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Localized time-periodic solutions of nonlinear wave equations

Wolfgang Reichel (financially supported by DFG through CRC 1173)

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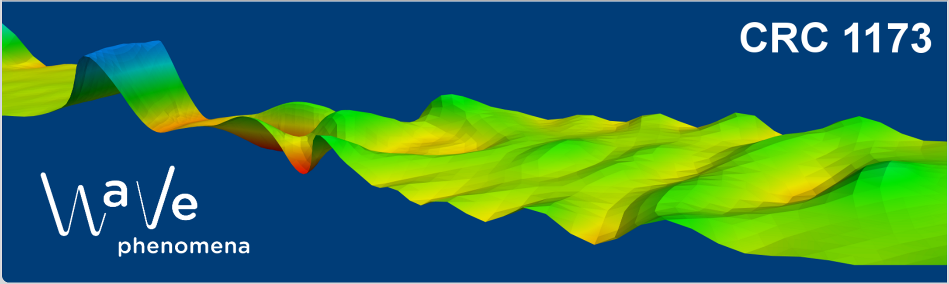
Institute for Analysis



The problem

Find spatially localized, time-periodic $E : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ such that

$$\text{(quasi)} \qquad \nabla \times \nabla \times E + \partial_t^2\Big(V(x)E + \Gamma(x)|E|^{p-1}E\Big) = 0$$

$$\text{(semi)} \qquad \nabla \times \nabla \times E + V(x)\partial_t^2 E + \Gamma(x)|E|^{p-1}E = 0$$

with $p > 1$ & suitable conditions on $V, \Gamma : \mathbb{R}^3 \to \mathbb{R}$

Outline:

(A) physical background

(B) results for time-harmonic/monochromatic solutions

(C) results for real-valued periodic/polychromatic solutions

(A): The quasilinear Curl-Curl wave equation

$$\nabla \times E + \partial_t B = 0, \qquad \nabla \cdot D = 0,$$
$$\nabla \times H - \partial_t D = 0, \qquad \nabla \cdot B = 0.$$

Material laws:

$$B = \mu_0 H, \quad D = \epsilon_0 E + P(x, E) = \epsilon_0 (1 + \chi_1(x) + \chi_3(x)|E|^2 + \ldots) E$$

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Quasilinear wave-equation for E:

$$\hookrightarrow \quad \nabla \times \nabla \times E + \partial_t^2 \Big(\underbrace{\mu_0 \epsilon_0 (1 + \chi_1(x))}_{= V(x) \geq 0} E + \underbrace{\mu_0 \epsilon_0 \chi_3(x) |E|^2 E + \ldots}_{= f(x, |E|^2) E} \Big) = 0$$

(B) Time-harmonic/monochromatic approach

$$\nabla \times \nabla \times E + \partial_t^2 \Big(V(x)E + f(x, |E|^2)E \Big) = 0$$

Time-harmonic/monochromatic ansatz: $E(x,t) = U(x)e^{i\omega t}$ leads to

$$(*) \qquad \nabla \times \nabla \times U + \tilde{V}(x)U = \tilde{f}(x, |U|^2)U \text{ in } \mathbb{R}^3$$

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(0) $U(x_1, x_2, x_3) = (0, 0, u(x_1, x_2))^T$ leads to scalar NLS (many results!)

(1) Benci-Fort.('04) & Azzollini-B.-d'Aprile-F.('06) & d'A.-Siciliano('11), Bartsch-Mederski ('14,'15), Mederski('14)

(2) Bartsch-Dohnal-Plum-R. ('14) & Hirsch-R. ('16) ... next

(3) $E(x,t) = U(x) \cos(\omega t)$ works for time-averaged material response

$$f\left(x, \frac{1}{T} \int_0^T |E|^2 \, dt\right) E$$

(B) Common variational set-up

$$J[U] = \int_{\mathbb{R}^3} |\nabla \times U|^2 + \tilde{V}(x)|U|^2 - \tilde{F}(r, z, |U|^2)\, dx,$$

$$U \in X = H(\text{curl}; \mathbb{R}^3) \cap L^{p+1}(\mathbb{R}^3)$$

Here is the problem: $\|\nabla U\|_{L^2}^2 = \|\nabla \times U\|_{L^2}^2 + \|\nabla \cdot U\|_{L^2}^2$.

