Localized time-periodic solutions of nonlinear wave equations

W. Reichel

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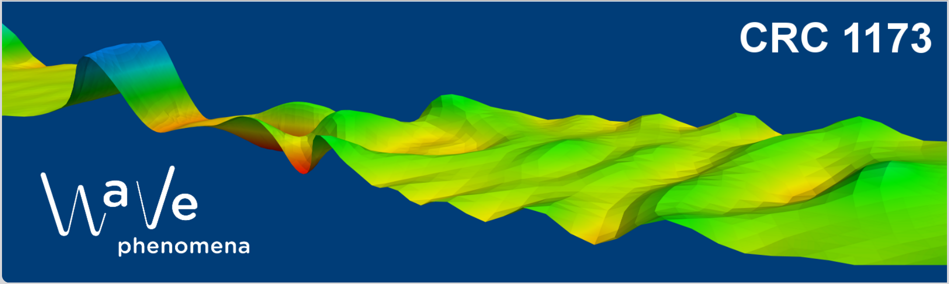
Institute for Analysis



The problem

Find spatially localized, time-periodic $E : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ such that

$$\text{(quasi)} \qquad \nabla \times \nabla \times E + \partial_t^2\Big(V(x)E + \Gamma(x)|E|^{p-1}E\Big) = 0$$

$$\text{(semi)} \qquad \nabla \times \nabla \times E + V(x)\partial_t^2 E + \Gamma(x)|E|^{p-1}E = 0$$

with $p > 1$ & suitable conditions on $V, \Gamma : \mathbb{R}^3 \to \mathbb{R}$

Outline:

(A.1) Physical background

(A.2) Time-harmonic solutions: previous results/our results

(A.3) Some details

(B.1) Real-valued periodic solutions: previous results/our resuts

(B.2) Some details

(A): Time-harmonic sol'ns of nonlin. Maxwell

$$\nabla \times E + \partial_t B = 0, \qquad \nabla \cdot D = 0,$$
$$\nabla \times H - \partial_t D = 0, \qquad \nabla \cdot B = 0.$$

Material laws:

$$B = \mu_0 H, \quad D = \epsilon_0 E + P(x, E) = \epsilon_0 (1 + \chi_1(x) + \chi_3(x)|E|^2) E$$

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Quasilinear wave-equation for E:

$$\hookrightarrow \qquad \nabla \times \nabla \times E + \partial_t^2 \Big(\underbrace{\mu_0\epsilon_0 \underbrace{(1+\chi_1(x))}_{=n^2(x)\geq 0}}_{=\tilde{V}(x)\geq 0} E + \underbrace{\mu_0\epsilon_0\chi_3(x)}_{=\tilde{\Gamma}(x)} |E|^2 E \Big) = 0$$

Ansatz: $E(x,t) = U(x)e^{i\omega t}$ leads to

$$\nabla \times \nabla \times U - \omega^2 \tilde{V}(x)U - \omega^2 \tilde{\Gamma}(x)|U|^2 U = 0 \text{ in } \mathbb{R}^3$$

i.e. stationary, nonlinear Schrödinger-type problem

Results - Part I

$$(*) \qquad \nabla\times\nabla\times U + V(x)U = f(x,|U|^2)U \quad \text{in} \quad \mathbb{R}^3$$

(0) $U(x_1,x_2,x_3) = (0,0,u(x_1,x_2))^T$ leads to NLS (many results!)

$$-\Delta u + V(x)u = f(x,|u|^2)u \text{ in } \mathbb{R}^2$$

(1) Benci-Fort.('04) & Azzollini-B.-d'Aprile-F.('06) & d'A.-Siciliano('11):

$$\nabla\times\nabla\times U = f(|U|^2)U \text{ in } \mathbb{R}^3$$

Existence of ground-states in subspaces of cylindrical symmetry

(2) Bartsch-Mederski ('14,'15):

$$\nabla\times\mu(x)^{-1}\nabla\times U - \omega^2\epsilon(x)U = \partial_U F(x,U) \text{ in } \Omega, \quad \nu\times U = 0 \text{ on } \partial\Omega.$$

