$$

So by (3. 9), for $Y < X$,

$$l \in D_X(\lambda) \Rightarrow \int_a^X (\theta+\phi l)^* C_\lambda(\theta+\phi l)dx \le \mathcal{A}(l) \Rightarrow \int_a^Y (\theta+\phi l)^* C_\lambda(\theta+\phi l)dx \le \mathcal{A}(l) \Rightarrow l \in D_Y(\lambda).$$

(ii) We have from (3. 20) that

$$k \in D_X - \mathcal{C}(X) \Leftrightarrow k^* P(X) k \le \mathcal{R}(X).$$

By Lemma 3.5 (i),(iii), P is non-decreasing and $\mathcal{R}$ is non-increasing in X. Thus, for $Y < X$,

$$k^* P(Y) k \le \mathcal{R}(Y)$$

and hence $k \in D_Y - \mathcal{C}(Y)$.

(iii) Let $X \ge X_0$ where X_0 is as in Lemma 3.5. Obviously, $D_X(\lambda)$ is closed, and by (3. 20) and Lemma 3.5 (i), $D_X(\lambda)$ is bounded. The compactness follows since dim $\mathbf{C^{n,n}} < \infty$.

To prove convexity let $l_1, l_2 \in D_X$. Then $l_i - \mathcal{C} = P^{-1/2} V_i \mathcal{R}^{1/2}$, $\quad V_i^* V_i \le I_n$, by (3. 21). For $\alpha \in [0,1]$,

$$\| \alpha V_1 + (1-\alpha)V_2 \| \le \alpha \| V_1 \| + (1-\alpha) \| V_2 \| \le 1.$$

Thus $(\alpha V_1 + (1-\alpha)V_2)^*(\alpha V_1 + (1-\alpha)V_2) \le I_n$ and so $\alpha l_1 + (1-\alpha) l_2 \in D_X$.

(iv) For $Y > X \geq X_0$, let $l \in D_Y$, so that, by (3. 21),

$$l = \mathcal{C}(Y) + (P^{-1/2} V_Y \mathcal{R}^{1/2})(Y)$$

for some matrix V_Y with $V_Y^* V_Y \leq I_n$. By the nesting property, $D_Y \subseteq D_X$; hence $l \in D_X$ and so, by (3. 21),

$$l = \mathcal{C}(X) + (P^{-1/2} V_X \mathcal{R}^{1/2})(X),$$

for some V_X such that $V_X^* V_X \leq I_n$. Hence

$$\mathcal{C}(Y) - \mathcal{C}(X) = (P^{-1/2} V_X \mathcal{R}^{1/2})(X) - (P^{-1/2} V_Y \mathcal{R}^{1/2})(Y).$$

By Lemma 3.5 (i) and (iii), the function $F : V_Y \mapsto V_X$ is well-defined and is a continuous map from the unit ball in $\mathbf{C^{n,n}}$ into itself. So it has a fixed point V by Brouwer's fixed point theorem, and replacing V_Y and V_X by V gives

$$\begin{aligned} & \| \mathcal{C}(X) - \mathcal{C}(Y) \| = \| P^{-1/2}(X) V \mathcal{R}^{1/2}(X) - P^{-1/2}(Y) V \mathcal{R}^{1/2}(Y) \| \\ \leq & \| P^{-1/2}(X) V \mathcal{R}^{1/2}(X) - P^{-1/2}(Y) V \mathcal{R}^{1/2}(X) \| + \| P^{-1/2}(Y) V \mathcal{R}^{1/2}(X) - P^{-1/2}(Y) V \mathcal{R}^{1/2}(Y) \| \\ \leq & \| P^{-1/2}(X) - P^{-1/2}(Y) \| \cdot \| \mathcal{R}^{1/2}(X) \| + \| P^{-1/2}(Y) \| \cdot \| \mathcal{R}^{1/2}(X) - \mathcal{R}^{1/2}(Y) \| . \end{aligned}$$

Both $P^{-1/2}(X)$ and $\mathcal{R}^{1/2}(X)$ have limits as $X \to b$, so the centres $\mathcal{C}$ form a Cauchy sequence and converge.

(v) Let $p \in \bigcap_{X \in [X_0, b)} (D_X - \mathcal{C}(X))$. Then for all $X > X_0$,

$$p = l_X - \mathcal{C}(X)$$

for some $l_X \in D_X$. Since $\mathcal{C}(X) \to \mathcal{C}_b$ as $X \to b$ by (iv) we obtain $l_X \to l_b := p + \mathcal{C}_b$, and $l_b \in \bigcap_{X > X_0} D_X$ in view of (i) and the closedness of D_X. Thus

$$p = l_b - \mathcal{C}_b \in D_b - \mathcal{C}_b.$$

Conversely let $p = l_b - \mathcal{C}_b, \quad l_b \in \bigcap_{X > X_0} D_X$. Then $q(X) := l_b - \mathcal{C}(X) \to p$ as $X \to b$, and $q(X) \in D_X - \mathcal{C}(X)$. By (ii) (and the closeness of D_X), we obtain $p \in \bigcap_{X \in [X_0, b)} (D_X - \mathcal{C}(X))$.

(vi) By (3. 20) and Lemma 3.5 (iii), the left-hand side of (3. 36) contains $0 \in \mathbf{C^{n,n}}$, whence (vi) follows from (v).

□

4 Square integrable solutions

Let $L^2_A(a,b)$ denote the Hilbert space of measurable $\mathbf{C^n}$-valued functions y for which

$$\| y \|_A^2 := \int_a^b y^* A y dx < \infty. \tag{4. 1}$$

Since $A(x)$ is merely assumed to be positive semi-definite for any $x \in [a,b)$, elements of $L^2_A(a,b)$ are equivalent if they are equal almost everywhere outside the set $\{x : A(x) \text{ is singular }\} =: \mathcal{N}_A$ and for $x \in \mathcal{N}_A$ their difference lies in the kernel of $A(x)$.

In the proof of Theorem 3.2 (i), we saw that $l \in D_X(\lambda)$ if and only if

$$\int_a^X (\theta + \phi l)^* C_\lambda (\theta + \phi l) dx \leq \mathcal{A}(a, \mathcal{U}_{2n}, l) := -\frac{1}{2}[(\theta + \phi l)^* (\mathcal{U}_{2n} J)(\theta + \phi l)](a).$$

Hence, if $l = l(\lambda) \in D_b(\lambda)$, we have that

$$\int_a^b (\theta + \phi l)^* C_\lambda (\theta + \phi l) dx \leq \mathcal{A}(a, \mathcal{U}_{2n}, l(\lambda)). \tag{4. 2}$$

If $\lambda \in \Lambda(k, \mathcal{U}_{2n})$ for some $(k, \mathcal{U}_{2n}) \in \mathcal{S}_a$ it follows that, with

$$\psi(x,\lambda) := \theta(x,\lambda) + \phi(x,\lambda) l(\lambda) \quad \in \mathbf{C^{2n,n}} \tag{4. 3}$$

and $\hat{A} := \mathcal{U}_{2n} A \mathcal{U}_{2n}^*$,

$$\int_a^b \psi^* C_k \psi dx + \int_a^b \psi^* \hat{A} \psi dx < \infty. \tag{4. 4}$$

or $\psi \in L^2_{C_k}(a,b) \cap L^2_{\hat{A}}(a,b)$.

Theorem 4.1 *There are $m+n$ linearly independent $L^2_{C_\lambda}(a,b)$-solutions of (2. 1) for $\lambda \in \Lambda(k, \mathcal{U}_{2n})$, with $(k, \mathcal{U}_{2n}) \in \mathcal{S}_a$, if and only if there are m linearly independent $L^2_{C_\lambda}(a,b)$-solutions of the form ϕK, where $K \in \mathbf{C^{n,m}}$ is of rank m, that is if and only if m eigenvalues of $P(X)$ remain bounded as $X \to b$.*

There are $m+n$ linearly independent $L^2_A(a,b)$-solutions of (2. 1) for $\lambda \in \Lambda(k, \mathcal{U}_{2n})$, with $(k, \mathcal{U}_{2n}) \in \mathcal{S}_a$, if and only if there are m linearly independent $L^2_A(a,b)$-solutions of the form $\phi K'$, where $K' \in \mathbf{C^{n,m}}$ is of rank m.

Proof Let there be $m+n$ linearly independent $L^2_{C_\lambda}$-solutions. Thus besides the columns of $\theta + \phi l$ (which form n linearly independent $L^2_{C_\lambda}$ -solutions for any fixed $l \in D_b(\lambda)$, by (4. 2)), m additional $L^2_{C_\lambda}$-solutions are given by the columns of $\theta\alpha + \phi\beta$, for some $\alpha, \beta \in \mathbf{C^{n,m}}$ such that

$$\operatorname{rank} \begin{pmatrix} I_n & \alpha \\ l & \beta \end{pmatrix} = n + m$$

Multiplying the latter matrix by $\begin{pmatrix} \alpha \\ -I_m \end{pmatrix} \in \mathbf{C^{m+n,m}}$ (which has rank m) we obtain that

$$\operatorname{rank}\begin{pmatrix} 0 \\ l\alpha - \beta \end{pmatrix} = \operatorname{rank}\left[\begin{pmatrix} I_n & \alpha \\ l & \beta \end{pmatrix}\begin{pmatrix} \alpha \\ -I_m \end{pmatrix}\right] = m. \tag{4. 5}$$

Since the columns of both $(\theta + \phi l)\alpha$ and $\theta\alpha + \phi\beta$ are $L^2_{C_\lambda}$-solutions, we obtain by subtraction that the columns of

$$\phi \cdot (l\alpha - \beta)$$

are $L^2_{C_\lambda}$-solutions. By (4. 5), this yields m linearly independent solutions of the form ϕK.

Suppose conversely that there are m linearly independent $L^2_{C_\lambda}$-solutions of that form, i.e., some $K \in \mathbf{C^{n,m}}$ with rank m exists such that the columns of ϕK are $L^2_{C_\lambda}$-solutions, then the columns of $\theta + \phi l$ and ϕK altogether give $m + n$ linearly independent $L^2_{C_\lambda}$-solutions, since

$$\operatorname{rank}\begin{pmatrix} I_n & 0 \\ l & K \end{pmatrix} = n + m.$$

The last part of the theorem concerning L^2_A-solutions follows by the same argument.

