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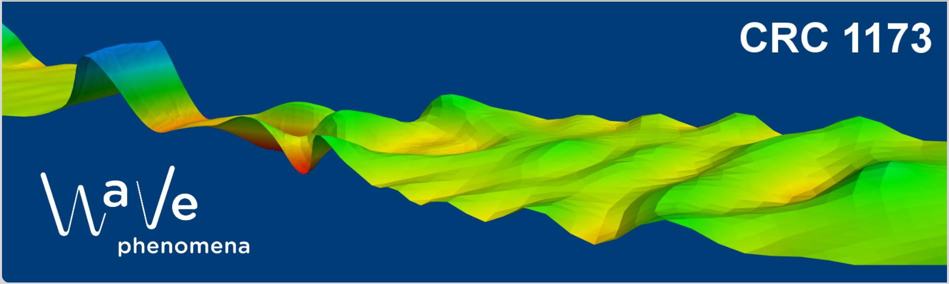
A variational approach to real-valued breathers for a class of cubic nonlinear wave equations

W. Reichel

PDEs at the Grand Paradis, Cogne • June 20–24, 2016

Institute for Analysis



Me and the Italian language

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But here in Cogne:

bon anniversairé Filomena!

Semilinear wave equations – the problem

Find solutions $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that

$$(*) \qquad \begin{cases} s(x)u_{tt} - u_{xx} + V(x)u &= \Gamma(x)|u|^{p-1}u \\ u(x,t) &\to 0 \text{ as } |x| \to \infty \\ u(x,t+T) &= u(x,t) \end{cases}$$

with $p > 1$ & suitable conditions on $s, V, \Gamma : \mathbb{R} \to \mathbb{R}$.
u (real-valued, time-periodic & spatially localized) is called „breather"

Outline:

1. The famous Sine-Gordon breather and other examples
2. A vector-valued breather problem
3. Our example by a variational approach

Sine-Gordon breather and other examples

$$\begin{cases} u_{tt} - u_{xx} + \sin u & = & 0 \\ u(x,t) & \to & 0 \text{ as } |x| \to \infty \\ u(x,t+T) & = & u(x,t) \end{cases}$$

Explixit solution family:

$$u(x,t) = 4\arctan\left(\frac{m\sin(\omega t)}{\omega\cosh(mx)}\right), \qquad m^2 + \omega^2 = 1$$

Replace $\sin(u)$ by $g(u)$ with $g(0) = 0, g'(0) = 1$

$\Rightarrow$ breathers disappear [Denzler, Kichenassamy, Sigal, Vuillermont]

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But – $\exists$ many examples on bounded intervals with Dirichlet b.-c.:

Bambusi, Berti, Biasco, Bolle, Bourgain, Brezis, Craig, Kuksin, Palleari, Procesi, Rabinowitz, Yamaguchi, Wayne, A. Weinstein

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Examples of breathers in periodic lattices:

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Examples of breathers in water-waves:

Buffoni, Groves, Haragus, Plotnikov, Sun, Toland, Wahlén

Sine-Gordon breather and other examples

For a different equation:

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$$s(x) = 1 + 15\chi_{[6/13,7/13)}(x), \quad x \bmod 1$$

$$V(x) = \left(\left(\frac{13\pi}{16}\right)^2 - \left(\frac{13\arccos((9+\sqrt{1881})/100))}{8}\right)^2 - \epsilon^2\right)s(x),$$

$$\Gamma(x) = 1$$

$\exists$ breather-solutions with minimal period $T = \frac{32}{13}$ for all $\epsilon \in (0, \epsilon_0]$.
Method: center-manifold reduction; spatial dynamics; bifurcation theory

A vector-valued breather example in $\mathbb{R}^3 \times \mathbb{R}$

$$(*_{\text{vec}}) \qquad s(x)\partial_t^2 U + \nabla \times \nabla \times U + V(x)U \pm \Gamma(x)|U|^{p-1}U = 0$$

ansatz: $U(x,t) = \psi(r,t)\frac{x}{r}, \quad r = |x|$.

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Theorem (Plum, R. 2016)

Let $T = 2\pi\sqrt{\frac{s(0)}{V(0)}}$.

