Karlsruhe Institute of Technology

Localized time-periodic solutions of nonlinear wave equations

Wolfgang Reichel (financially supported by DFG through CRC 1173)

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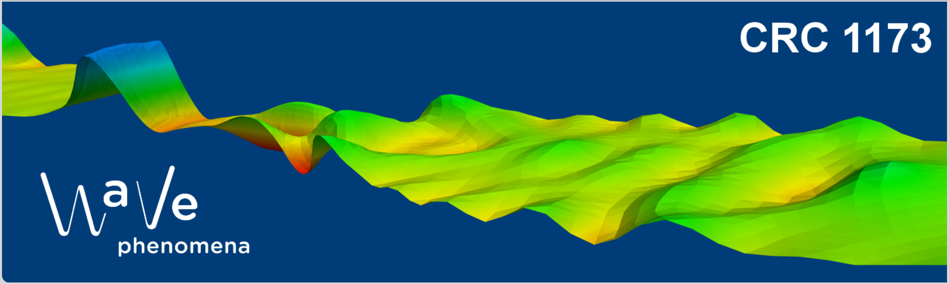
Institute for Analysis



The problem

Find spatially localized, time-periodic $E : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ such that

(quasi) $$\nabla \times \nabla \times E + \partial_t^2\Big(V(x)E + \Gamma(x)|E|^{p-1}E\Big) = 0$$

(semi) $$\nabla \times \nabla \times E + V(x)\partial_t^2 E + \Gamma(x)|E|^{p-1}E = 0$$

with $p > 1$ & suitable conditions on $V, \Gamma : \mathbb{R}^3 \to \mathbb{R}$

Outline:

(A) physical background

(B) results for semilinear wave equation

(C) results for quasilinear wave equation

(A): Motivation for Curl-Curl wave equations

$$\nabla \times E + \partial_t B = 0, \qquad \nabla \cdot D = 0,$$
$$\nabla \times H - \partial_t D = 0, \qquad \nabla \cdot B = 0.$$

Material laws:

$$B = \mu_0 H, \quad D = \epsilon_0 E + P(x, E) = \epsilon_0 (1 + \chi_1(x) + \chi_3(x)|E|^2 + \ldots)E$$

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$\hookrightarrow$

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Quasilinear wave-equation:

$$\hookrightarrow \quad \nabla \times \nabla \times E + \partial_t^2 \Big(\underbrace{\mu_0 \epsilon_0 (1 + \chi_1(x))}_{=V(x) \geq 0} E + \underbrace{\mu_0 \epsilon_0 \chi_3(x) |E|^2 E + \ldots}_{=f(x,|E|^2)E} \Big) = 0$$

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Semilinear (approximative/toy-model) variant:

$$\hookrightarrow \qquad \nabla \times \nabla \times E + V(x)\partial_t^2 E + f(x, |E|^2)E = 0$$

(B): Semilinear wave-equations

Find solutions $U : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ such that

$$(*) \qquad \begin{cases} \nabla \times \nabla \times U + V(x)U_{tt} + f(x,|U|^2)U &= 0 \\ U(x,t) &\to 0 \text{ as } |x| \to \infty \\ U(x,t+T) &= U(x,t) \end{cases}$$

under suitable conditions on V, f.

U (real-valued, time-periodic & spatially localized) is called „breather“

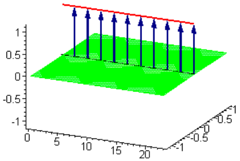
Motivation:

i. The famous Sine-Gordon breather

$$u_{tt} - u_{xx} + \sin u = 0$$

ii. The example by Blank, Chirilus-Bruckner, Lescaret, Schneider ('11) for

$$V(x)u_{tt} - u_{xx} + q(x)u = \Gamma(x)u^3$$



Source: Wikipedia

$u(x,t) = 4\arctan\left(\frac{m\sin(\omega t)}{\omega\cosh(mx)}\right)$

$m^2 + \omega^2 = 1$

(B) A vector-valued breather example

$$(*) \quad V(x)\partial_t^2 U + \nabla \times \nabla \times U + q(x)U \pm \Gamma(x)|U|^{p-1}U = 0 \quad \text{in} \quad \mathbb{R}^3 \times \mathbb{R}$$

ansatz: $U(x,t) = \psi(r,t)\frac{x}{r}, \quad r = |x|.$

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Theorem 1 (Plum, R. JEPE 2017)

Let $T = 2\pi\sqrt{\frac{V(0)}{q(0)}}$.