The assertion about the eigenvalues of $P(X)$ in the first part is now obtained as follows. The identity (3. 30) implies that, for $K \in \mathbf{C^{n,m}}$ with rank m, the m columns of ϕK being in $L^2_{C_\lambda}(a, b)$ is equivalent to

$$K^* P(X) K = K^* P(a) K + \int_a^X K^*(\phi^* C_\lambda \phi)(x) K dx$$

being bounded as $X \to b$. This implies, by the min-max principle, that the smallest m eigenvalues of $P(X)$ remain bounded as $X \to b$. Conversely, let $\lambda_1(X) \le \lambda_2(X) \le ... \le \lambda_m(X)$ be eigenvalues of $P(X)$, with $\lambda_m(X)$ bounded as $X \to b$, and let $K(X) \in \mathbf{C}^{n,m}$ be formed column-wise by corresponding orthonormal eigenvectors. By compactness of the unit sphere in $\mathbf{C}^n$, $K(X_j) \to K \in \mathbf{C}^{n,m}$ for some sequence $X_j \to b$, and K has orthonormal columns and therefore rank m. Furthermore, by Lemma 3.5 (i),

$$\begin{aligned} K(X_j)^* P(Y) K(X_j) &\le K(X_j)^* P(X_j) K(X_j) = \operatorname{diag}(\lambda_1(X_j), ..., \lambda_m(X_j)) \\ &\le \text{const.} \cdot I_m \ \text{ for all } j \text{ and all } Y \in [a, X_j]. \end{aligned}$$

Letting $j \to \infty$, we obtain the boundedness of $K^* P(Y) K$ as $Y \to b$. □

Define

$$\mathcal{L}(\lambda) := D_b(\lambda) - \mathcal{C}_b(\lambda) \tag{4. 6}$$

$$\mathcal{N}(\lambda) := \bigcup_{N \in \mathcal{L}(\lambda)} \operatorname{Range}(N) \tag{4. 7}$$

$$r := \max\{m \in \{0, ..., n\} : \mathcal{N}(\lambda) \text{ contains } m \text{ linearly independent vectors}\} \tag{4. 8}$$

Theorem 4.2 *Let $\lambda \in \Lambda(k,\mathcal{U}_{2n})$, $(k,\mathcal{U}_{2n}) \in \mathcal{S}_a$. Then*

(i) *there are at least $n+r$ linearly independent solutions of (2. 1) in $L^2_{C_\lambda}(a,b)$;*

(ii) *if $\mathcal{R}(X) \not\to 0_n$ as $X \to b$, then there are exactly $n+r$ linearly independent solutions of (2. 1) in $L^2_{C_\lambda}(a,b)$.*

Proof

(i) Let $\zeta_1, ...\zeta_r \in \mathcal{N}$ be linearly independent. Then, $\zeta_i = N_i \eta_i$ where $N_i \in \mathcal{L}$, $\eta_i \in \mathbf{C^n}$ $(i = 1, ..., r)$. By Theorem 3.6 (v), for each X there exist $l_1(X), ..., l_r(X) \in D_X$ such that

$$(l_i - \mathcal{C})(X) = N_i \quad (i = 1, ...r)$$

and therefore,

$$(l_i - \mathcal{C})(X)\eta_i = \zeta_i \quad (i = 1, ...r).$$

Thus, since $l_i(X) \in D_X$, (3. 20) implies that

$$\zeta_i^* P(X)\zeta_i \leq \eta_i^* \mathcal{R}(X)\eta_i \quad \text{is bounded as } X \to b \;\; (i = 1, ..., r). \tag{4. 9}$$

.

Let $V_r(X) = [\zeta_1, ..., \zeta_r]$, the linear span of $\zeta_1, ..., \zeta_r$ and , let $\lambda_1(X), ..., \lambda_n(X)$ denote the eigenvalues of $P(X)$ in non-decreasing order. Then, by the min-max principle,

$$\lambda_r(X) = \min_{\substack{W \subseteq C^n \\ \dim W = r}} \max_{\substack{\zeta \in W \\ \zeta \neq 0}} \frac{\zeta^* P(X)\zeta}{\zeta^*\zeta} \leq \max_{\substack{\zeta \in V_r(X) \\ \zeta \neq 0}} \frac{\zeta^* P(X)\zeta}{\zeta^*\zeta} \leq \frac{\sum_{i=1}^{r} \zeta_i^* P(X)\zeta_i}{\lambda_{\min}(\mathrm{Gram}(\zeta_1, \ldots, \zeta_r))}$$

which is bounded as $X \to b$ in view of (4. 9). Consequently, at least r eigenvalues of $P(X)$ are bounded as $X \to b$, which proves the assertion, by Theorem 4.1.

(ii) Let $n+s$ be the exact number of linearly independent $L^2_{C_\lambda}(a,b)$-solutions. By Theorem 4.1, the s smallest eigenvalues $\lambda_1(X), ..., \lambda_s(X)$ of $P(X)$ are therefore bounded as $X \to b$. Let $\zeta_1(X), ..., \zeta_s(X)$ denote orthonormal eigenvectors corresponding to $\lambda_1(X), ..., \lambda_s(X)$. We now need Lemma 4.3 below. Then extracting subsequences inductively, a common subsequence (X_{m_j}) (independent of i) can be extracted satisfying

$$\zeta_i(X_{m_j}) \longrightarrow \hat{\zeta}_i \quad \text{for } i = 1, ..., k,$$

as $j \to \infty$. This ensures that $\hat{\zeta}_1, ..., \hat{\zeta}_s \in \mathcal{N}$ are orthogonal and in particular linearly independent. By the definition of r, this provides $s \leq r$. Since part (i) of the theorem yields $s \geq r$, we obtain $s = r$ and thus the assertion. □

Lemma 4.3 *Under the hypothesis of Theorem 4.2 (ii) we have that for each* $i \in \{1, ..., s\}$ *and any sequence* (X_m) *converging to* b*, there exists a subsequence* $(X_{m_j}^{(i)})_{j\in\mathbf{N}}$ *such that* $(\zeta_i(X_{m_j}^{(i)}))_{j\in\mathbf{N}}$ *converges to some* $\hat{\zeta}_i \in \mathcal{N}$*, where* $\zeta_i(X), i = 1, ..., s$*, are the orthonormal eigenvectors in part (ii) of the proof of Theorem 4.2.*

Proof Fix $i \in \{1, ...s\}$ and let (X_m) denote some sequence tending to b. For the moment, fix m, and regard all X-dependent terms as being evaluated at X_m. Let $U_1 \in \mathbf{C^{n,n}}$ denote a unitary matrix with columns formed by orthonormal eigenvectors of P, with ζ_i in the first column.

Moreover, by assumption, at least one eigenvalue ν of $\mathcal{R}$ does not tend to zero (and is therefore, due to monotonicity, bounded away from zero) as $X \to b$. Let U_2 denote a unitary matrix with columns formed by orthonormal eigenvectors of $\mathcal{R}$ with an eigenvector ψ corresponding to ν in the first column.

Let

$$l := \mathcal{C} + P^{-1/2}U_1U_2^*\mathcal{R}^{1/2}, \tag{4. 10}$$

so that, by (3. 21),

$$l \in D_X \text{ for } X = X_m. \tag{4. 11}$$

Furthermore, on choosing

$$\eta := \sqrt{\frac{\lambda_i}{\nu}}\psi \tag{4. 12}$$

we obtain from (4. 10), with e_1 denoting the first canonical unit vector

$$(l - \mathcal{C})\eta = P^{-1/2}U_1U_2^*(\sqrt{\nu} \cdot \sqrt{\frac{\lambda_i}{\nu}}\psi) = \sqrt{\lambda_i}P^{-1/2}U_1 \cdot e_1 = \sqrt{\lambda_i}P^{-1/2}\zeta_i = \zeta_i.$$

Now let m vary again. Due to (4. 11) and the nesting property Theorem 3.6 (i), the sequence $l(X_m)$ is bounded. From (4. 12) and the fact that $\lambda_i(X_m)$ is bounded, $\nu(X_m)$ is bounded away from zero and $\mid \psi \mid= 1$, it follows that the sequence $\eta(X_m)$ is bounded. Finally, the sequence $\zeta_i(X_m)$ is bounded since $\mid \zeta_i \mid\equiv 1$. Thus, along a subsequence $(X_{m_j}^{(i)})_{j\in N}$, l, η, ζ_i tend to limits $\hat{l}, \hat{\eta}, \hat{\zeta}_i$, respectively. Moreover, since $\mathcal{C}(X)$ tends to $\mathcal{C}_b$ as $X \to b$, we obtain from (4. 13)

$$(\hat{l} - C_b)\hat{\eta} = \hat{\zeta}_i. \tag{4. 13}$$

The nesting property Theorem 3.6 (i) and (4. 11) give $\hat{l} \in D_b$, so that $\hat{l} - \mathcal{C}_b \in \mathcal{L}$. Thus, by (4. 13), $\hat{\zeta}_i \in \mathcal{N}$, which proves the lemma. □

Corollary 4.4 *If* $r \geq 1$*, there are exactly* $n + r$ *linearly independent solutions of (2. 1) in* $L^2_{C_\lambda}(a, b)$*.*

Proof This follows from Theorem 4.2 since $\mathcal{R}(X)$ tending to 0_n implies in turn $\mathcal{L} = \{0\}$, $\mathcal{N} = \{0\}$ and $r = 0$. □

The case when $\mathcal{R}(X) \to 0_n$ as $X \to b$ remains to be investigated. To analyse this we turn to the equation which is formally adjoint to (2. 1), namely

$$Jz' = (\bar{\lambda}A + B^*)z. \tag{4. 14}$$

Let $Z = (\eta \mid \chi)$ be the fundamental matrix for (4. 14) satisfying (2. 13) and set (cf (3. 10))

$$Z^*\hat{\mathcal{U}}_{2n}JZ =: 2\begin{pmatrix} \hat{S} & \hat{T} \\ \hat{T}^* & \hat{P} \end{pmatrix} \tag{4. 15}$$

for some $\hat{\mathcal{U}}_{2n}$ satisfying corresponding conditions (3. 2) and (3. 3) . The preceding analysis applied to (4. 14) now requires (3. 5)-(3. 7) to be replaced by

$$\hat{C}_k(x) := \mathbf{Re}[(kA(x) + B(x))\hat{\mathcal{U}}_{2n}^*] \geq 0 \tag{4. 16}$$

$$\hat{\Lambda}(k, \hat{\mathcal{U}}_{2n}) := \{\lambda \in \mathbf{C} : \text{ for some } \delta > 0,\ \mathbf{Re}[(\lambda - k)A(x)\hat{\mathcal{U}}_{2n}^*] \geq \delta\hat{\mathcal{U}}_{2n}A(x)\hat{\mathcal{U}}_{2n}^* \text{for all } x \in [a, b)\}. \tag{4. 17}$$

The corresponding set of admissible pairs $(k, \hat{\mathcal{U}}_{2n})$ is denoted by $\hat{\mathcal{S}}$, and $\hat{\mathcal{S}}_a$ is the subset of $\hat{\mathcal{S}}$ for which

$$\hat{P}(a) = (-\alpha_2^* \mid \alpha_1^*)(\hat{\mathcal{U}}_{2n}J)\begin{pmatrix} -\alpha_2 \\ \alpha_1 \end{pmatrix} \geq 0. \tag{4. 18}$$