(3) Mederski('14): $f(s)\approx|s|^{\frac{p-1}{2}}$ near 0, $f(s)\approx|s|^{\frac{q-1}{2}}$ near ∞, $1<p<5<q$.

$$\nabla\times\nabla\times U + V(x)U = f(|U|^2)U \text{ in } \mathbb{R}^3$$

(4) Bartsch-Dohnal-Plum-R. ('14) & Hirsch-R. ('16) ... next

Results - Part II (Bartsch-Dohnal-Plum-R., NoDeA 2016)

$(*) \qquad \nabla \times \nabla \times U + V(x)U = \Gamma(x)|U|^{p-1}U \quad \text{in} \quad \mathbb{R}^3$

General assumption: $V = V(r, x_3), \Gamma = \Gamma(r, x_3), r = \sqrt{x_1^2 + x_2^2}$

Theorem (Defocusing case)

- $\Gamma(x) \le -C(1 + |x|^\alpha)$, $\alpha > \frac{3}{2}(p-1)$, $p > 1$,
- $V \in L^\infty(\mathbb{R}^3)$, $\sup V < 0$.

Then $(*)$ has a (restricted) ground-state.

Theorem (Focusing case)

- $\inf \Gamma > 0$, $V, \Gamma \in L^\infty(\mathbb{R}^3)$ *are* 1*-periodic in* x_3,
- $1 < p < 5$
- $0 \notin \sigma(L)$ *with* $L = \nabla \times \nabla \times + V(x)$.

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Example of cylind. symm. V with $0 \notin \sigma(L)$. Unfortunately: $\sup V > 0$!

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Results - Part III (Hirsch-R., ArXiv 2016)

$$(*) \qquad \nabla \times \nabla \times U + V(r,z)U = f(r,z,|U|^2)U \quad \text{in} \quad \mathbb{R}^3$$

Theorem (Positive definite case)

- $0 < \min \sigma(\nabla \times \nabla \times + V)$
- $V(r,z)$ *reverse Steiner-symmetric in* z
- $0 \leq f(r,z,s) \leq C(1 + s^{\frac{p-1}{2}})$, $1 < p < 5$,
- $f(r,z,s) = o(1)$ *as* $s \to 0$ *uniformly in* r, z
- $s \mapsto f(r,z,s)$ *strictly increasing in* s
- $F(r,z,s)/s \to \infty$ *as* $s \to \infty$ *uniformly in* r, z
- $\phi_\sigma(r,z,s) := f(r,z,(s+\sigma)^2)(s+\sigma)^2 - f(r,z,s^2)s^2$ *is symmetrically decreasing in* z *for all* $s \geq 0$, $\sigma \geq 0$

Then $(*)$ *has a (restricted) ground-state.*

Ex.: $f(z,s) = \Gamma(z)s^{\frac{p(z)-1}{2}}$, $1 < \inf p \leq \sup p < 5$, Γ, p Steiner symmetric.

Variational set-up

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$$J[U] = \int_{\mathbb{R}^3} |\nabla \times U|^2 + V(x)|U|^2 - F(r, z, |U|^2)\, dx,$$

$$U \in X = H(\text{curl}; \mathbb{R}^3) \cap L^{p+1}(\mathbb{R}^3)$$

Here is the problem: $\|\nabla U\|_{L^2}^2 = \|\nabla \times U\|_{L^2}^2 + \|\nabla \cdot U\|_{L^2}^2$.

Constraint $\{U : \text{div } U = 0\}$ does not solve it $\Rightarrow$ Lagrange-multiplier!

Symmetries! Look for cylindrical symmetry in coordinates (r, z):

$$U(r, z) := u(r, z) \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}. \quad \Rightarrow \text{div } U = 0.$$

$$-\Delta_5 u(r, z) + V(r, z)u = f(r, z, r^2 u^2)u \text{ for } r > 0, z \in \mathbb{R}.$$

This is a NLS-type equation in $\mathbb{R}^5$!