- $V, q, \Gamma > 0$ *radially symmetric* C^2*-functions,*
- $\sup\frac{q}{\Gamma} < \infty$,
- $\frac{q(r)}{V(r)} \lneqq \frac{q(0)}{V(0)}$ *on* $\mathbb{R}^3\setminus\{0\}$,
- $$\left|\frac{q(r)}{V(r)} - \frac{q(0)}{V(0)}\right|^{\frac{1}{p-1}} = \begin{cases} O(e^{-\alpha r}) \text{ as } r\to\infty, \\ o(1) \text{ in } C^2\text{-sense as } r\to 0.\end{cases}$$

Then $\exists$ T*-periodic, real-valued, exponentially decaying solution.*

(B) The proof – solving an ODE

$$U(r,t) = \psi(r,t)\frac{x}{r}, \qquad V(r)\ddot{\psi} + q(r)\psi \pm \Gamma(r)|\psi|^{p-1}\psi = 0$$

ODE in time with $r =$ parameter

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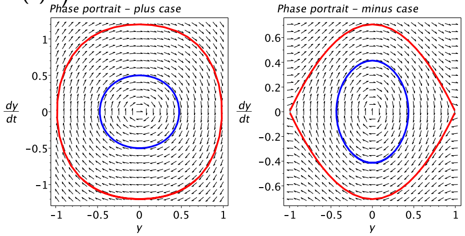
$$U(r,t) = \psi(r,t)\frac{x}{r}, \qquad V(r)\ddot{\psi} + q(r)\psi \pm \Gamma(r)|\psi|^{p-1}\psi = 0$$

ODE in time with $r =$ parameter

Rescale: $\psi(r,t) = \left(\frac{q(r)}{\Gamma(r)}\right)^{1/(p-1)} y\left(\sqrt{\frac{q(r)}{V(r)}}t\right)$

$$\ddot{y} + y \pm |y|^{p-1}y = 0$$



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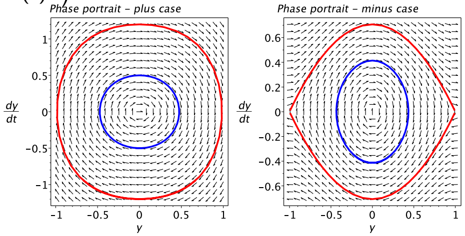
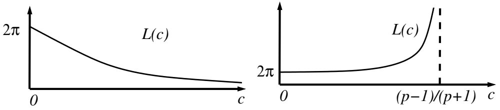
periodic orbits $y(t;c)$, period $L(c)$

- c= value of first integral
- How to choose $c = c(r)$?
- Answer: $\underbrace{\sqrt{\frac{q(r)}{V(r)}}}_{\leqslant 2\pi} T = L(c)$





(B) The proof – solving an ODE

$$U(r,t)=\psi(r,t)\frac{x}{r}, \qquad V(r)\ddot{\psi}+q(r)\psi\pm\Gamma(r)|\psi|^{p-1}\psi=0$$

ODE in time with $r=$ parameter

Rescale: $\psi(r,t)=\left(\frac{q(r)}{\Gamma(r)}\right)^{1/(p-1)} y\left(\sqrt{\frac{q(r)}{V(r)}}t\right)$

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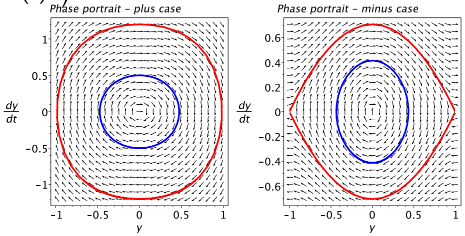

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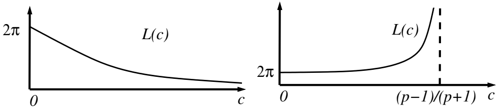
by assumptions $\Rightarrow$ result



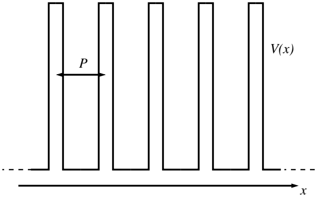
(B) Scalar breather examples via calc.var.