We now make the special choice

$$\hat{\mathcal{U}}_{2n} := \mathcal{U}_{2n}^{-1}, \tag{4. 19}$$

in which case we obtain

$$C_\lambda = \mathcal{U}_{2n}\hat{C}_\lambda\mathcal{U}_{2n}^* \text{ for all } \lambda \in \mathbf{C}, \tag{4. 20}$$

$$(k, \mathcal{U}_{2n}) \in \mathcal{S} \Leftrightarrow (k, \mathcal{U}_{2n}^{-1}) \in \hat{\mathcal{S}}, \tag{4. 21}$$

$$\hat{\Lambda}(k, \mathcal{U}_{2n}^{-1}) = \{\lambda \in \mathbf{C} : \text{ for some } \delta > 0, \mathbf{Re}[(\lambda - k)\mathcal{U}_{2n}A(x)] \geq \delta A(x) \text{ for all } x \in [a, b)\}. \tag{4. 22}$$

From (2. 15),

$$I_{2n} = -(JZ^*)(JY) = -(JY)(JZ^*) = -(JYJZ^*)^* = -ZJY^*J \tag{4. 23}$$

and

$$JZJY^* = -I_{2n}.$$

Consequently,

$$J = Z^*JY = -(Z^*\hat{\mathcal{U}}_{2n}JZ)J(Y^*\mathcal{U}_{2n}JY) = -4\begin{pmatrix} \hat{S} & \hat{T} \\ \hat{T}^* & \hat{P} \end{pmatrix} J \begin{pmatrix} S & T \\ T^* & P \end{pmatrix}, \tag{4. 24}$$

and hence, with e, f from (3. 34),

$$J\begin{pmatrix} e \\ f \end{pmatrix} = -4\begin{pmatrix} \hat{S} & \hat{T} \\ \hat{T}^* & \hat{P} \end{pmatrix} J \begin{pmatrix} I_n \\ 0_n \end{pmatrix} = -4\begin{pmatrix} \hat{T} \\ \hat{P} \end{pmatrix}.$$

A straightforward calculation and use of (3. 35) now gives, in the notation of (3. 18) and its analogue for the adjoint problem, that, for X sufficiently large and $\lambda \in \Lambda(k, \mathcal{U}_{2n}) \cap \hat{\Lambda}(k, \mathcal{U}_{2n}^{-1})$,

$$\hat{\mathcal{C}} = \mathcal{C}^*, \tag{4. 25}$$

$$\hat{P} = \frac{1}{4}\mathcal{R}^{-1}, \tag{4. 26}$$

and similarly,

$$P = \frac{1}{4}\hat{\mathcal{R}}^{-1} \tag{4. 27}$$

(c.f. [8, Lemma 4.3] when $B = B^*$). For reasons of symmetry between problems (2. 1) and its formal adjoint (4. 14), one would expect an analogous definiteness condition to that of (3. 14) to hold for (4. 14). We now prove that this condition

$$\left[\hat{P}(a)\hat{\zeta} = 0 \text{ and } (\hat{C}_\lambda \chi)(x)\hat{\zeta} = 0 \text{ for } a.e.\ x \in [a, b)\right] \Rightarrow \hat{\zeta} = 0 \tag{4. 28}$$

is in fact equivalent to (3. 14).

Lemma 4.5 *For each $\lambda \in \mathbf{C}$, (3. 14) and (4. 28) are equivalent.*

Proof By symmetry it is sufficient to show that (4. 28) implies (3. 14). So let (4. 28) hold, and let $\zeta \in \mathbf{C^n}$ satisfy the premise of (3. 14). Then, by Remark 3.1 (2), $\mathcal{U}_{2n}^*\phi\zeta$ solves (4. 14), whence

$$\mathcal{U}_{2n}^*\phi(x)\zeta = Z(x)\eta \ \ (x \in [a, b)) \text{ for some } \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \in \mathbf{C^{2n}}.$$

On multiplying by $Y(x)^*J$ from the left and using (3. 2), (3. 10), (2. 15), we obtain $\eta_1 = 0$, $\eta_2 = 2T(a)\zeta =: \hat{\zeta}$, whence

$$\mathcal{U}_{2n}^*\phi(x)\zeta = \chi(x)\hat{\zeta} \ \ (x \in [a, b)).$$

From (4. 20) and $C_\lambda\phi\zeta = 0$ a.e., we obtain

$$\hat{C}_\lambda(x)\chi(x)\hat{\zeta} = 0 \text{ for } a.e.\ x \in [a, b).$$

Moreover, since $\hat{P}T = \hat{T}^*P$ by (3. 18) and (4. 25), we obtain

$$\hat{P}(a)\hat{\zeta} = 2\hat{P}(a)T(a)\zeta = 2\hat{T}^*(a)P(a)\zeta = 0.$$

Thus, $\hat{\zeta}$ satisfies the premise of (4. 28), whence $\hat{\zeta} = 0$. Consequently,

$$\begin{pmatrix} T(a) \\ P(a) \end{pmatrix} \zeta = 0,$$

implying $\zeta = 0$ since $\begin{pmatrix} T(a) \\ P(a) \end{pmatrix}$ has rank n by (3. 10). □

Denoting the Weyl-Sims sets associated with (4. 14) by $\hat{D}_X(\lambda)$, we have from (3. 20), (4. 25)-(4. 27) (and the fact that $K^*K \le I_n \Leftrightarrow KK^* \le I_n$ for all $K \in \mathbf{C}^{\mathbf{n}\cdot\mathbf{n}}$) for $\lambda \in \Lambda(k, \mathcal{U}_{2n}) \cap \hat{\Lambda}(k, \mathcal{U}_{2n}^{-1})$ and X sufficiently large

$$\begin{aligned} l \in D_X(\lambda) \;&\Leftrightarrow\; (l - \mathcal{C}(X))^* \hat{\mathcal{R}}(X)^{-1} (l - \mathcal{C}(X)) \le \hat{P}(X)^{-1} \\ &\Leftrightarrow\; [\hat{\mathcal{R}}(X)^{-1/2}(l - \mathcal{C}(X))\hat{P}(X)^{1/2}]^* [\hat{\mathcal{R}}(X)^{-1/2}(l - \mathcal{C}(X))\hat{P}(X)^{1/2}] \le I_n \\ &\Leftrightarrow\; [\hat{\mathcal{R}}(X)^{-1/2}(l - \mathcal{C}(X))\hat{P}(X)^{1/2}][\hat{\mathcal{R}}(X)^{-1/2}(l - \mathcal{C}(X))\hat{P}(X)^{1/2}]^* \le I_n \\ &\Leftrightarrow\; (l^* - \hat{\mathcal{C}}(X))^* \hat{P}(X)(l^* - \hat{\mathcal{C}}(X)) \le \hat{\mathcal{R}}(X) \\ &\Leftrightarrow\; l^* \in \hat{D}_X(\lambda) \end{aligned} \qquad (4.\ 29)$$

$$(4.\ 30)$$

Hence, if $\hat{D}_X(\lambda)^* := \{l^* : l \in \hat{D}_X(\lambda)\}$, we have, for $\lambda \in \Lambda(k, \mathcal{U}_{2n}) \cap \hat{\Lambda}(k, \mathcal{U}_{2n}^{-1})$,

$$D_b(\lambda) \equiv \bigcap_{X>a} D_X(\lambda) = \bigcap_{X>a} \hat{D}_X(\lambda)^* =: \hat{D}_b(\lambda)^*. \qquad (4.\ 31)$$

Moreover,

$$\mathcal{C}_b(\lambda) = \lim_{X \to b} \mathcal{C}(X) = \lim_{X \to b} \hat{\mathcal{C}}(X)^* =: \hat{\mathcal{C}}_b(\lambda)^*, \qquad (4.\ 32)$$

and from (4. 25) and (4. 31),

$$\mathcal{L}(\lambda) = D_b(\lambda) - \mathcal{C}_b(\lambda) = \left(\hat{D}_b(\lambda) - \hat{\mathcal{C}}_b(\lambda)\right)^* =: \hat{\mathcal{L}}(\lambda)^*. \qquad (4.\ 33)$$

It is clear from (4. 33) that $r = 0$ if and only if $\hat{r} = 0$, where $\hat{r}$ is the analogue of r in (4. 8) for problem (4. 14). It remains an open question if (4. 33) implies any other useful relation between $\mathcal{N}(\lambda)$ and $\hat{\mathcal{N}}(\lambda)$ (defined analogously to (4. 7)), and thus between r and $\hat{r}$.

While Theorem 4.2 and Corollary 4.4 give full information in the cases $\mathcal{R}(X) \not\to 0_n$ (as $X \to b$) and $r \ge 1$ respectively, the following theorem treats the complementary cases.

Theorem 4.6 *Let $\lambda \in \Lambda(k, \mathcal{U}_{2n}) \cap \hat{\Lambda}(k, \mathcal{U}_{2n}^{-1})$, $(k, \mathcal{U}_{2n}) \in \mathcal{S}_a$, $(k, \mathcal{U}_{2n}^{-1}) \in \hat{\mathcal{S}}_a$. Then,*

(i) *if $\mathcal{R}(X) \to 0_n$ as $X \to b$, (4. 14) has exactly n linearly independent solutions in $L^2_{\hat{C}_\lambda}(a, b)$;*

(ii) *if $r = 0$, then $\hat{r} = 0$ and at least one of the equations (2. 1), (4. 14) has exactly n linearly independent solutions in $L^2_{C_\lambda}(a, b)$, $L^2_{\hat{C}_\lambda}(a, b)$ respectively.*

Proof

(i) This is an immediate consequence of (4. 26) and Theorem 4.1.

(ii) We have already noted that $r = 0$ if and only if $\hat{r} = 0$. If $\mathcal{R}(X) \to 0_n$, then (4. 14) has exactly n linearly independent solutions in $L^2_{\hat{C}_\lambda}(a,b)$ by (i). If $\mathcal{R}(X) \not\to 0_n$, then (2. 1) has exactly n linearly independent $L^2_{\hat{C}_\lambda}(a,b)$ solutions by Theorem 4.2 (ii). □

Suppose that (2. 1) has exactly m linearly independent solutions in $L^2_{C_\lambda}(a,b)$. Then, for $\lambda \in \Lambda(k, \mathcal{U}_{2n})$, $(k, \mathcal{U}_{2n}) \in \mathcal{S}_a$, we have proved that

$$n \leq m \leq 2n, \tag{4. 34}$$

where the first inequality follows from the n columns of $\psi = \theta + \phi l$, $l \in D_b(\lambda)$, being in $L^2_{C_\lambda}(a,b)$. If

$$C_\lambda \geq \delta A \text{ for some } \delta > 0, \tag{4. 35}$$

then

$$L^2_{C_\lambda}(a,b) \subseteq L^2_A(a,b). \tag{4. 36}$$

Thus, there are $s \in \{m, m+1, ...\}$ solutions of (2. 1) in $L^2_A(a,b)$. For the Sturm-Liouville case when $n = 1$, it was proved in [11] and [3] that all these possibilities for s can be realized and they constitute the Sims limit-point-limit-circle characterisation.