Constraint $\{U : \text{div}\, U = 0\}$ does not solve it $\Rightarrow$ Lagrange-multiplier!

Symmetries! Look for cylindrical symmetry in coordinates (r, z):

$$U(r, z) := u(r, z) \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}. \quad \Rightarrow \text{div}\, U = 0.$$

$$-\Delta_5 u(r, z) + \tilde{V}(r, z)u = \tilde{f}(r, z, r^2u^2)u \text{ for } r > 0, z \in \mathbb{R}.$$

This is a NLS-type equation in $\mathbb{R}^5$!

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(B) Results I - (Bartsch-Dohnal-Plum-R., NoDeA 2016)

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$$(*) \qquad \nabla \times \nabla \times U + \tilde{V}(x)U = \tilde{\Gamma}(x)|U|^{p-1}U \quad \text{in} \quad \mathbb{R}^3$$

General assumption: $\tilde{V} = \tilde{V}(r, x_3), \tilde{\Gamma} = \tilde{\Gamma}(r, x_3), r = \sqrt{x_1^2 + x_2^2}$

Theorem 1 (Defocusing case)

- $\tilde{\Gamma}(x) \le -C(1 + |x|^\alpha), \alpha > \frac{3}{2}(p-1), p > 1,$
- $\tilde{V} \in L^\infty(\mathbb{R}^3)$, $\sup \tilde{V} < 0$.

Then $(*)$ has a (restricted) ground-state.

Theorem 2 (Focusing case)

- $\inf \tilde{\Gamma} > 0$, $\tilde{V}, \tilde{\Gamma} \in L^\infty(\mathbb{R}^3)$ *are* 1*-periodic in* x_3,
- $1 < p < 5$
- $0 \notin \sigma(L)$ *with* $L = \nabla \times \nabla \times + \tilde{V}(x)$.

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(B) Results II - statements (Hirsch-R., ZAA 2017)

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$$(*) \qquad \nabla \times \nabla \times U + \tilde{V}(r,z)U = \tilde{\Gamma}(r,z)|U|^{p-1}U \quad \text{in} \quad \mathbb{R}^3$$

Theorem 3 (Positive definite, focusing case)

- $0 < \min \sigma(\nabla \times \nabla \times + \tilde{V})$
- $\tilde{V}(r,z)$ *reverse Steiner-symmetric in* z
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Remarks:

- the theorem also covers nonlinearities $\tilde{f}(r,z,|U^2)U$
- Ex.: $\tilde{f}(r,z,|U|^2) = \tilde{\Gamma}(r,z)\log(1+|U|^2)$
- Ex.: $\tilde{f}(r,z,|U|^2) = \tilde{\Gamma}(r,z)|U|^{p(z)-1}$, $\overline{\mathrm{Rg}(p)} \subset (1,5)$, p Steiner symm.

(B) Sketch of variational existence proof

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$$J[u] = \int_{\mathbb{R}^5} |\nabla u|^2 + \tilde{V}(r,z)u^2 - \frac{2}{p+1}\tilde{\Gamma}(r,z)r^{p-1}|u|^{p+1}\,dx^5, \quad u \in H^1_{cyl}(\mathbb{R}^5)$$

Defocusing case – Theorem 1: minimize J directly

Focusing cases – Theorem 2 & 3: spectral splitting $H^1_{cyl}(\mathbb{R}^5) = H^+ \oplus H^-$

minimize J over the Nehari-manifold [cf. Szulkin-Weth, '10]:

$$\mathcal{N} = \left\{ u \neq 0; J'[u]\phi = 0 \ \forall \phi \in [u] + H^- \right\}$$

Take a minimizing sequence $u_k \rightharpoonup u_0$.