- $V, s, \Gamma > 0$ *radially symmetric* C^2*-functions,*
- $\sup \frac{V}{\Gamma} < \infty$,
- $T\sqrt{\frac{V(r)}{s(r)}} \lessgtr 2\pi$ *on* $\mathbb{R}^3 \setminus \{0\}$,
- $$\left|2\pi - T\sqrt{\frac{V(r)}{s(r)}}\right|^{\frac{1}{p-1}} = \begin{cases} O(e^{-\alpha r}) \text{ as } r \to \infty, \\ o(1) \text{ in } C^2\text{-sense as } r \to 0. \end{cases}$$

Then $\exists$ T*-periodic, real-valued, exponentially decaying solution.*

The proof in the plus case – solving an ODE

$$U(r,t) = \psi(r,t)\frac{x}{r}, \qquad s(r)\ddot{\psi} + V(r)\psi + \Gamma(r)|\psi|^{p-1}\psi = 0$$

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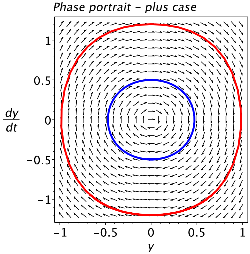
$$U(r,t) = \psi(r,t)\frac{x}{r}, \qquad s(r)\ddot{\psi} + V(r)\psi + \Gamma(r)|\psi|^{p-1}\psi = 0$$

Rescale: $\psi(r,t) = \left(\frac{V(r)}{\Gamma(r)}\right)^{1/(p-1)} y\left(\sqrt{\frac{V(r)}{s(r)}}t\right)$

$$\ddot{y} + y + |y|^{p-1}y = 0$$

$$\dot{y}^2 + y^2 + \frac{2}{p+1}|y|^{p+1} = \text{cst.} = c$$



periodic orbits $y(t;c)$

- period $L(c) = 2\pi - O(c^{\frac{p-1}{2}})$
- $\max_{\mathbb{R}} |y(t;c)| \leq \sqrt{c}$
- How to choose $c = c(r)$?

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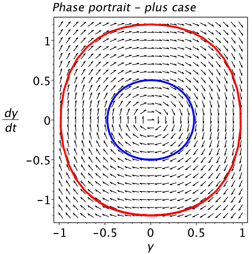

Answer:
$$\sqrt{\frac{V(r)}{s(r)}}T = L(c),\ c := L^{-1}\left(\sqrt{\frac{V(r)}{s(r)}}T\right)$$

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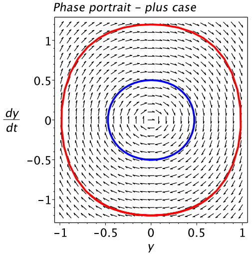

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$$|\psi(r,t)| \le \text{cst.}\sqrt{c(r)} \le \text{cst.}\left|2\pi - \sqrt{\frac{V(r)}{s(r)}}T\right|^{1/(p-1)} = \begin{cases} \to 0 \text{ as } r \to 0 \\ O(e^{-\alpha r}) \text{ as } r \to \infty \end{cases}$$

Remarks on real-valued curl-curl breathers

$$(*_{\text{vec}}) \qquad s(x)\partial_t^2 U + \nabla \times \nabla \times U + V(x)U \pm \Gamma(x)|U|^{p-1}U = 0$$

- Use radial symmetry → it is easy to construct real-valued breathers $U(r,t) = \psi(r,t)\frac{x}{r}$
- Under exactly the same assumptions on s, V, Γ: time-harmonic complex exponentially decaying solutions exist:

$$U(x,t) = e^{i\frac{2\pi}{T}t}\psi(r)\frac{x}{r}$$

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$$|\psi|^{p-1} = \underbrace{\left(\left(\frac{2\pi}{T}\right)^2 \frac{s(r)}{V(r)} - 1\right)}_{\text{positive, } \to 0 \text{ as } r\to 0,\infty} \cdot \underbrace{\frac{V(r)}{\Gamma(r)}}_{\text{bounded}}$$

A scalar breather example via calc.var.

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$$(*) \qquad \begin{cases} s(x)u_{tt} - u_{xx} + V(x)u & = & \gamma\delta_0 u^3 \text{ in } \mathbb{R}\times\mathbb{R} \\ u(x,t) & \to & 0 \text{ as } |x|\to\infty \\ u(x,t+T) & = & u(x,t) \end{cases}$$

where δ_0 is the δ-distribution in x-direction centered at 0.

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Assume that $u(-x,t) = u(x,t)$.

$$(**) \qquad \begin{cases} s(x)u_{tt} - u_{xx} + V(x)u &= 0 \text{ in } (0,\infty)\times\mathbb{R}, \\ -2\partial_x u(0,t) &= \gamma u(0,t)^3, \\ u(x,t) &\to 0 \text{ as } x\to\infty \\ u(x,t+T) &= u(x,t) \end{cases}$$

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The choice of the periodic linear operator with periodicity cell $[0, P]$:

$$L = s(x)\partial_t^2 - \partial_x^2 + V(x)$$

with

$$s(x) = \alpha + \beta\delta^{per,P}, \quad V(x) = \epsilon s(x), \qquad \alpha,\beta > 0.$$

$\delta^{per,P}$ is the P-periodic extension of the $\delta_{P/2}$-distribution on x-axis.