Ansatz:

$$E(x,t) = \begin{pmatrix} 0 \\ 0 \\ u(x_1,t) \end{pmatrix} \quad \hookrightarrow \quad \nabla \times \nabla \times E(x,t) = \begin{pmatrix} 0 \\ 0 \\ -\partial_{x_1}^2 u(x_1,t) \end{pmatrix}$$

Scalar semilinear wave equation:

$$(*) \quad \begin{cases} V(x)u_{tt} - u_{xx} &= \Gamma(x)|u|^{p-1}u \text{ in } \mathbb{R}\times\mathbb{R} \\ u(x,t) &\to 0 \text{ as } |x| \to \infty \\ u(x,t+T) &= u(x,t) \end{cases}$$



Two approaches via calc.var.:

- A. Hirsch & W.R.: $\Gamma \equiv \gamma$, $1 < p < \frac{5}{3}$, $V(x) = \alpha + \beta\delta^{per,P}(x)$, $\beta = \frac{8\alpha P}{\pi}$
- W.R.: $\Gamma(x) = \gamma\delta_0(x)$, $p = 3$, $V(x) = \alpha + \beta\delta^{per,P}(x)$, $\beta > \frac{4\alpha P}{\pi}$

(B) Scalar breather examples via calc.var.

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$$(*) \qquad \left\{ \begin{array}{rcl} V(x)u_{tt} - u_{xx} & = & \Gamma(x)|u|^{p-1}u \text{ in } \mathbb{R}\times\mathbb{R} \\ u(x,t) & \to & 0 \text{ as } |x|\to\infty \\ u(x,t+T) & = & u(x,t) \end{array} \right.$$

with $V(x) = \alpha + \beta\delta^{per,P}$, $\alpha, \beta > 0$

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Common features:

- $\delta^{per,P}$ is the P-periodic extension of the $\delta_{P/2}$-distribution on x-axis
- time-period $T = 4P\sqrt{\alpha}$
- Fourier-decomp. of solution $u(x,t) = \sum_{k \text{odd}} u_k(x)e^{ik\omega t}, u_{-k} = \bar{u}_k$.
- Fourier-decomp. of wave operator $L = V(x)\partial_t^2 - \partial_x^2$:

$$\sigma(L) = \bigcup_{k \text{odd}} \sigma(L_k) = \bigcup_{k \text{odd}} \sigma\Big(-\partial_x^2 - k^2\omega^2 V(x)\Big)$$

- choice of $\alpha, P, \beta > 4\alpha P/\pi \Rightarrow 0 \notin \sigma(L)$

(B) The case of δ_0 right-hand side

$$(*) \quad \begin{cases} V(x)u_{tt} - u_{xx} &= \gamma\delta_0(x)u^3 \text{ in } \mathbb{R}\times\mathbb{R} \\ u(x,t) &\to 0 \text{ as } |x|\to\infty \\ u(x,t+T) &= u(x,t) \end{cases}$$

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Even solutions $u(x,t) = u(-x,t) \Rightarrow$ nonlinear Neumann problem

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(nN) \qquad \left\{ \begin{array}{rcl} V(x)u_{tt} - u_{xx} & = & 0 \text{ in } (0,\infty) \times \mathbb{R}, \\ -2u_x(0,t) & = & \gamma u(0,t)^3, \\ u(x,t) & \to & 0 \text{ as } x \to \infty \\ u(x,t+T) & = & u(x,t) \end{array} \right.
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Theorem 2 (R. 2016)

Let $V(x) = \alpha + \beta\delta^{per,P}(x)$ where $\alpha, P > 0$, $\beta > 4\alpha P/\pi$ and $\gamma \neq 0$. Then $\exists$ a real-valued breather which is even in x, $T/2$-antiperiodic in t with $T = 4P\sqrt{\alpha}$.