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Sketch of variational existence proof

$$-\Delta_5 u(r,z) + V(r,z)u = f(r,z,r^2u^2)u \text{ for } r>0, z\in\mathbb{R}.$$

$$J[u] = \int_{\mathbb{R}^5} |\nabla u|^2 + V(r,z)u^2 - \frac{F(r,z,r^2u^2)}{r^2}\,dx^5, \quad u \in H^1_{cyl}(\mathbb{R}^5)$$

Minimize J over the Nehari-manifold [cf. Szulkin-Weth, '10]:

$$\mathcal{N} = \left\{ u \neq 0; N[u] = \int_{\mathbb{R}^5} |\nabla u|^2 + V(r,z)u^2 - f(r,z,r^2u^2)u^2\,dx^5 = 0 \right\}$$

Changing from u to $|u|^*$ (Steiner symmetrization w.r.t. z) we get

$$J[|u|^*] \le J[u], \quad N[|u|^*] \le N[u]$$

because of the condition [Brock, '00]

$$\phi_\sigma(r,z,s) := f(r,z,(s+\sigma)^2)(s+\sigma)^2 - f(r,z,s^2)s^2 \quad \searrow_{symm} z.$$

Moreover: weak sequ. cont.'y along $(|u_k|^*)_{k\in\mathbb{N}}$ [inspired by Lions,'81,'82].

(B): Real-valued time-periodic solutions

Find solutions $U : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ such that

$$(*) \quad \begin{cases} \nabla \times \nabla \times U + V(x)U_{tt} + \Gamma(x)|U|^{p-1}U = 0 & \\ U(x,t) & \to \quad 0 \text{ as } |x| \to \infty \\ U(x,t+T) & = \quad u(x,t) \end{cases}$$

with $p > 1$ & suitable conditions on $V, \Gamma : \mathbb{R}^3 \to \mathbb{R}$.
U (real-valued, time-periodic & spatially localized) is called „breather“

Outline:

i. The famous Sine-Gordon breather and other examples

ii. A vector-valued example

iii. A scalar example by a variational approach

Sine-Gordon breather and other examples

$$\begin{cases} u_{tt} - u_{xx} + \sin u &= 0 \\ u(x,t) &\to 0 \text{ as } |x| \to \infty \\ u(x,t+T) &= u(x,t) \end{cases}$$

Explixit solution family:

$$u(x,t) = 4 \arctan\left(\frac{m \sin(\omega t)}{\omega \cosh(mx)}\right), \qquad m^2 + \omega^2 = 1$$

Replace $\sin(u)$ by $f(u)$ with $f(0) = 0, f'(0) = 1$

$\Rightarrow$ breathers disappear [Denzler, Kichenassamy, Sigal, Vuillermont]

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Examples of breathers in water-waves:

Buffoni, Groves, Haragus, Plotnikov, Sun, Toland, Wahlén

Sine-Gordon breather and other examples

For a different equation:

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$$V(x) = 1 + 15\chi_{[6/13,7/13)}(x), \quad x \bmod 1$$

$$q(x) = \left(\left(\frac{13\pi}{16}\right)^2 - \left(\frac{13\arccos((9+\sqrt{1881})/100))}{8}\right)^2 - \epsilon^2\right)V(x),$$

$$\Gamma(x) = 1$$

$\exists$ breather-solutions with minimal period $T = \frac{32}{13}$ for all $\epsilon \in (0, \epsilon_0]$.
Method: center-manifold reduction; spatial dynamics; bifurcation theory

A vector-valued breather example in $\mathbb{R}^3 \times \mathbb{R}$

$$(*_{\mathrm{vec}}) \qquad V(x)\partial_t^2 U + \nabla \times \nabla \times U + q(x)U \pm \Gamma(x)|U|^{p-1}U = 0$$

ansatz: $U(x,t) = \psi(r,t)\frac{x}{r}, \quad r = |x|.$

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Theorem (Plum, R. 2016)

Let $T = 2\pi\sqrt{\frac{V(0)}{q(0)}}$.