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Take a minimizing sequence $u_k \rightharpoonup u_0$.
To get $u_0 \neq 0$ modify minimizing sequences

- Theorem 2: by using shifts along periodicity structure (concentration compactness of P.L.Lions)
- Theorem 3: by using Steiner-symmetrization u_k^* in z-direction and weak sequential cont.'y of u_k^*

(C): Real-valued time-periodic solutions

Find solutions $U : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ such that

$$(*) \quad \begin{cases} \nabla \times \nabla \times U + V(x)U_{tt} + f(x, |U|^2)U = 0 & \\ \qquad\qquad\qquad U(x,t) & \to \quad 0 \text{ as } |x| \to \infty \\ \qquad\qquad\qquad U(x,t+T) & = \quad u(x,t) \end{cases}$$

under suitable conditions on V, f.

U (real-valued, time-periodic & spatially localized) is called „breather“

Motivation:

i. The famous Sine-Gordon breather

$$u_{tt} - u_{xx} + \sin u = 0$$

ii. The example by Blank, Chirilus-Bruckner, Lescaret, Schneider ('11) for

$$V(x)u_{tt} - u_{xx} + q(x)u = \Gamma(x)u^3$$

A vector-valued breather example in $\mathbb{R}^3 \times \mathbb{R}$

$$(*_{\text{vec}}) \qquad V(x)\partial_t^2 U + \nabla \times \nabla \times U + q(x)U \pm \Gamma(x)|U|^{p-1}U = 0$$

ansatz: $U(x,t) = \psi(r,t)\frac{x}{r}, \quad r = |x|.$

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Theorem 4 (Plum, R. JEPE 2017)

Let $T = 2\pi\sqrt{\frac{V(0)}{q(0)}}$.

- $V, q, \Gamma > 0$ *radially symmetric* C^2*-functions,*
- $\sup \frac{q}{\Gamma} < \infty$,
- $\frac{q(r)}{V(r)} \lessgtr \frac{q(0)}{V(0)}$ *on* $\mathbb{R}^3 \setminus \{0\}$,
- $$\left|\frac{q(r)}{V(r)} - \frac{q(0)}{V(0)}\right|^{\frac{1}{p-1}} = \begin{cases} O(e^{-\alpha r}) \text{ as } r \to \infty, \\ o(1) \text{ in } C^2\text{-sense as } r \to 0. \end{cases}$$

Then $\exists$ T*-periodic, real-valued, exponentially decaying solution.*

The proof – solving an ODE

$$U(r,t) = \psi(r,t)\frac{x}{r}, \qquad V(r)\ddot{\psi} + q(r)\psi \pm \Gamma(r)|\psi|^{p-1}\psi = 0$$

ODE in time with $r =$ parameter

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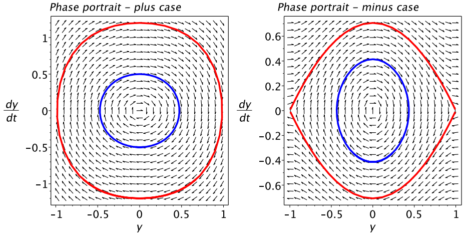
$$\ddot{y} + y \pm |y|^{p-1}y = 0$$



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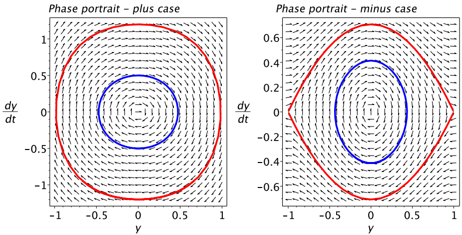
periodic orbits $y(t;c)$, period $L(c)$

- c= value of first integral
- How to choose $c = c(r)$?
- Answer: $\sqrt{\frac{q(r)}{V(r)}}T = L(c)$



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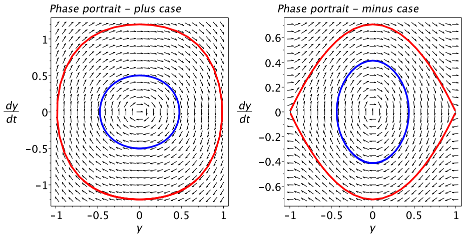

& assumptions $\Rightarrow$ result

A scalar breather example via calc.var.