Remark 4.7 *The condition (4. 35) holds in the following cases:*

1. *by (3. 6) and (3. 7), in the case*

$$\hat{A} := \mathcal{U}_{2n} A \mathcal{U}^*_{2n} \geq \gamma A \text{ for some } \gamma > 0, \tag{4. 37}$$

2. *by (4. 22), (4. 21) and (3. 6), in the case* $\lambda \in \hat{\Lambda}(k, \mathcal{U}^{-1}_{2n})$.

Remark 4.8 *If* $\lambda \in \Lambda(k, \mathcal{U}_{2n})$, *then* $\hat{C}_\lambda \geq \delta A$ *for some* $\delta > 0$, *and hence* $L^2_{\hat{C}_\lambda}(a,b) \subseteq L^2_A(a,b)$. *Thus, if* $\lambda \in \Lambda(k, \mathcal{U}_{2n}) \cap \hat{\Lambda}(k, \mathcal{U}^{-1}_{2n})$,

$$L^2_{C_\lambda}(a,b) \cup L^2_{\hat{C}_\lambda}(a,b) \subseteq L^2_A(a,b).$$

Remark 4.9 *The condition (4. 37) implies, by (3. 7), (4. 22), that*

$$\Lambda(k, \mathcal{U}_{2n}) \subseteq \hat{\Lambda}(k, \mathcal{U}^{-1}_{2n}) \tag{4. 38}$$

for $(k, \mathcal{U}_{2n}) \in \mathcal{S}$ *(implying* $(k, \mathcal{U}^{-1}_{2n}) \in \hat{\mathcal{S}}$ *by (4. 21)), whence Theorem 4.6 holds indeed for all* $\lambda \in \Lambda(k, \mathcal{U}_{2n})$. *If, in addition to (4. 37), also a reverse inequality* $A \geq \tilde{\gamma}\hat{A}$ *holds for some* $\tilde{\gamma} > 0$, *i.e., if*

$$\mathcal{U}_{2n} A \mathcal{U}^*_{2n} \asymp A,$$

equality holds in (4. 38).

5 Operator theoretic implications

Let $\lambda \in \Lambda(k,\mathcal{U}_{2n}) \cap \hat{\Lambda}(k,\mathcal{U}_{2n}^{-1})$, $M(\lambda) \in D_b(\lambda)$, and set

$$\begin{aligned} \psi(x,\lambda) &= \theta(x,\lambda)+\phi(x,\lambda)M(\lambda) \qquad (5.\ 1)\\ \zeta(x,\lambda) &= \eta(x,\lambda)+\chi(x,\lambda)M^*(\lambda) \qquad (5.\ 2)\end{aligned}$$

where θ, ϕ are the solutions of (2. 1) satisfying (2. 7), and η, χ are the solutions of (2. 11) satisfying (2. 12).Define

$$G(x,y;\lambda) := \begin{cases} \phi(x,\lambda)\zeta^*(y,\lambda) & a<x<y<b \\ \psi(x,\lambda)\chi^*(y,\lambda) & a<y<x<b \end{cases} \qquad (5.\ 3)$$

$$\begin{aligned}\tilde{G}(x,y;\lambda) &:= \begin{cases} \chi(x,\lambda)\psi^*(y,\lambda) & a<x<y<b \\ \zeta(x,\lambda)\phi^*(y,\lambda) & a<y<x<b \end{cases} \qquad (5.\ 4)\\ &= G^*(y,x;\lambda) \qquad (5.\ 5)\end{aligned}$$

and, for $f \in L^2_A(a,b)$,

$$R_\lambda f(x) := \int_a^b G(x,y;\lambda)A(y)f(y)dy, \qquad (5.\ 6)$$

$$\tilde{R}_\lambda f(x) := \int_a^b \tilde{G}(x,y;\lambda)A(y)f(y)dy. \qquad (5.\ 7)$$

It follows from (4. 23) that $JYJZ^* = -I_{2n}$, where $Y = (\theta \mid \phi)$ and $Z = (\eta \mid \chi)$, i.e.

$$J(\theta\chi^* - \phi\eta^*)(x) = I_{2n} \qquad (5.\ 8)$$

and this readily yields

$$J(R_\lambda f)' = (\lambda A + B)(R_\lambda f) + Af. \qquad (5.\ 9)$$

Also, by (2. 7) and (2. 9),

$$(\alpha_1^* \mid \alpha_2^*)(R_\lambda f)(a) = 0_n \qquad (5.\ 10)$$

or, using (2. 12) and the notation of (2. 5),

$$[R_\lambda f, \chi(\cdot,\lambda)](a) \equiv \chi^*(a,\lambda)JR_\lambda f(a) = 0_n. \qquad (5.\ 11)$$

Since $\psi = Y \begin{pmatrix} I_n \\ M \end{pmatrix}$ and $\zeta = Z \begin{pmatrix} I_n \\ M^* \end{pmatrix}$ we derive from (2. 15) that

$$[\psi(\cdot,\lambda),\zeta(\cdot,\lambda)](X) = (I_n \mid M(\lambda))(Z^*JY)(X)\begin{pmatrix} I_n \\ M(\lambda) \end{pmatrix} = 0_n \qquad (5.\ 12)$$

and

$$[\phi(\cdot,\lambda),\zeta(\cdot,\lambda)](X) = (I_n \mid M(\lambda))(Z^*JY)(X)\begin{pmatrix} 0_n \\ I_n \end{pmatrix} = -I_n. \tag{5. 13}$$

Hence

$$[R_\lambda f,\zeta(\cdot,\lambda)](b) := \lim_{X\to b}[R_\lambda f,\zeta(\cdot,\lambda)](X) = -\lim_{X\to b}\int_X^b \zeta^*(y,\lambda)A(y)f(y)dy = 0, \tag{5. 14}$$

since $\zeta(\cdot,\lambda) \in L^2_{\hat{C}_\lambda}(a,b) \subseteq L^2_A(a,b)$.

For the adjoint problem we obtain similarly

$$J(\tilde{R}_\lambda f)' = (\overline{\lambda}A + B^*)(\tilde{R}_\lambda f) + Af \tag{5. 15}$$

$$[\tilde{R}_\lambda f,\phi(\cdot,\lambda)](a) = 0, \tag{5. 16}$$

$$[\tilde{R}_\lambda f,\psi(\cdot,\lambda)](b) = 0. \tag{5. 17}$$

Theorem 5.1 *Let $f \in L^2_A(a,b)$ and $\lambda \in \Lambda(k,\mathcal{U}_{2n}) \cap \hat{\Lambda}(k,\mathcal{U}_{2n}^{-1})$. Then, with $\Phi \equiv R_\lambda f$ and $\hat{A} = \mathcal{U}_{2n}A\mathcal{U}_{2n}^*$,*

$$\| \Phi \|^2_{C_k} + \delta \| \Phi \|^2_{\hat{A}} \leq \epsilon \| \Phi \|^2_{\hat{A}} + \frac{1}{4\epsilon} \| f \|^2_A \tag{5. 18}$$

for any $0 < \epsilon < \delta$, with $\delta = \delta(\lambda)$ as in (3. 7), and

$$\| \Phi \|_{\hat{A}} \leq \frac{1}{\delta} \| f \|_A . \tag{5. 19}$$

Thus, if $\hat{A} \geq \gamma A$ for some $\gamma > 0$, R_λ is bounded on $L^2_A(a,b)$.

Proof Let $f_X = f$ on $[a,X]$, $f_X = 0$ on (X,b), and $\Phi_X = R_\lambda f_X$. Then, from (2. 4) and (5. 9) (c.f. (3. 4))

$$2\int_a^X \Phi_X^* C_\lambda \Phi_X dx = \Phi_X^*\mathcal{U}_{2n}J\Phi_X \big|_a^X - 2\int_a^X \mathbf{Re}[\Phi_X^*\mathcal{U}_{2n}Af_X]dx. \tag{5. 20}$$

Let $T_1(x), T_2(x)$ be the quantities

$$T_1(x) := \int_a^x \chi^* A f_X dy, \quad T_2(x) := \int_x^X \zeta^* A f_X dy.$$

Then, by (5. 3),

$$\Phi_X = \psi T_1 + \phi T_2 = Y\begin{pmatrix} T_1 \\ MT_1 + T_2 \end{pmatrix},$$

so that, by (3. 10),

$$\Phi_X^*\left(\mathcal{U}_{2n}J\right)\Phi_X = 2\begin{pmatrix} T_1 \\ MT_1 + T_2 \end{pmatrix}^*\begin{pmatrix} S & T \\ T^* & P \end{pmatrix}\begin{pmatrix} T_1 \\ MT_1 + T_2 \end{pmatrix}.$$

Since $T_2(X)=0$, we have, for X sufficiently large,

$$\begin{aligned}
\left[\Phi_X^*\left(\mathcal{U}_{2n}J\right)\Phi_X\right](X) &= 2T_1^*(X)\left[S+TM+M^*T^*+M^*PM\right](X)T_1(X)\\
&= 2T_1^*(X)\left[(M+P^{-1}T^*)^*P(M+P^{-1}T^*)-TP^{-1}T^*+S\right](X)T_1(X)\\
&\leq 0
\end{aligned}$$

by (3. 20) and (3. 18), since $M\in D_b(\lambda)\subseteq D_X(\lambda)$. Moreover, since $T_1(a)=0$,

$$\left[\Phi_X^*\left(\mathcal{U}_{2n}J\right)\Phi_X\right](a)=2\left(T_2^*PT_2\right)(a)\geq 0$$

from (3. 12). On substituting in (5. 20) and noting that

$$\begin{aligned}
\mid \Phi_X^*\mathcal{U}_{2n}Af_X\mid &\equiv |< Af_X,\mathcal{U}_{2n}^*\Phi_X >|\leq < Af_X, f_X >^{1/2}< A\mathcal{U}_{2n}^*\Phi_X,\mathcal{U}_{2n}^*\Phi_X >^{1/2}\\
&= (f_X^*Af_X)^{1/2}(\Phi_X^*\hat{A}\Phi_X)^{1/2},
\end{aligned}$$

we obtain

$$\begin{aligned}
0 &\leq \int_a^X \Phi_X^*C_\lambda\Phi_X dx \leq \left|\int_a^X \mathbf{Re}[\Phi_X^*\mathcal{U}_{2n}Af]dx\right|\\
&\leq \left(\int_a^X f^*Afdx\right)^{1/2}\left(\int_a^X \Phi_X^*\hat{A}\Phi_X dx\right)^{1/2}\\
&\leq \frac{1}{4\epsilon}\parallel f\parallel_A^2+\epsilon\parallel \Phi_X\parallel_{\hat{A}}^2 .
\end{aligned}$$