- $V, q, \Gamma > 0$ *radially symmetric* C^2*-functions,*
- $\sup \frac{q}{\Gamma} < \infty$,
- $T\sqrt{\frac{q(r)}{V(r)}} \lneqq 2\pi$ *on* $\mathbb{R}^3 \setminus \{0\}$,
- $$\left|2\pi - T\sqrt{\frac{q(r)}{V(r)}}\right|^{\frac{1}{p-1}} = \begin{cases} O(e^{-\alpha r}) \text{ as } r \to \infty, \\ o(1) \text{ in } C^2\text{-sense as } r \to 0. \end{cases}$$

Then $\exists$ T-periodic, real-valued, exponentially decaying solution.

The proof in the plus case – solving an ODE

$$U(r,t) = \psi(r,t)\frac{x}{r}, \qquad V(r)\ddot{\psi} + q(r)\psi + \Gamma(r)|\psi|^{p-1}\psi = 0$$

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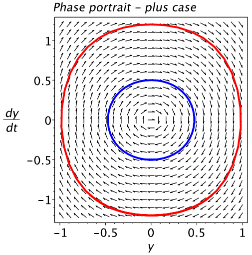
Rescale: $\psi(r,t) = \left(\frac{q(r)}{\Gamma(r)}\right)^{1/(p-1)} y\left(\sqrt{\frac{q(r)}{V(r)}}t\right)$

$$\ddot{y} + y + |y|^{p-1}y = 0$$

$$\dot{y}^2 + y^2 + \frac{2}{p+1}|y|^{p+1} = \text{cst.} = c$$

periodic orbits $y(t;c)$

- period $L(c) = 2\pi - O(c^{\frac{p-1}{2}})$
- $\max_{\mathbb{R}} |y(t;c)| \leq \sqrt{c}$
- How to choose $c = c(r)$?



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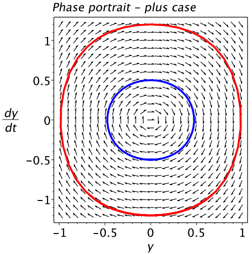

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Answer:
$\sqrt{\frac{q(r)}{V(r)}}T = L(c)$, $c := L^{-1}\left(\sqrt{\frac{q(r)}{V(r)}}T\right)$

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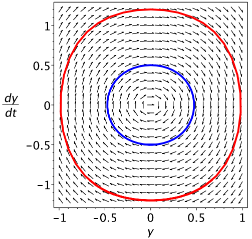

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$$|\psi(r,t)| \le \text{cst.}\sqrt{c(r)} \le \text{cst.}\left|2\pi - \sqrt{\frac{q(r)}{V(r)}}T\right|^{1/(p-1)} = \begin{cases} \to 0 \text{ as } r \to 0 \\ O(e^{-\alpha r}) \text{ as } r \to \infty \end{cases}$$

Remarks on real-valued curl-curl breathers

$$(*_{\text{vec}}) \qquad V(x)\partial_t^2 U + \nabla \times \nabla \times U + q(x)U \pm \Gamma(x)|U|^{p-1}U = 0$$

- Use radial symmetry $\rightarrow$ it is easy to construct real-valued breathers $U(r,t) = \psi(r,t)\frac{x}{r}$
- Under exactly the same assumptions on q, V, Γ:
  time-harmonic complex exponentially decaying solutions exist:

$$U(x,t) = e^{i\frac{2\pi}{T}t}\psi(r)\frac{x}{r}$$

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$$|\psi|^{p-1} = \underbrace{\left(\left(\frac{2\pi}{T}\right)^2 \frac{V(r)}{q(r)} - 1\right)}_{\text{positive, } \to 0 \text{ as } r\to 0,\infty} \cdot \underbrace{\frac{q(r)}{\Gamma(r)}}_{\text{bounded}}$$

A scalar breather example via calc.var.

$$(*) \qquad \begin{cases} V(x)u_{tt} - u_{xx} &= \gamma\delta_0 u^3 \text{ in } \mathbb{R}\times\mathbb{R} \\ u(x,t) &\to 0 \text{ as } |x|\to\infty \\ u(x,t+T) &= u(x,t) \end{cases}$$

where δ_0 is the δ-distribution in x-direction centered at 0.