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Steps:

- $u(x,t) = \sum_{k \text{odd}} a_k\phi_k(x)e^{ik\omega t}$
- ϕ_k = Bloch-mode
  $L_k\phi_k = 0$ on $(0,\infty)$, exp. decaying at $+\infty$, $\phi_k(0) = 1$
- variational problem for coefficients $(a_k)_{k\text{odd}}$, $a_{-k} = \bar{a}_k$

(B) The variational problem

$$(nN) \quad \begin{cases} V(x)u_{tt} - u_{xx} &= 0 \text{ in } (0,\infty)\times\mathbb{R}, \\ -2u_x(0,t) &= \gamma u(0,t)^3, \\ u(x,t) &\to 0 \text{ as } x\to\infty \\ u(x,t+T) &= u(x,t) \end{cases}$$

with Fourier-Bloch-decomposition $u(x,t) = \sum_{k \text{odd}} a_k \phi_k(x) e^{ik\omega t}$

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with Fourier-Bloch-decomposition $u(x,t) = \sum_{k\,\mathrm{odd}} a_k\phi_k(x)e^{ik\omega t}$

$$u_x(0,t) = \sum_{k\,\mathrm{odd}} \phi_k'(0)a_k e^{ik\omega t}, \quad u(0,t)^3 = \sum_{k\,\mathrm{odd}} (a*a*\bar{a})_k e^{ik\omega t}$$

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u weakly solves (nN) $\Leftrightarrow J'[a] = 0$, $a \in H$ with

$$J[a] := \sum_{k=2l+1} \underbrace{\phi_k'(0)}_{\approx -(-1)^l |k|} |a_k|^2 - \frac{\gamma}{4}|(a*a)_k|^2, \quad H := \Big\{ a_{-k} = \bar{a}_k, \sum_{k \text{odd}} |k||a_k|^2 < \infty \Big\}$$

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abstract critical point theorem (Szulkin-Weth 2010):

$\Rightarrow \exists$ truly polychromatic ground state

(B) Summary on semilinear wave-equations

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Polychromatic waves $E(x,t) = \sum_k U_k(x) e^{ik\omega t}$, $U_k = \bar{U}_{-k}$ for

(semi) $$\nabla \times \nabla \times E + V(x)\partial_t^2 E = \Gamma(x)|E|^{p-1}E$$

- vector case: radial symmetry, $E(x,t) = \psi(|x|,t)\frac{x}{|x|}$, ODE in time
- scalar case: $p = 3$, $V(x) =$ cst.+periodic delta, $\Gamma(x) = \gamma\cdot$ delta at 0
  - $0 \notin \sigma(-\partial_x^2 + V(x)\partial_t^2)$
  - use Fourier-Bloch decomposition $\sum_{k \text{odd}} a_k \phi_k(x) e^{ik\omega t}$
  - indefinite variational problem for $(a_k)_{k \text{odd}}$
  - solve by abstract critical point theorem of Szulkin, Weth 2010
- scalar case: $1 < p < \frac{5}{3}$, $V(x) =$ cst.+periodic delta, $\Gamma(x) =$cst.
  -> details in talk by Andreas Hirsch, Tuesday, 11:25

(C) Quasilinear wave-equations

$$\nabla \times \nabla \times E + \partial_t^2 \Big(V(x)E + f(x, |E|^2)E \Big) = 0$$

Our approaches

- monochrom. $\mathbb{C}$-valued waves: $E(x,t) = U(x)e^{i\omega t}$

  $$\nabla \times \nabla \times U + \tilde{V}(x)U = \tilde{f}(x, |U|^2)U \text{ in } \mathbb{R}^3$$

  with $\tilde{V} = -\omega^2 V$, $\tilde{f} = \omega^2 f$. elliptic, variational

(C) Quasilinear wave-equations

$$\nabla \times \nabla \times E + \partial_t^2 \Big(V(x)E + f(x, |E|^2)E \Big) = 0$$

Our approaches

- $E(x,t) = U(x)\cos(\omega t)$ works for time-averaged material response

$$f\Big(x, \frac{1}{T}\int_0^T |E|^2\,dt\Big)E$$

(C) Quasilinear wave-equations

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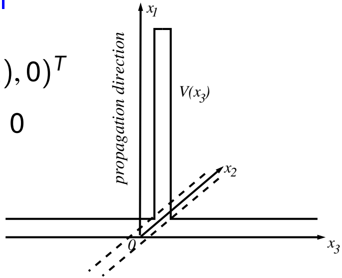
- traveling waves: $E(x,t) = (0, u(x_1 - \sqrt{\lambda}t, x_3), 0)^T$

  $$\Big(-\Delta_{x_1,x_3} + \lambda V(x_3)\Big)u + \lambda\Big(f(x_3,|u|^2)u\Big)_{x_1x_1} = 0$$

  hyperbolic, bifuraction w.r.t. λ from guided modes. Talk by Piotr Idzik, Tuesday, 11:50