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Theorem (R. 2016)

For given $\alpha, P, \gamma > 0$ assume $\beta > 4\alpha P/\pi$. Then there exists $\epsilon_0 > 0$ s.t.:

$|\epsilon| \le \epsilon_0 \Rightarrow \exists$breather: even in x, $T/2$-antiperiodic in t with $T = 4P\sqrt{\alpha}$.

Sketch of the proof – overview

Fourier-decomposition of solution:

$$u(x,t) = \sum_{k \text{odd}} u_k(x) e^{ik\omega t}, \quad u_{-k} = \bar{u}_k.$$

Fourier-decomposition of operator L:

$$\sigma(L) = \bigcup_{k \text{odd}} \sigma(L_k) = \bigcup_{k \text{odd}} \sigma\Big(-\partial_x^2 + V(x) - k^2\omega^2 s(x)\Big)$$

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Steps:

- for each odd k check: $0 \notin \sigma(L_k)$
- determine Bloch mode ϕ_k, Floquet-multiplier ρ_k:

$$L_k \phi_k = 0, \qquad \phi_k(0) = 1, \qquad \phi_k(x + jP) = \rho_k^j \phi_k(x), \qquad |\rho_k| < 1$$

- $u(x,t) = \sum_{k \text{odd}} a_k \phi_k(x) e^{ik\omega t}, \quad a_k \in \mathbb{C},\ a_{-k} = \bar{a}_k$
- solve the variational problem for $(a_k)_{k \text{odd}}$ in a sequence-space

The spectral non-resonance

Recall: $0 \notin \sigma(L_k) = \sigma\Big(-\partial_x^2 + V(x) - k^2\omega^2 s(x)\Big)$, i.e.,

$$\underbrace{(k^2\omega^2 - \epsilon)\alpha}_{=:\lambda} \notin \sigma\Big(\underbrace{-\partial_x^2 + \underbrace{(\epsilon - k^2\omega^2)\beta}_{-\tau}\,\delta^{per,P}}_{=:\tilde{L}}\Big)$$

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Bloch-discriminant: $D(\lambda) = \phi_1(P) + \phi_2'(P)$, where ϕ_1, ϕ_2 are a fundamental system of $\tilde{L}\phi = \lambda\phi$ with

$$\phi_1(0) = 1, \phi_1'(0) = 0, \qquad \phi_2(0) = 0, \phi_2'(0) = 1.$$

Then

$$\lambda \notin \sigma(\tilde{L}) \Leftrightarrow |D(\lambda)| > 2$$

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and here

$$D(\lambda) = \begin{cases} -\tau\frac{\sin(\sqrt{\lambda}P)}{\sqrt{\lambda}} + 2\cos(\sqrt{\lambda}P), & \lambda \geq 0, \\ -\tau\frac{\sinh(\sqrt{-\lambda}P)}{\sqrt{-\lambda}} + 2\cosh(\sqrt{-\lambda}P), & \lambda < 0. \end{cases}$$

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For us ($\epsilon = 0$): $\lambda = k^2\omega^2\alpha$, k odd, and τ from above:

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$$|D(\lambda)| = \left|\frac{\beta}{\sqrt{\alpha}}|k|\omega\sin(|k|\omega\sqrt{\alpha}P) + 2\cos(|k|\underbrace{\omega\sqrt{\alpha}P}_{=\pi/2})\right| = \frac{\beta\omega}{\sqrt{\alpha}}\underbrace{|k|}_{\geq 1} > 2$$

Finding the Bloch-modes of L_k

For $\tilde{L} = -\partial_x^2 - \tau\delta^{per,P}$ and $\lambda \notin \sigma(\tilde{L})$:

Floquet-multiplier: $\rho = \operatorname{sign} D(\lambda)\left(\frac{|D(\lambda)|}{2} - \sqrt{\frac{D(\lambda)^2}{4} - 1}\right) \in (-1, 1)$

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Bloch-mode:

$$\begin{aligned}\phi_k(x) = &\Big(\sin(|k|\frac{\pi}{4}) + O(\frac{1}{k})\Big)\cos(|k|\omega\sqrt{\alpha}x) \\ &- \Big(\cos(|k|\frac{\pi}{4}) + O(\frac{1}{k})\Big)\sin(|k|\omega\sqrt{\alpha}x), \; 0 \le x \le \frac{P}{2}\end{aligned}$$