Hence, by (3. 7),

$$\int_a^X \Phi_X^*C_k\Phi_X dx+(\delta-\epsilon)\int_a^X \Phi_X^*\hat{A}\Phi_X dx\leq \frac{1}{4\epsilon}\parallel f\parallel_A^2 .$$

As $X\to b$, $\Phi_X\to\Phi\equiv R_\lambda f$, and when $\epsilon<\delta$, (5. 18) follows by Fatou's Lemma. This yields (5. 19) by choosing $\epsilon=\frac{1}{2}\delta$. □

Hereafter we shall assume that

$$\mathcal{U}_{2n}A\mathcal{U}_{2n}^*\asymp A \tag{5. 21}$$

so that by Remark 4.9, $\Lambda(k,\mathcal{U}_{2n})=\hat{\Lambda}(k,\mathcal{U}_{2n}^{-1})$. We note that (5. 21) is true if

$$\mathcal{U}_{2n}=\begin{pmatrix} -e^{i\eta}I_n & 0_n\\ 0_n & e^{-i\eta}I_n\end{pmatrix}\ \text{ and }\ A=\begin{pmatrix} A_{11} & 0_n\\ 0_n & A_{22}\end{pmatrix}, \tag{5. 22}$$

as in all our examples in section 3.

Lemma 5.2 *Let $\lambda,\mu\in\Lambda(k,\mathcal{U}_{2n})$, $(k,\mathcal{U}_{2n})\in\mathcal{S}_a$. Then*

$$\psi(x,\mu)=\psi(x,\lambda)-\phi(x,\lambda)C+(\mu-\lambda)\left(R_\lambda\psi(\cdot,\mu)\right)(x) \tag{5. 23}$$

where $C=[\psi(\cdot,\mu),\zeta(\cdot,\lambda)](b)$. If (2. 1) has precisely n linearly independent solutions in $L_A^2(a,b)$, then

$$[\psi(\cdot,\mu),\zeta(\cdot,\lambda)](b)=0. \tag{5. 24}$$

Proof By (2. 1),

$$J\psi'(\cdot,\mu) = (\lambda A + B)\psi'(\cdot,\mu) + (\mu-\lambda)\psi'(\cdot,\mu),$$

whence (5. 9) implies

$$\psi(\cdot,\mu) = \psi(\cdot,\lambda)C_1 + \phi(\cdot,\lambda)C_2 + (\mu-\lambda)R_\lambda\psi(\cdot,\mu).$$

Pre-multiplication by $(\alpha_1^* \mid \alpha_2^*)$ yields $C_1 = I_n$, by (2. 7)-(2. 9) and (5. 10). Pre-multiplying byby $\zeta(X,\lambda)J$ and using (5. 12)-(5. 14), we obtain $C_2 = -[\psi(\cdot,\mu),\zeta(\cdot,\lambda)](b)$, whence (5. 23). The last part follows since $\psi, R_\lambda\psi(\cdot,\mu) \in L^2_A(a,b)$, whereas $\phi(\cdot,\lambda)C$ only does if $C=0$, on account of Theorem 4.1. □

In the case where (2. 1) has more than n linearly independent solutions in $L^2_A(a,b)$, the set $D_b(\lambda)$ may contain more than one element, and (5. 24) can then only be true if the right selection of $M(\lambda) \in D_b(\lambda)$ and $M(\mu) \in D_b(\mu)$ (forcing C in (5. 23) to be zero) has been made. Theorem 5.4 below shows that such a selection is always possible. For proving it, we first need Lemma 5.3.

Lemma 5.3 *Let $\lambda,\mu \in \Lambda(k,\mathcal{U}_{2n})$, $(k,\mathcal{U}_{2n}) \in \mathcal{S}_a$, and $X_0 = X_0(\lambda)$ from Lemma 3.5. Moreover, let $M_0 \in D_b(\mu)$ and $\zeta(\cdot,\mu) := \eta(\cdot,\mu) + \chi(\cdot,\mu)M_0^*$. Then, for $X \geq X_0$, the (regular) boundary value problem*

$$Jy' = (\lambda A + B)y \text{ on } (a,X),\ \chi^*(a,\mu)Jy(a) = 0,\ \ \zeta^*(X,\mu)Jy(X) = 0 \qquad (5.\ 25)$$

has only the trivial solution $y \equiv 0$.

Proof Let y be a solution of (5. 25), whence $y \equiv \theta(\cdot,\lambda)\tilde{v} + \phi(\cdot,\lambda)v$ for some $\tilde{v}, v \in \mathbf{C^n}$. Since $\chi^*(a,\mu)J\theta(a,\lambda) = I_n$ and $\chi^*(a,\mu)J\phi(a,\lambda) = 0_n$ by (2. 7)-(2. 12), the boundary condition for y at a yields $\tilde{v} = 0$. Moreover, with $\psi(\cdot,\mu) := \theta(\cdot,\mu) + \phi(\cdot,\mu)M_0$, (5. 12) yields $\zeta^*(X,\mu)J\psi(X,\mu) = 0_n$. Since both $\zeta(X,\mu)$ and $\psi(X,\mu)$ have full rank, the boundary condition for y at X therefore shows that

$$y(X) = \psi(X,\mu)w \text{ for some } w \in \mathbf{C}^n.$$

Consequently, since $M_0 \in D_b(\mu) \subset D_X(\mu)$, (3. 16) yields

$$(y^*\mathcal{U}_{2n}Jy)(X) \leq 0,$$

whence, by (3. 10),

$$0 \geq v^*\phi^*(X,\lambda)\mathcal{U}_{2n}J\phi(X,\lambda)v = 2v^*P(X,\lambda)v,$$

giving $v=0$ by Lemma 3.5 (i). □

Theorem 5.4 *Let $\mu \in \Lambda(k,\mathcal{U}_{2n})$, $(k,\mathcal{U}_{2n}) \in \mathcal{S}_a$ and $M_0 \in D_b(\mu)$ be fixed. Then, there exists a function $M : \Lambda(k,\mathcal{U}_{2n}) \to \mathbf{C}^{n,n}$ such that $M(\mu) = M_0$ and, for all $\lambda \in \Lambda(k,\mathcal{U}_{2n})$, $M(\lambda) \in D_b(\lambda)$ and*

$$M(\lambda) - M_0 = (\lambda-\mu)\int_a^b \zeta^*(x,\mu)A(x)\psi(x,\lambda)dx = (\lambda-\mu)\int_a^b \zeta^*(x,\lambda)A(x)\psi(x,\mu)dx, \qquad (5.\ 26)$$

where $\psi(\cdot,\lambda) := \theta(\cdot,\lambda) + \phi(\cdot,\lambda)M(\lambda)$ and $\zeta(\cdot,\lambda) := \eta(\cdot,\lambda) + \chi(\cdot,\lambda)M(\lambda)^*$. Moreover,

$$[\psi(\cdot,\lambda),\zeta(\cdot,\mu)](b) = [\psi(\cdot,\mu),\zeta(\cdot,\lambda)](b) = 0 \text{ for all } \lambda \in \Lambda(k,\mathcal{U}_{2n}). \qquad (5.\ 27)$$

Proof Let $\psi(\cdot,\mu) := \theta(\cdot,\mu) + \phi(\cdot,\mu)M_0$ and $\zeta(\cdot,\mu) := \eta(\cdot,\mu) + \chi(\cdot,\mu)M_0^*$. For all $\lambda \in \Lambda(k,\mathcal{U}_{2n})$ and $X \geq X_0$, with $X_0 = X_0(\lambda)$ from Lemma 3.5, the matrix $\zeta^*(X,\mu)J\phi(X,\lambda)$ is invertible, since otherwise some $v \in \mathbf{C}^n\backslash\{0\}$ would exist such that $y := \phi(\cdot,\lambda)v$ solves the boundary value problem (5. 25), contradicting Lemma 5.3. Consequently

$$l_X(\lambda) := -[\zeta^*(X,\mu)J\phi(X,\lambda)]^{-1}[\zeta^*(X,\mu)J\theta(X,\lambda)] \tag{5. 28}$$

is well defined for these X, and

$$\psi_X(\cdot,\lambda) := \theta(\cdot,\lambda) + \phi(\cdot,\lambda)l_X(\lambda) \tag{5. 29}$$

satisfies the condition

$$\zeta^*(X,\mu)J\psi_X(X,\lambda) = 0_n. \tag{5. 30}$$

Moreover, (5. 12) yields

$$\zeta^*(X,\mu)J\psi(X,\mu) = 0_n. \tag{5. 31}$$

All three matrices $\zeta(X,\mu), \psi_X(X,\lambda), \psi(X,\mu)$ have rank n, so (5. 30) and (5. 31) together imply that the ranges of $\psi_X(X,\lambda)$ and $\psi(X,\mu)$ coincide, i.e., that

$$\psi_X(X,\lambda) = \psi(X,\mu)\Omega$$

for some (invertible) $\Omega \in \mathbf{C}^{n,n}$. This yields

$$\psi_X^*(X,\lambda)(\mathcal{U}_{2n}J)\psi_X(X,\lambda) = \Omega^*[\psi^*(X,\mu)(\mathcal{U}_{2n}J)\psi(X,\mu)]\Omega. \tag{5. 32}$$

By (3. 16), the matrix in brackets on the right-hand side in (5. 32) is non-positive, since $M_0 \in D_b(\mu) \subset D_X(\mu)$. Thus, the left-hand side of (5. 32) is non-positive, which in turn, again by (3. 16), shows that

$$l_X(\lambda) \in D_X(\lambda). \tag{5. 33}$$

Defining, analogously to (5. 29),

$$\zeta_X(\cdot,\lambda) := \eta(\cdot,\lambda) + \chi(\cdot,\lambda)l_X(\lambda)^* \tag{5. 34}$$

we obtain from(2. 15) that

$$\zeta_X^*(X,\lambda)J\psi_X(X,\lambda) = 0_n. \tag{5. 35}$$

Since all three matrices $\zeta(X,\lambda), \zeta_X(X,\lambda), \psi_X(X,\lambda)$ have rank n, (5. 30) and (5. 35) show that the ranges of $\zeta(X,\mu)$ and $\zeta_X(X,\lambda)$ coincide. In particular, the (n-dimensional) range of $\zeta_X(X,\lambda)$ is independent of λ (as long as $X \geq X_0(\lambda)$), whence (5. 35) implies

$$\zeta_X^*(X,\tilde{\lambda})J\psi_X(X,\lambda) = 0_n \text{ for all } \lambda,\tilde{\lambda} \in \Lambda(k,\mathcal{U}_{2n}), \text{ and } X \geq \max\{X_0(\lambda), X_0(\tilde{\lambda})\}. \tag{5. 36}$$