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Our approaches

- monochrom. $\mathbb{C}$-valued waves: $E(x,t) = U(x)e^{i\omega t}$

  $$\nabla \times \nabla \times U + \tilde V(x)U = \tilde f(x,|U|^2)U \text{ in } \mathbb{R}^3$$

  with $\tilde V = -\omega^2 V$, $\tilde f = \omega^2 f$. elliptic, variational

- traveling waves: $E(x,t) = (0, u(x_1 - \sqrt{\lambda} t, x_3), 0)^T$

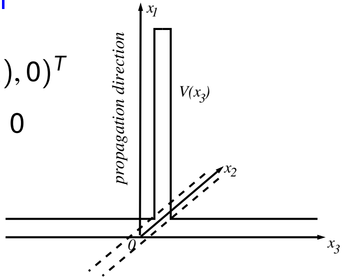
  $$\Big(-\Delta_{x_1,x_3} + \lambda V(x_3)\Big)u + \lambda\Big(f(x_3,|u|^2)u\Big)_{x_1x_1} = 0$$

  hyperbolic, bifuraction w.r.t. λ from guided modes. Talk by Piotr Idzik, Tuesday, 11:50

- standing waves: $E(x,t) = (0, u_t(x_1,t), 0)^T$

  $$-u_{xx} + V(x)u_{tt} + \Big(f(x_1,|u_t|^2)u_t\Big)_t = 0$$

  hyperbolic, variational



(C) Elliptic Curl-Curl problem

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$$\nabla \times \nabla \times U + V(x)U = f(x, |U|^2)U \quad \text{in} \quad \mathbb{R}^3$$

(0) $U(x_1, x_2, x_3) = (0, 0, u(x_1, x_2))^T$ leads to NLS (many results!)

(1) Benci-Fort.('04) & Azzollini-B.-d'Aprile-F.('06) & d'A.-Siciliano('11), Zeng ('16):

$$\nabla \times \nabla \times U = f(|U|^2)U \text{ in } \mathbb{R}^3$$

Existence of ground-states in subspaces of cylindrical symmetry

(2) Bartsch-Mederski ('15,'17), survey ('17 J. Fixed Point Theory Appl.):

$$\nabla \times \mu(x)^{-1}\nabla \times U - \omega^2\epsilon(x)U = \partial_U F(x, U) \text{ in } \Omega, \quad \nu \times U = 0 \text{ on } \partial\Omega.$$

(3) Mederski('15): $f(s) \approx |s|^{\frac{p-1}{2}}$ near 0, $f(s) \approx |s|^{\frac{q-1}{2}}$ near ∞, $1 < p < 5 < q$.

$$\nabla \times \nabla \times U + V(x)U = f(|U|^2)U \text{ in } \mathbb{R}^3$$

(4) Mederski ('16): Brezis-Nirenberg Curl-Curl problem: Tuesday, 14:00

(5) Bartsch-Dohnal-Plum-R. ('14) & Hirsch-R. ('16) ... next

(C) Common variational set-up

$$J[U] = \int_{\mathbb{R}^3} |\nabla \times U|^2 + \tilde{V}(x)|U|^2 - \tilde{F}(r, z, |U|^2)\, dx,$$

$$U \in X = H(\mathrm{curl}; \mathbb{R}^3) \cap L^{p+1}(\mathbb{R}^3)$$

Here is the problem: $\|\nabla U\|_{L^2}^2 = \|\nabla \times U\|_{L^2}^2 + \|\nabla \cdot U\|_{L^2}^2$.

Constraint $\{U : \mathrm{div}\, U = 0\}$ does not solve it $\Rightarrow$ Lagrange-multiplier!

Symmetries! Look for cylindrical symmetry in coordinates (r, z):

$$U(r, z) := u(r, z) \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} \quad \Rightarrow \mathrm{div}\, U = 0.$$

$$-\Delta_5 u(r, z) + \tilde{V}(r, z) u = \tilde{f}(r, z, r^2 u^2) u \text{ for } r > 0, z \in \mathbb{R}.$$

This is a NLS-type equation in $\mathbb{R}^5$!