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Assume that $u(-x,t) = u(x,t)$.

$$(**) \qquad \begin{cases} V(x)u_{tt} - u_{xx} &= 0 \text{ in } (0,\infty)\times\mathbb{R}, \\ -2\partial_x u(0,t) &= \gamma u(0,t)^3, \\ u(x,t) &\to 0 \text{ as } x\to\infty \\ u(x,t+T) &= u(x,t) \end{cases}$$

A scalar breather example via calc.var.

$$(*) \quad \begin{cases} V(x)u_{tt} - u_{xx} &= \gamma\delta_0 u^3 \text{ in } \mathbb{R}\times\mathbb{R} \\ u(x,t) &\to 0 \text{ as } |x|\to\infty \\ u(x,t+T) &= u(x,t) \end{cases}$$

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The choice of the periodic linear operator with periodicity cell $[0,P]$:

$$L = V(x)\partial_t^2 - \partial_x^2$$

with

$$V(x) = \alpha + \beta\delta^{per,P}, \qquad \alpha,\beta > 0.$$

$\delta^{per,P}$ is the P-periodic extension of the $\delta_{P/2}$-distribution on x-axis.

A scalar breather example via calc.var.

$$(**) \quad \begin{cases} V(x)u_{tt} - u_{xx} = 0 \text{ in } (0,\infty)\times\mathbb{R}, \\ -2\partial_x u(0,t) = \gamma u(0,t)^3, \\ u(x,t) \to 0 \text{ as } x\to\infty \\ u(x,t+T) = u(x,t) \end{cases}$$

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Theorem (R. 2016)

Let $\alpha, P, \gamma > 0$ be given with $\beta > 4\alpha P/\pi$. Then there exists infinitely many breathers which are even in x, $T/2$-antiperiodic in t with $T = 4P\sqrt{\alpha}$.

Sketch of the proof – overview

Fourier-decomposition of solution:

$$u(x,t) = \sum_{k \text{odd}} u_k(x) e^{ik\omega t}, \quad u_{-k} = \bar{u}_k.$$

Fourier-decomposition of operator L:

$$\sigma(L) = \bigcup_{k \text{odd}} \sigma(L_k) = \bigcup_{k \text{odd}} \sigma\Big(-\partial_x^2 - k^2\omega^2 V(x)\Big)$$

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Steps:

- for each odd k check: $0 \notin \sigma(L_k)$
- determine Bloch mode ϕ_k, Floquet-multiplier ρ_k:

$$L_k\phi_k = 0, \qquad \phi_k(0) = 1, \qquad \phi_k(x + jP) = \rho_k^j \phi_k(x), \qquad |\rho_k| < 1$$

- $u(x,t) = \sum_{k \text{odd}} a_k \phi_k(x) e^{ik\omega t}, \quad a_k \in \mathbb{C},\ a_{-k} = \bar{a}_k$
- solve the variational problem for $(a_k)_{k \text{odd}}$ in a sequence-space

The spectral non-resonance

Recall for odd k: $0 \notin \sigma(L_k) = \sigma\Big(-\partial_x^2 - k^2\omega^2 V(x)\Big)$, i.e.,

$$\Leftrightarrow k^2\omega^2\alpha \notin \sigma\Big(-\partial_x^2 - k^2\omega^2\beta\delta^{per,P}\Big)$$

$$\Leftrightarrow \Big|\frac{\beta}{\sqrt{\alpha}}|k|\omega\sin(|k|\underbrace{\omega\sqrt{\alpha}P}_{=\pi/2}) + 2\cos(|k|\underbrace{\omega\sqrt{\alpha}P}_{=\pi/2})\Big| = \frac{\beta\omega}{\sqrt{\alpha}}\underbrace{|k|}_{\geq 1} > 2$$

Floquet-multiplier:

$$\rho_k = (-1)^l\left(\frac{\beta|k|\pi}{4P\alpha} - \sqrt{(\ldots)^2 - 1}\right) = O(\frac{1}{k})$$