On using (2. 1) and (2. 11), we obtain

$$(\zeta_X^*(\cdot,\tilde{\lambda})J\psi_X(\cdot,\lambda))' = (\lambda - \tilde{\lambda})\zeta_X^*(\cdot,\tilde{\lambda})A\psi_X(\cdot,\lambda).$$

Integrating from a to X, and using (5. 36) and (2. 10), (2. 13), (2. 15), we obtain

$$l_X(\lambda) - l_X(\tilde{\lambda}) = (\lambda - \tilde{\lambda}) \int_a^X \zeta_X^*(x, \tilde{\lambda}) A(x) \psi_X(x, \lambda) dx. \tag{5. 37}$$

A second formula of this kind is obtained by interchanging λ and $\tilde{\lambda}$. Putting then $\tilde{\lambda} := \mu$ in both these formulas yields, on noting that $l_X(\mu) = M_0$ by (5. 28) and (5. 13) (and a corresponding calculation for θ),

$$l_X(\lambda) - M_0 = (\lambda - \mu) \int_a^X \zeta^*(x, \mu) A(x) \psi_X(x, \lambda) dx = (\lambda - \mu) \int_a^X \zeta_X^*(x, \lambda) A(x) \psi(x, \mu) dx \tag{5. 38}$$

for each $\lambda \in \Lambda(k, \mathcal{U}_{2n})$, kept fixed for the rest of the proof, and all $X \geq \max\{X_0(\lambda), X_0(\mu)\}$.

To achieve convergence in (5. 38) as $X \to b$ (at least along a sequence), we define $\tilde{\psi}_X(\cdot, \lambda) := \psi_X(\cdot, \lambda)$ on $[a, X]$, $\tilde{\psi}_X(\cdot, \lambda) := 0$ on (X, b), and show that $(\tilde{\psi}_X(\cdot, \lambda))_{X \in (X_0, b)}$ is bounded in $L_A^2(a, b)$. Indeed, (3. 6), (3. 7), (5. 21) yield a constant $k = k(\lambda)$ such that

$$\begin{aligned} \int_a^b \tilde{\psi}_X^*(x, \lambda) A(x) \tilde{\psi}_X(x, \lambda) dx &= \int_a^X \psi_X^*(x, \lambda) A(x) \psi_X(x, \lambda) dx \\ &\leq k \int_a^X \psi_X^*(x, \lambda) C_\lambda(x) \psi_X(x, \lambda) dx \\ &= \frac{k}{2} [\psi_X^*(X, \lambda)(\mathcal{U}_{2n} J) \psi_X(X, \lambda) - \psi_X^*(a, \lambda)(\mathcal{U}_{2n} J) \psi_X(a, \lambda)], \end{aligned}$$

by (3. 4). Here, the first boundary term is non-positive by (3. 16) and (5. 33), and the second is bounded (with respect to $X \in [X_0, b)$) by (5. 29), since $l_X(\lambda) \in D_X(\lambda) \subset D_{X_0}(\lambda)$ and the latter set is bounded by Theorem 3.6 (iii). This yields the desired boundedness property.

Therefore, along a sequence $X_m \to b$, $\tilde{\psi}_{X_m}(\cdot, \lambda)$ converges weakly in $L_A^2(a, b)$ to some $F \in L_A^2(a, b)$, i.e., for every $g \in L_A^2(a, b)$,

$$\int_a^b g^*(x) A(x) \tilde{\psi}_{X_m}(x, \lambda) dx \to \int_a^b g^*(x) A(x) F(x) dx \text{ as } m \to \infty. \tag{5. 39}$$

Moreover, since $(l_X(\lambda))_{X \in [X_0, b)}$ is bounded in $\mathbf{C}^{n,n}$, the sequence (X_m) can be chosen such that simultaneously

$$l_{X_m}(\lambda) \to M(\lambda) \text{ as } m \to \infty \tag{5. 40}$$

for some $M(\lambda) \in \mathbf{C}^{n,n}$, which by (5. 33) and Theorem 3.6 lies in $D_b(\lambda)$.

Taking compact support functions g in (5. 39) and noting that, by (5. 40) and (5. 29), $\tilde{\psi}_{X_m}(\cdot, \lambda)$ converges uniformly to $\psi(\cdot, \lambda) = \theta(\cdot, \lambda) + \phi(\cdot, \lambda) M(\lambda)$ on compact subintervals of $[a, b)$, we obtain

$$F = \psi(\cdot, \lambda) \text{ in } \mathrm{L}_\mathrm{A}^2(\mathrm{a}, \mathrm{b}).$$

Thus, choosing $g := \zeta(\cdot,\mu)$ in (5. 39),

$$\int_a^{X_m} \zeta^*(x,\mu)A(x)\psi_{X_m}(x,\lambda)dx \to \int_a^b \zeta^*(x,\mu)A(x)\psi(x,\lambda)dx \quad \text{as } m\to\infty. \tag{5. 41}$$

In a completely analogous way, we can extract a sequence (which we may assume to be (X_m) again, by successive subsequence extraction) such that

$$\int_a^{X_m} \zeta^*_{X_m}(x,\lambda)A(x)\psi(x,\mu)dx \to \int_a^b \zeta^*(x,\lambda)A(x)\psi(x,\mu)dx \quad \text{as } m\to\infty.$$

Together with (5. 40) and (5. 41), this gives (5. 26) from (5. 38).

To show (5. 27), we use (2. 1) and (2. 11) again to obtain

$$(\zeta^*(\cdot,\mu)J\psi(\cdot,\lambda))' = (\lambda-\mu)\zeta^*(\cdot,\mu)A\psi(\cdot,\lambda),$$

whence integration from a to X provides, by (2. 10), (2. 13) and (2. 15),

$$[\psi(\cdot,\lambda),\zeta(\cdot,\mu)](X) + M(\lambda) - M_0 = (\lambda-\mu)\int_a^X \zeta^*(x,\mu)A(x)\psi(x,\lambda)dx.$$

Here, the right-hand side converges as $X\to b$ since $\zeta(\cdot,\mu)$ and $\psi(\cdot,\lambda)$ are in $L^2_A(a,b)$. This establishes the existence of the limit $[\psi(\cdot,\lambda),\zeta(\cdot,\mu)](b)$, and comparison with (5. 26) provides $[\psi(\cdot,\lambda),\zeta(\cdot,\mu)](b)=0$. The second equality in (5. 27) follows analogously. □

Hereafter in this section we assume that $(k,\mathcal{U}_{2n})\in\mathcal{S}_a$ is fixed and that the function M and the selected value μ are as in Theorem 5.4. Moreover, we assume now that R_μ and $\tilde{R}_\mu$ defined in (5. 6) and (5. 7) are one-to-one as operators from $L^2_A(a,b)$ into itself, i.e., that

$$f\in L^2_A(a,b),\ A(R_\mu f)=0 \ a.e. \text{ on } (a,b) \Rightarrow Af=0 \ a.e. \text{ on } (a,b), \tag{5. 42}$$

$$g\in L^2_A(a,b),\ A(\tilde{R}_\mu g)=0 \ a.e. \text{ on } (a,b) \Rightarrow Ag=0 \ a.e. \text{ on } (a,b). \tag{5. 43}$$

Later, it will become clear that the same conditions then hold when μ is replaced by any $\lambda\in\Lambda(k,\mathcal{U}_{2n})$. We are now able to define the natural operators associated with our analysis. Set

$$\begin{aligned}\mathcal{D}(L) := \quad & \{y\in L^2_A(a,b) : y\in AC_{loc}[a,b),\\ & ly\equiv Jy' - By = Af \text{ a.e. for some } f\in L^2_A(a,b),\ [y,\chi(\cdot,\mu)](a)=0,\ [y,\zeta(\cdot,\mu)](b)=0\},\end{aligned} \tag{5. 44}$$

$$Ly := f \text{ for } y\in D(L) \text{ satisfying } ly = Af \ a.e. \text{ on } (a,b),$$

$$\begin{aligned}\mathcal{D}(\tilde{L}) := \quad & \{z\in L^2_A(a,b) : z\in AC_{loc}[a,b),\\ & \tilde{l}z\equiv Jz' - B^*z = Ag \text{ a.e. for some } g\in L^2_A(a,b),\ [z,\phi(\cdot,\mu)](a)=0,\ [z,\psi(\cdot,\mu)](b)=0\},\end{aligned} \tag{5. 45}$$

$$\tilde{L}z := g \text{ for } z \in D(\tilde{L}) \text{ satisfying } \tilde{l}z = Ag \text{ a.e. on } (a,b).$$

More precisely, $D(L)$ and $D(\tilde{L})$ consist of all equivalence-classes in $L^2_A(a,b)$ such that at least one representative of the class satisfies the conditions after the colons in $D(L)$ and $D(\tilde{L})$, respectively. In fact, conditions (5. 42) and (5. 43) ensure that this representative is always unique, and so, in particular, L and $\tilde{L}$ are well defined: Suppose that y_1 and y_2 are two representatives in the same equivalence-class (i.e., $A(y_1 - y_2) = 0$ a.e. on (a,b)), both satisfying the conditions after the colon in $D(L)$, with two functions $f_1, f_2 \in L^2_A(a,b)$. For $y := y_1 - y_2$ and $f := f_1 - f_2$ we then obtain $Ay = 0$ a.e. on (a,b) and $Jy' = By + Af = (\mu A + B)y + Af$, whence $y = R_\mu f + \theta(\cdot,\mu)\tilde{v} + \phi(\cdot,\mu)v$ for some $\tilde{v}, v \in \mathbf{C}^n$. The boundary condition $[y, \chi(\cdot,\mu)](a)$ yields $\tilde{v} = 0$, by (5. 11) and (2. 7)-(2. 12), and the boundary condition $[y, \zeta(\cdot,\mu)](b)$ provides $v = 0$, by (5. 14) and (5. 13). Thus, $y = R_\mu f$, and $Ay = 0$ a.e. on (a,b), whence $Af = 0$ a.e. on (a,b), by (5. 42). Hence (5. 6) shows that $y = R_\mu f = 0$, providing the desired uniqueness.