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(C) Results I - (Bartsch-Dohnal-Plum-R., NoDeA 2016)

$$(*) \qquad \nabla \times \nabla \times U + \tilde{V}(x)U = \tilde{\Gamma}(x)|U|^{p-1}U \quad \text{in} \quad \mathbb{R}^3$$

General assumption: $\tilde{V} = \tilde{V}(r, x_3), \tilde{\Gamma} = \tilde{\Gamma}(r, x_3), r = \sqrt{x_1^2 + x_2^2}$

Theorem 3 (Defocusing case)

- $\tilde{\Gamma}(x) \le -C(1 + |x|^\alpha)$, $\alpha > \frac{3}{2}(p-1)$, $p > 1$,
- $\tilde{V} \in L^\infty(\mathbb{R}^3)$, $\sup \tilde{V} < 0$.

Then $(*)$ has a (restricted) ground-state.

Theorem 4 (Focusing case)

- $\inf \tilde{\Gamma} > 0$, $\tilde{V}, \tilde{\Gamma} \in L^\infty(\mathbb{R}^3)$ *are 1-periodic in* x_3,
- $1 < p < 5$
- $0 \notin \sigma(L)$ *with* $L = \nabla \times \nabla \times + \tilde{V}(x)$.

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(C) Results II - statements (Hirsch-R., ZAA 2017)

$$(*) \qquad \nabla\times\nabla\times U + \tilde V(r,z)U = \tilde f(r,z,|U|^2)U \quad \text{in} \quad \mathbb{R}^3$$

Theorem 5 (Positive definite case, focusing)

- $0 < \min \sigma(\nabla\times\nabla\times + \tilde V)$
- $0 \le \tilde f(r,z,s) \le C(1+s^{\frac{p-1}{2}})$, $1 < p < 5$,
- $\tilde f(r,z,s) = o(1)$ *as* $s \to 0$ *uniformly in* r, z
- $s \mapsto \tilde f(r,z,s)$ *strictly increasing in* s
- $\tilde F(r,z,s)/s \to \infty$ *as* $s \to \infty$ *uniformly in* r, z
- $\tilde V(r,z)$ *reverse Steiner-symmetric in* z
- $\phi_\sigma(r,z,s) := \tilde f(r,z,(s+\sigma)^2)(s+\sigma)^2 - \tilde f(r,z,s^2)s^2$ *is symmetrically decreasing in* z *for all* $s \ge 0$, $\sigma \ge 0$

Then $(*)$ *has a (restricted) ground-state.*

Ex.: $\tilde f(r,z,|U|^2) = \tilde\Gamma(r,z)\log(1+|U|^2), = \tilde\Gamma(r,z)|U|^{p(z)-1}$, $\overline{\mathrm{Rg}(p)} \subset (1,5)$

(C) Results II - statements (Hirsch-R., ZAA 2017)

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(C) Sketch of variational existence proof

$$-\Delta_5 u(r,z) + V(r,z)u = f(r,z,r^2u^2)u \text{ for } r>0, z\in\mathbb{R}.$$

$$J[u] = \int_{\mathbb{R}^5} |\nabla u|^2 + V(r,z)u^2 - \frac{F(r,z,r^2u^2)}{r^2}\,dx^5, \quad u \in H^1_{cyl}(\mathbb{R}^5)$$

Minimize J over the Nehari-manifold [cf. Szulkin-Weth, '10]:

$$\mathcal{N} = \left\{ u \neq 0; N[u] = \int_{\mathbb{R}^5} |\nabla u|^2 + V(r,z)u^2 - f(r,z,r^2u^2)u^2\,dx^5 = 0 \right\}$$

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Changing from u to $|u|^*$ (Steiner symmetrization w.r.t. z) we get

$$J[|u|^*] \leq J[u], \quad N[|u|^*] \leq N[u]$$

because of the condition [Brock, '00]

$$\phi_\sigma(r,z,s) := f(r,z,(s+\sigma)^2)(s+\sigma)^2 - f(r,z,s^2)s^2 \quad \searrow_{symm} z.$$

Moreover: weak sequ. cont.'y along $(|u_k|^*)_{k\in\mathbb{N}}$ [inspired by Lions,'81,'82].