Bloch-mode:

$$\phi_k(0) = 1, \phi_k'(0) = -|k|\omega\sqrt{\alpha}\Big(1 + O(\frac{1}{k})\Big)\cdot(-1)^l$$

The variational problem – Part I

$$(**) \qquad \begin{cases} V(x)u_{tt} - u_{xx} &= 0 \text{ in } (0,\infty)\times\mathbb{R}, \\ -2\partial_x u(0,t) &= \gamma u(0,t)^3, \\ u(x,t) &\to 0 \text{ as } x\to\infty \\ u(x,t+T) &= u(x,t) \end{cases}$$

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Moreover:

$$u(0,t) = \sum_{k=2l+1} \underbrace{\phi_k(0)}_{=1} a_k e^{ik\omega t}, \quad u(0,t)^3 = \sum_{k=2l+1} (a*a*\bar{a})_k e^{ik\omega t}$$

$$u_x(0,t) = \sum_{k=2l+1} \phi_k'(0)a_k e^{ik\omega t} = \sum_{k=2l+1} \Big(-|k|\underbrace{\omega\sqrt{\alpha}(-1)^l}_{=:g_k} + O(1)\Big)a_k e^{ik\omega t}$$

The variational problem – Part II

The nonlinear Neumann boundary condition:

(*nN*) $$-2\partial_x u(0,t) = \underbrace{\gamma}_{=1} u(0,t)^3$$

becomes

(*nN*) $$2|k|g_k a_k + O(1)a_k = (a * a * \bar{a})_k$$

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Work in the sequence Hilbert-space

$$H = \Big\{(a_k)_{k\in\mathbb{Z}} : a_{-k} = \bar{a}_k, a_k = 0 \text{ for } k \text{ even s.t. } \|a\|^2 := \sum_{k\in\mathbb{Z}} |k||a_k|^2 < \infty\Big\}$$

functional

$$J[a] = \sum_{k\in\mathbb{Z}} |k|g_k|a_k|^2 + O(1)|a_k|^2 - \frac{1}{4}|(a*a)_k|^2, \qquad a \in H$$

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Note:

- H embedds compactly into l^q, $1 < q \leq \infty$
- $\|a * a\|_2 \leq \text{cst.} \|a\|_{4/3}^2 \leq \text{cst.} \|a\|^2$ by Young's inequality
- $J'[a] = 0$ if and only if $(a)_{k\in\mathbb{Z}}$ solves (nN)

Solving the variational problem

Finding a critical point of

$$J[a] = \sum_{k\in\mathbb{Z}} |k| g_k |a_k|^2 + O(1)|a_k|^2 - \frac{1}{4}|(a*a)_k|^2$$
$$= Q(a,a) - \frac{1}{4}\sum_{k\in\mathbb{Z}} |(a*a)_k|^2$$

is done by spectral splitting

$$H = H^- \oplus H^+$$

and minimizing J on the generalized Nehari-manifold

$$N = \{a \in H \setminus \{0\} : J'[a]b = 0 \ \forall b \in [a] + H^-\}$$

Szulkin-Weth('10): existence of minimizer & infinitely many critical points

□

Some concluding remarks/open questions

Karlsruhe Institute of Technology

- By construction we get „polychromatic" waves $\sum_k a_k \phi_k(x) e^{ik\omega t}$ with $a_k \neq 0$ for infinitely many k
- Even the „ground states" are polychromatic
- A pure monochromatic wave $a_k \phi_k(x) e^{ik\omega t}$ is a critical point of J if

$$\tilde{H} := \left\{ (a_k)_{k\in\mathbb{Z}} : a_{-k} = \bar{a}_k, a_k = 0 \text{ for } k \text{ even} \right\}$$

- What are the „ground states" on $\tilde{H}$? Pure monochromatic wave $a_1 \phi_1 e^{i\omega t}$?
- What about nonlinearities $|u(x,t)|^{p-1} u(x,t)$?
- What about other operators $L = V(x)\partial_t^2 - \partial_x^2 + q(x)$ with $0 \notin \sigma(L)$?

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