Theorem 5.5 *Let*

$$\mathcal{D}_1 := \left\{ y \in L^2_A(a,b) : y \in AC_{loc}[a,b), ly \equiv Jy' - By = Af \text{ for some } f \in L^2_A(a,b),\ [y, \chi(\cdot,\mu)](a) = 0 \right\} \tag{5. 46}$$

$$\tilde{\mathcal{D}}_1 := \left\{ z \in L^2_A(a,b) : z \in AC_{loc}[a,b), \tilde{l}z \equiv Jz' - B^*z = Ag \text{ for some } g \in L^2_A(a,b),\ [z, \phi(\cdot,\mu)](a) = 0 \right\} \tag{5. 47}$$

Then,

$$\begin{aligned} \mathcal{D}_1 &= \mathcal{D}(L) \dot{+} [\phi(\cdot,\mu)]_{L^2_A}, \\ \tilde{\mathcal{D}}_1 &= \mathcal{D}(\tilde{L}) \dot{+} [\chi(\cdot,\mu)]_{L^2_A} \end{aligned} \tag{5. 48}$$

where $[\phi(\cdot,\mu)]_{L^2_A}$ denotes the space of functions $\phi(\cdot,\mu)c$ (with $c \in \mathbf{C}^n$) which are in $L^2_A(a,b)$. In particular $\mathcal{D}(L) = \mathcal{D}_1$ if (2. 1) (for $\lambda = \mu$) has precisely n linearly independent solutions in $L^2_A(a,b)$; in other words, there are no boundary conditions at b in this case.

Proof Let $u \in \mathcal{D}_1$ so that $lu = Af$ for some $f \in L^2_A(a,b)$. Then $(l - \mu A)u = Ag$ for $g := -\mu u + f$. Furthermore, $v := R_\mu g \in \mathcal{D}(L)$ satisfies

$$(l - \mu A)(v - u) = 0, \quad [v - u, \chi(\cdot,\mu)](a) = 0,$$

whence $v - u = \theta(\cdot,\mu)\tilde{c} + \phi(\cdot,\mu)c$ for some $\tilde{c}, c \in \mathbf{C}^n$, and $\tilde{c} = 0$ since $[\theta(\cdot,\mu), \chi(\cdot,\mu)](a) = I_n$ and $[\phi(\cdot,\mu), \chi(\cdot,\mu)](a) = 0_n$. Moreover, $v - u \in L^2_A(a,b)$, implying $v - u \in [\phi(\cdot,\mu)]_{L^2_A}$, whence the result for $\mathcal{D}_1$ since $\mathcal{D}(L) \subseteq \mathcal{D}_1$ and $[\phi(\cdot,\mu)]_{L^2_A} \subset \mathcal{D}_1$; the sum in (5. 48) is indeed direct since $\phi(\cdot,\mu)c \in \mathcal{D}(L)$ implies $c = 0$, by (5. 13). The result concerning $\tilde{\mathcal{D}}_1$ follows similarly. □

Lemma 5.6 *Denoting the resolvent sets of L and $\tilde{L}$ by $\rho(L)$ and $\rho(\tilde{L})$, respectively, we have $\mu \in \rho(L), \bar{\mu} \in \rho(\tilde{L})$, $(L - \mu)^{-1} = R_\mu$ and $(\tilde{L} - \bar{\mu})^{-1} = \tilde{R}_\mu$. Moreover, L and $\tilde{L}$ are closed.*

Proof $L-\mu$ is one-to-one, since $y \in D(L)$, $(L-\mu)y = 0$ implies $Jy' = (\mu A + B)y$, whence $y = \theta(\cdot,\mu)\tilde{v} + \phi(\cdot,\mu)v$ for some $\tilde{v}, v \in \mathbf{C}^n$. By (2. 7)-(2. 12) and (5. 13), the boundary conditions for y yield $\tilde{v} = v = 0$ and thus, $y = 0$.

For $f \in L^2_A(a,b)$, (5. 9),(5. 11) and (5. 14) give $R_\mu f \in D(L)$, and $(L-\mu)R_\mu f = f$. Thus, the range of $L-\mu$ is $L^2_A(a,b)$, and $(L-\mu)^{-1} = R_\mu$. Therefore, $(L-\mu)^{-1}$ is bounded by Theorem 5.1, whence $\mu \in \rho(L)$.

In particular, $(L-\mu)^{-1}$ is closed, implying that L is closed. The statements for $\tilde{L}$ and $\tilde{R}_\mu$ follow analogously. □.

Lemma 5.7 *The space $\mathcal{D}(L)$ is dense in $L^2_A(a,b)$. Also $\tilde{L} = L^*$, the adjoint of L.*

Proof By (5. 5) and the boundedness of R_μ and $\tilde{R}_\mu$, we have $\tilde{R}_\mu = R^*_\mu$, the $(\cdot,\cdot)_{L^2_A}$-adjoint of R_μ. Thus, (5. 43) shows that R^*_μ is one-to-one, whence the range of R_μ is dense in $L^2_A(a,b)$. By Lemma 5.6, this range equals $D(L)$.

Moreover, Lemma 5.6 and $\tilde{R}_\mu = R^*_\mu$ together imply $(\tilde{L} = \bar{\mu})^{-1} = ((L-\mu)^{-1})^*$, whence $\tilde{L} = L^*$.
□

Theorem 5.8 *We have $\Lambda(k,\mathcal{U}_{2n}) \subset \rho(L)$ and, for $\lambda \in \Lambda(k,\mathcal{U}_{2n})$,*

$$R_\lambda = (L-\lambda)^{-1}.$$

A corresponding statement holds for $\tilde{R}_\lambda$ and $\tilde{L}$.

Proof Let $\lambda \in \Lambda(k,\mathcal{U}_{2n})$. For all $f \in L^2_A(a,b)$ we have, by (5. 9),

$$l(R_\lambda f) = A[\lambda(R_\lambda f) + f]. \tag{5. 49}$$

Furthermore, by (5. 11) (and $\chi(a,\lambda)$ being independent of λ),

$$[R_\lambda f, \chi(\cdot,\mu)](a) = 0. \tag{5. 50}$$

Next we prove that

$$[R_\lambda f, \zeta(\cdot,\mu)](b) = 0, \tag{5. 51}$$

first restricting ourselves to compact support functions f. With X_f chosen such that f vanishes for $X \geq X_f$, (5. 3) and (5. 6) give

$$(R_\lambda f)(X) = \psi(X,\lambda)\int_a^{X_f} \chi^*(y,\lambda)(Af)(y)dy \text{ for } X \geq X_f,$$

whence

$$[R_\lambda f, \zeta(\cdot,\mu)](b) = [\psi(\cdot,\lambda), \zeta(\cdot,\mu)](b)\int_a^{X_f} \chi^*(y,\lambda)(Af)(y) = 0,$$

by Theorem 5.4. To obtain (5. 51) for general $f \in L^2_A(a,b)$, we choose a sequence (f_m) of compact support functions $f_m \in L^2_A(a,b)$ converging to f in $L^2_A(a,b)$. Using (2. 4) (for $z := \zeta(\cdot,\mu)$ and $y := R_\lambda(f - f_m)$) we obtain

$$\begin{aligned}[R_\lambda f, \zeta(\cdot,\mu)](X) &= [R_\lambda f_m, \zeta(\cdot,\mu)](X) + [R_\lambda(f-f_m), \zeta(\cdot,\mu)](a) \\ &+ \int_a^X [(\lambda-\mu)\zeta^*(x,\mu)A(x)R_\lambda(f-f_m)(x) + \zeta^*(x,\mu)A(x)(f-f_m)(x)]dx.\end{aligned}$$

As $X \to b$, the first term on the right-hand side tends to 0, since (5. 51) holds for compact support functions. Thus,

$$\begin{aligned}[R_\lambda f, \zeta(\cdot,\mu)](b) &= [R_\lambda(f-f_m), \zeta(\cdot,\mu)](a) \\ &+ \int_a^b [(\lambda-\mu)\zeta^*(x,\mu)A(x)R_\lambda(f-f_m)(x) + \zeta^*(x,\mu)A(x)(f-f_m)(x)]dx.\end{aligned}$$

Since $f - f_m \to 0$ in $L^2_A(a,b)$ implies $R_\lambda(f - f_m) \to 0$ in $L^2_A(a,b)$ by Theorem 5.1, and $R_\lambda(f - f_m)(a) \to 0$ by (5. 6), (5. 51) follows.

Now, (5. 49),(5. 50) and (5. 51) yield $R_\lambda f \in D(L)$ and

$$(L-\lambda)(R_\lambda f) = f, \tag{5. 52}$$

implying that the range of $L - \lambda$ is $L^2_A(a,b)$. Analogously, the range of $\tilde{L} - \bar{\lambda}$ is $L^2_A(a,b)$. Since $\tilde{L} - \bar{\lambda} = (L-\lambda)^*$ by Lemma 5.7, this implies that $L - \lambda$ is one-to-one. Consequently, from (5. 52), $R_\lambda = (L-\lambda)^{-1}$, and since R_λ is bounded by Theorem 5.1, we have that $\lambda \in \rho(L)$. □

6 Properties of Q and M

In this final section, we require the conditions on $(k, \mathcal{U}_{2n})$ assumed so far (namely, (3. 1)-(3. 3),(3. 6),(3. 12),(3. 14), (4. 18),(4. 19),(5. 21),(5. 42),(5. 43)) to hold for *every* $(k, \mathcal{U}_{2n}) \in \mathcal{S}_a$ such that $\Lambda(k, \mathcal{U}_{2n}) \neq \emptyset$, and that values $\mu = \mu_{k,\mathcal{U}_{2n}}$ and functions $M = M_{k,\mathcal{U}_{2n}}$ have been selected according to Theorem 5.4, for all these $(k, \mathcal{U}_{2n})$. Furthermore, we will now need the following additional condition implying in particular that the sets $\Lambda(k, \mathcal{U}_{2n})$ are open: For every $(k, \mathcal{U}_{2n}) \in \mathcal{S}_a$ such that $\Lambda(k, \mathcal{U}_{2n}) \neq \emptyset$, some $\gamma = \gamma(k, \mathcal{U}_{2n}) > 0$ exists such that

$$\mathbf{Re}[e^{i\varphi}\mathcal{U}_{2n}A(x)] \geq -\gamma \cdot \mathcal{U}_{2n}A(x)\mathcal{U}_{2n}^* \text{ for all } \varphi \in [0,2\pi), x \in (a,b). \tag{6. 1}$$

We note that (6. 1) is satisfied in the case (5. 22).

Theorem 6.1 *For each $\lambda \in \mathbf{C}$, let $N(\lambda)$ denote the precise number of linearly independent solutions of (2. 1) in $L^2_A(a,b)$. Then, N is constant on each connected component of $\mathbf{C}\backslash Q$ (cf. (3. 8)).*

Proof Let Λ_c denote any of the connected components of $\mathbf{C}\backslash Q$. It is sufficient to prove that N is locally constant on Λ_c, since then the result follows by standard connectivity arguments.