Normalization:

$$\phi_k(0) = 1, \phi_k'(0) = -|k|\omega\sqrt{\alpha}\Big(1 + O(\frac{1}{k})\Big)\cdot(-1)^l$$

The variational problem – Part I

$$(**)\qquad \begin{cases} s(x)u_{tt} - u_{xx} + V(x)u &= 0 \text{ in } (0,\infty)\times\mathbb{R}, \\ -2\partial_x u(0,t) &= \gamma u(0,t)^3, \\ u(x,t) &\to 0 \text{ as } x\to\infty \\ u(x,t+T) &= u(x,t) \end{cases}$$

with

$$s(x) = \alpha + \beta\delta^{per,P}, \quad V(x) = \epsilon s(x), \qquad \alpha,\beta > 0.$$

Fourier-Bloch-decomposition of solution:

$$u(x,t) = \sum_{k \text{odd}} a_k \phi_k(x) e^{ik\omega t}, \quad a_{-k} = \bar{a}_k, \quad a_{k even} = 0.$$

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Moreover:

$$u(0,t) = \sum_{k=2l+1} \underbrace{\phi_k(0)}_{=1} a_k e^{ik\omega t}, \quad u(0,t)^3 = \sum_{k=2l+1} (a * a * \bar{a})_k e^{ik\omega t}$$

$$u_x(0,t) = \sum_{k=2l+1} \phi_k'(0) a_k e^{ik\omega t} = \sum_{k=2l+1} \Big(-|k|\underbrace{\omega\sqrt{\alpha}(-1)^l}_{=:g_k} + O(1)\Big) a_k e^{ik\omega t}$$

The variational problem – Part II

The nonlinear Neumann boundary condition:

$$(nN) \qquad -2\partial_x u(0,t) = \underbrace{\gamma}_{=1} u(0,t)^3$$

becomes

$$2|k|g_k a_k + O(1)a_k = (a * a * \bar{a})_k$$

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Work in the sequence Hilbert-space

$$H = \left\{ (a_k)_{k\in\mathbb{Z}} : a_{-k} = \bar{a}_k, a_k = 0 \text{ for } k \text{ even s.t. } \|a\|^2 := \sum_{k\in\mathbb{Z}} |k||a_k|^2 < \infty \right\}$$

functional

$$J[a] = \sum_{k\in\mathbb{Z}} |k||a_k|^2 + O(1)|a_k|^2 - \frac{1}{4}|(a*a)_k|^2 g_k^{-1}, \qquad a \in H$$

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Note:

- H embedds compactly into l^q, $1 < q \le \infty$
- $\|a * a\|_2 \le \text{cst.} \|a\|_{4/3}^2 \le \text{cst.} \|a\|^2$ by Young's inequality
- $J'[a] = 0$ if and only if $(a)_{k \in \mathbb{Z}}$ solves (nN)

Solving the variational problem

Finding a critical point of

$$J[a] = \sum_{k\in\mathbb{Z}} |k||a_k|^2 + O(1)|a_k|^2 - \frac{1}{4}|(a*a)_k|^2 g_k^{-1}$$
$$= Q(a,a) - \frac{1}{4}\sum_{k\in\mathbb{Z}} |(a*a)_k|^2 g_k^{-1}$$

is done by spectral splitting

$$H = H^- \oplus H^+$$

and minimizing J on the generalized Nehari-manifold

$$N = \{a \in H \setminus \{0\} : J'[a]b = 0 \ \forall b \in [a] + H^-\}$$

Szulkin-Weth (2010): existence of a minimizer □

Some concluding remarks/open questions

Karlsruhe Institute of Technology

- By construction we get „polychromatic" waves $\sum_k a_k \phi_k(x) e^{ik\omega t}$ with $a_k \neq 0$ for infinitely many k
- By construction they are „ground states"
- A pure monochromatic wave $a_k \phi_k(x) e^{ik\omega t}$ is a critical point of J if

$$\tilde{H} := \left\{ (a_k)_{k \in \mathbb{Z}} : \cancel{a_{-k} = \bar{a}_k}, a_k = 0 \text{ for } k \text{ even} \right\}$$

- What are the „ground states" on $\tilde{H}$? Pure monochromatic wave $a_1 \phi_1 e^{i\omega t}$?
- What about nonlinearities $|u(x,t)|^{p-1} u(x,t)$?
- What about other operators $L = s(x)\partial_t^2 - \partial_x^2 + V(x)$ with $0 \notin \sigma(L)$?

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