(C) Scalar breather with δ_0 right-hand side

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$$(*) \qquad \begin{cases} V(x)u_{tt} - u_{xx} &= \gamma\delta_0(x)({u_t}^3)_t \text{ in } \mathbb{R}\times\mathbb{R} \\ u(x,t) &\to 0 \text{ as } |x|\to\infty \\ u(x,t+T) &= u(x,t) \end{cases}$$

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Even solutions $u(x,t) = u(-x,t) \Rightarrow$ nonlinear Neumann problem

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(nN) \qquad \left\{ \begin{array}{rcl} V(x)u_{tt} - u_{xx} & = & 0 \text{ in } (0,\infty)\times\mathbb{R}, \\ -2u_x(0,t) & = & \gamma(u_t(0,t)^3)_t, \\ u(x,t) & \to & 0 \text{ as } x \to \infty \\ u(x,t+T) & = & u(x,t) \end{array} \right.
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Theorem 6 (R. 2016)

Let $V(x) = \alpha + \beta\delta^{per,P}(x)$ where $\alpha, P > 0$, $\beta > 4\alpha P/\pi$ and $\gamma \neq 0$. Then $\exists$ a real-valued breather which is even in x, $T/2$-antiperiodic in t with $T = 4P\sqrt{\alpha}$.

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Steps:

- $u(x,t) = \sum_{k \text{odd}} \frac{a_k}{k}\phi_k(x)e^{ik\omega t}$, $\phi_k =$ normalized Bloch-mode
- variational problem for coefficients $(a_k)_{k \text{odd}}$, $a_{-k} = -\bar{a}_k$

(C) The variational problem

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with Fourier-Bloch-decomposition $u(x,t) = \sum_{k \text{odd}} \frac{a_k}{k}\phi_k(x)e^{ik\omega t}$

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u weakly solves (nN) $\Leftrightarrow J'[a]=0$, $a\in H$ with

$$J[a] := \sum_{k=2l+1} \frac{\omega^4}{4}|(a*a)_k|^2 + \underbrace{\phi_k'(0)}_{\approx -(-1)^l|k|}\frac{|a_k|^2}{k^2\gamma}, \quad H := \left\{a_{-k} = -\bar{a}_k, \|a*a\|_{l^2} < \infty\right\}$$

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minimizer = polychromatic ground state

(C) Summary on quasilinear wave-equations

(quasi) $\nabla \times \nabla \times E + \partial_t^2\Big(V(x)E + \Gamma(x)|E|^{p-1}E\Big) = 0$

- $\exists$ monochromatic ground-state $E(x,t) = U(x)e^{i\omega t}$:
  - elliptic vector case in $\mathbb{R}^3$ with cylindrical symmetry
  - defocusing case: $\Gamma(x) \le -C(1+|x|^\alpha)$, $\alpha > \frac{3}{2}(p-1)$, $\inf V > 0$
  - focusing case: periodic structure in z, $0 \notin \sigma(\nabla \times \nabla \times -\omega^2 V(x))$
  - focusing case: Steiner symmetry in z, $0 < \sigma(\nabla \times \nabla \times -\omega^2 V(x))$

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  - focusing case: Steiner symmetry in z, $0 < \sigma(\nabla \times \nabla \times -\omega^2 V(x))$
- $\exists$ scalar $1+1$-dim $\mathbb{R}$-valued standing polychromatic waves:
  - $p = 3$, $V(x) =$ cst.+periodic delta, $\Gamma(x) = \gamma\cdot$ delta at 0
  - $0 \notin \sigma(-\partial_x^2 + V(x)\partial_t^2)$
  - use Fourier-Bloch decomposition $\sum_{k \text{odd}} \frac{a_k}{k}\phi_k(x)e^{ik\omega t}$
  - coercive variational problem for $(a_k)_{k\text{odd}} \hookrightarrow$ minimizer=breather

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- $\exists$ scalar $2+1$-dim $\mathbb{R}$-valued traveling waves:
  $p=3$, $V(x_3)=$ delta at 0, $\Gamma(x_3)\in L^\infty$: use bifurcation theory
  -> details in talk by Piotr Idzik, Tuesday, 11:50