Thus, let $\lambda_0 \in \Lambda_c$, and select any $(k, \mathcal{U}_{2n}) \in \mathcal{S}_a$ such that $\lambda_0 \in \Lambda(k, \mathcal{U}_{2n})$. Take $\delta_0 := \delta(\lambda_0)$ from (3. 7). Then, for each $\lambda \in \mathbf{C}$ such that $\mid \lambda - \lambda_0 \mid < \delta_0/(2\gamma)$ (with γ from (6. 1)), we have $\lambda \in \Lambda(k, \mathcal{U}_{2n})$, and $\delta(\lambda) := \delta_0/2$ can be chosen in (3. 7). Theorem 5.1 and (5. 21) therefore give

$$\parallel R_\lambda \parallel_{L^2_A} \leq k_0 \quad \text{for all } \lambda \in \mathbf{C}, \mid \lambda - \lambda_0 \mid < \frac{\delta_0}{2\gamma}, \tag{6. 2}$$

with k_0 independent of λ. Now, fix $\epsilon > 0$ such that

$$\epsilon < \min\{\frac{\delta_0}{2\gamma}, \frac{1}{2k_0}\}. \tag{6. 3}$$

We shall prove that $N(\lambda) = N(\lambda')$ for all λ, λ' in the $\epsilon-$neighbourhood of λ_0, which establishes the result.

Thus, let $\mid \lambda - \lambda_0 \mid < \epsilon, \quad \mid \lambda' - \lambda_0 \mid < \epsilon$. By symmetry, it is sufficient to prove that

$$N(\lambda') \geq N(\lambda). \tag{6. 4}$$

The case $N(\lambda) = n$ is trivial since Theorem 4.2 yields $N(\lambda') \geq n$. So let $s := N(\lambda) - n > 0$. By Theorem 4.1, we obtain s linearly independent L^2_A-solutions of (2. 1) as the columns of $\phi(\cdot, \lambda)K$, with $K \in \mathbf{C}^{n,s}$ having rank s. Since

$$J[\phi(\cdot, \lambda)K]' = (\lambda' A + B)\phi(\cdot, \lambda)K + (\lambda - \lambda')A\phi(\cdot, \lambda)K,$$

(5. 9) gives

$$\phi(\cdot, \lambda)K = \theta(\cdot, \lambda')\tilde{K}' + \phi(\cdot, \lambda')K' + (\lambda - \lambda')R_{\lambda'}[\phi(\cdot, \lambda)K], \tag{6. 5}$$

with $\tilde{K}', K' \in \mathbf{C}^{n,s}$. Evaluation at a and pre-multiplication by $(\alpha_1^* \mid \alpha_2^*)$ yields $\tilde{K}' = 0$, by (2. 7)-(2. 9) and (5. 10). Now (6. 5) shows that the columns of $\phi'(\cdot, \lambda')K'$ are in $L^2_A(a, b)$.

If K' had rank less than s, some $v \in \mathbf{C}^s\backslash\{0\}$ would exist such that $K'v = 0$ (but $Kv \neq 0$ since K has rank s), whence by (6. 5)

$$\phi(\cdot, \lambda)Kv = (\lambda - \lambda')R_{\lambda'}[\phi(\cdot, \lambda)Kv]. \tag{6. 6}$$

By (6. 2), this implies that

$$\parallel \phi(\cdot, \lambda)Kv \parallel_{L^2_A} \leq \mid \lambda - \lambda' \mid k_0 \parallel \phi(\cdot, \lambda)Kv \parallel_{L^2_A},$$

which contradicts (6. 3) and $\mid \lambda - \lambda' \mid < 2\epsilon$; note that $\phi(\cdot, \lambda)Kv \not\equiv 0$ since $Kv \neq 0$, whence (6. 6) and (5. 6) yield $\phi(\cdot, \lambda)Kv \neq 0$ in $L^2_A(a, b)$.

Therefore, K' has rank s, whence the columns of $\phi(\cdot,\lambda')K'$ provide s linearly independent L^2_A-solutions of problem (2. 1) at λ'. Using Theorem 4.1 again, we obtain (at least) $n+s=N(\lambda)$ linearly independent solutions of problem (2. 1) at λ', i.e., (6. 4). □

It is worth noting that, if there are precisely n linearly independent solutions of (2. 1) in $L^2_A(a,b)$, for $\lambda \in \Lambda(k,\mathcal{U}_{2n}) \cap \Lambda(k',\mathcal{U}'_{2n})$, then

$$M_{k,\mathcal{U}_{2n}}(\lambda) = M_{k',\mathcal{U}'_{2n}}(\lambda). \tag{6. 7}$$

For in this case

$$\theta(x,\lambda) + \phi(x,\lambda)M_{k,\mathcal{U}_{2n}}(\lambda) = \{\theta(x,\lambda) + \phi(x,\lambda)M_{k',\mathcal{U}'_{2n}}(\lambda)\}K(\lambda)$$

for some $K(\lambda) \in \mathbf{C}^{n,n}$, and (6. 7) follows from (2. 7)-(2. 9). Thus, if there are precisely n linearly independent L^2_A-solutions of (2. 1) in a connected component Λ_c of $\mathbf{C}\backslash Q$ (note Theorem 6.1), then we can define M on Λ_c by

$$M(\lambda) = M_{k,\mathcal{U}_{2n}}(\lambda), \quad \lambda \in \Lambda(k,\mathcal{U}_{2n}) \cap \Lambda_c. \tag{6. 8}$$

.

We now address the question of analyticity of the function M.

Theorem 6.2 *For each $(k,\mathcal{U}_{2n}) \in \mathcal{S}_a$, $M = M_{k,\mathcal{U}_{2n}}$ is analytic throughout $\Lambda(k,\mathcal{U}_{2n})$. If there are precisely n solutions of (2. 1) in $L^2_A(a,b)$ on a connected component Λ_c of $\mathbf{C}\backslash Q$, then M defined by (6. 8) is analytic on Λ_c.*

Proof Theorem 5.4, Lemma 5.2 and Theorem 5.8 readily yield

$$M(\lambda) = M_0 + (\lambda-\mu)\int_a^b \zeta^*(x,\mu)A(x)\psi(x,\mu)dx + (\lambda-\mu)^2((L-\lambda)^{-1}\psi(\cdot,\mu),\zeta(\cdot,\mu))_{L^2_A}, \tag{6. 9}$$

whence the analyticity of M since $(L-\lambda)^{-1}$ is analytic on $\rho(L) \supset \Lambda(k,\mathcal{U}_{2n})$ (cf. [7, III Theorem 6.7]). The last statement follows from the fact that, for $(k,\mathcal{U}_{2n}), (k',\mathcal{U}'_{2n}) \in \mathcal{S}_a$ such that $V := \Lambda_c \cap \Lambda(k,\mathcal{U}_{2n}) \cap \Lambda(k',\mathcal{U}'_{2n}) \neq \emptyset$, the analytic functions $M_{k,\mathcal{U}_{2n}}$ and $M_{k',\mathcal{U}'_{2n}}$ coincide on the open set V.□

Corollary 6.3 *For each $(k,\mathcal{U}_{2n}) \in \mathcal{S}_a$, (6. 9) defines an analytic extension of $M = M_{k,\mathcal{U}_{2n}}$ to the resolvent set of $L = L_{k,\mathcal{U}_{2n}}$.*

Theorem 6.4 *Let $(k,\mathcal{U}_{2n}) \in \mathcal{S}_a$ and $\mu = \mu_{k,\mathcal{U}_{2n}}$, $M = M_{k,\mathcal{U}_{2n}}$, $L = L_{k,\mathcal{U}_{2n}}$. Suppose that all solutions of (2. 1) (at $\lambda = \mu$) are in $L^2_A(a,b)$. Then, (6. 9) defines a meromorphic extension of M to the whole of $\mathbf{C}$. The poles of M are eigenvalues of L.*

Proof Since all solutions of (2. 1) at μ are in $L^2_A(a,b)$, (5. 3) and (5. 6) show that R_μ is a Hilbert-Schmidt operator, and thus compact. By Theorem 5.8, $(L-\mu)^{-1}$ is compact, whence the spectrum of L consists only of isolated eigenvalues with finite algebraic multiplicity, and $(L-\lambda)^{-1}$ is compact for all $\lambda \in \mathbf{C}$ except at these eigenvalues. The meromorphicity of $(L-\lambda)^{-1}$, with poles precisely at the eigenvalues of L, follows e.g. from [7, III, Section 5], and (6. 9) gives the corresponding statement for M. □

Lemma 6.5 *If all solutions of (2. 1) and (2. 11) are in $L^2_A(a,b)$ for some $\lambda' \in \mathbf{C}$, then all solutions of (2. 1) are in $L^2_A(a,b)$ for all $\lambda \in \mathbf{C}$.*

Proof By the variation of constants formula, any solution of

$$\begin{aligned} Jy'(x,\lambda) &= (\lambda A(x) + B(x))y(x,\lambda) \\ &= (\lambda' A(x) + B(x))y(x,\lambda) + (\lambda - \lambda')A(x)y(x,\lambda) \end{aligned}$$

can be written as

$$\begin{aligned} y(x,\lambda) &= \theta(x,\lambda')C + \phi(x,\lambda')D - (\lambda-\lambda')Y(x,\lambda')\int_c^x Y^{-1}(s,\lambda')JA(s)y(s,\lambda)ds \\ &= \theta(x,\lambda')C + \phi(x,\lambda')D - (\lambda-\lambda')Y(x,\lambda')J\int_c^x Z^*(s,\lambda')A(s)y(s,\lambda)ds, \end{aligned}$$

for some $C, D \in \mathbf{C}^n$ and any $c \in [a,b)$, where Y and Z are the fundamental matrices of (2. 1) and (2. 11) respectively.: for this we have used [4, Chapter 3, (3.2)] and (2. 14). A standard argument now yields the lemma (cf.[4, Chapter 9, Theorem 2.1]). □

An immediate consequence of Theorem 5.4 is

Theorem 6.6 *If all solutions of (2. 1) are in $L^2_A(a,b)$, then, for any $(k, \mathcal{U}_{2n}) \in \mathcal{S}_a$, $M = M_{k,\mathcal{U}_{2n}}$ satisfies*

$$\begin{aligned} M(\lambda) = \quad & [I_n - (\lambda-\mu)\int_a^b \zeta^*(x,\mu)A(x)\phi(x,\lambda)dx]^{-1} \\ & [M_0 + (\lambda-\mu)\int_a^b \zeta^*(x,\mu)A(x)\theta(x,\lambda)dx]. \end{aligned}$$

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