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$$(*) \qquad \left\{ \begin{array}{rcl} V(x)u_{tt} - u_{xx} & = & \gamma\delta_0 u^3 \text{ in } \mathbb{R}\times\mathbb{R} \\ u(x,t) & \to & 0 \text{ as } |x|\to\infty \\ u(x,t+T) & = & u(x,t) \end{array} \right.$$

where δ_0 is the δ-distribution in x-direction centered at 0.

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The choice of the periodic linear operator with periodicity cell $[0,P]$:

$$L = V(x)\partial_t^2 - \partial_x^2$$

with

$$V(x) = \alpha + \beta\delta^{per,P}, \qquad \alpha,\beta > 0.$$

$\delta^{per,P}$ is the P-periodic extension of the $\delta_{P/2}$-distribution on x-axis.

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Theorem 5 (R. 2016)

Let $\alpha, P >, \gamma \neq 0$ be given with $\beta > 4\alpha P/\pi$. Then there exists a real-valued breather which is even in x, $T/2$-antiperiodic in t with $T = 4P\sqrt{\alpha}$.

Sketch of the proof – overview

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Even solutions $u(x,t) = u(-x,t) \Rightarrow$ nonlinear Neumann problem

$$(nN) \qquad \begin{cases} V(x)u_{tt} - u_{xx} & = & 0 \text{ in } (0,\infty)\times\mathbb{R}, \\ -2u_x(0,t) & = & \gamma u(0,t)^3, \\ u(x,t) & \to & 0 \text{ as } x\to\infty \\ u(x,t+T) & = & u(x,t) \end{cases}$$

Fourier-decomp. of solution $u(x,t) = \sum_{k \text{odd}} u_k(x) e^{ik\omega t}, u_{-k} = \bar{u}_k$.
Fourier-decomp. of wave operator L:

$$\sigma(L) = \bigcup_{k \text{odd}} \sigma(L_k) = \bigcup_{k \text{odd}} \sigma\Big(-\partial_x^2 - k^2\omega^2 V(x)\Big)$$

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Steps:

- choice of $\alpha, \beta, P \Rightarrow 0 \notin \sigma(L)$
- $u(x,t) = \sum_{k\text{odd}} a_k\phi_k(x)e^{ik\omega t}$
- $\phi_k =$ Bloch-mode, $L_k\phi_k = 0$ on $(0,\infty)$, exp. decaying at $+\infty$
- find $(a_k)_{k\text{odd}}$, $a_{-k} = \bar{a}_k$

The variational problem

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$$u_x(0,t) = \sum_{k \text{odd}} \phi_k'(0) a_k e^{ik\omega t}, \quad u(0,t)^3 = \sum_{k \text{odd}} (a * a * \bar{a})_k e^{ik\omega t}$$

u weakly solves (nN) $\Leftrightarrow J'[a] = 0$, $a \in H$ with

$$J[a] := \sum_{k=2l+1} \underbrace{\phi_k'(0)}_{\approx -(-1)^l |k|} |a_k|^2 - \frac{\gamma}{4} |(a*a)_k|^2, \;\; H := \Big\{ a_{-k} = \bar{a}_k, \sum_{k \text{odd}} |k||a_k|^2 < \infty \Big\}$$

abstract critical point theorem: $\Rightarrow \exists$ truly polychromatic ground state

Summary

Monochromatic waves $E(x,t) = U(x)e^{i\omega t}$ for

(quasi) $$\nabla \times \nabla \times E + \partial_t^2\big(V(x)E + \Gamma(x)|E|^{p-1}E\big) = 0$$

- cylindrical symmetry
- existence of ground states for
  - defocusing case: $\Gamma(x) \le -C(1+|x|^\alpha)$, $\alpha > \frac{3}{2}(p-1)$, $\inf V > 0$
  - focusing case: periodic structure in z, $0 \notin \sigma(\nabla \times \nabla \times -\omega^2 V(x))$
  - focusing case: Steiner symmetry in z, $0 < \sigma(\nabla \times \nabla \times -\omega^2 V(x))$

Polychromatic waves $E(x,t) = \sum_k U_k(x)e^{ik\omega t}$, $U_k = \bar{U}_{-k}$ for

(semi) $$\nabla \times \nabla \times E + V(x)\partial_t^2 E + \Gamma(x)|E|^{p-1}E = 0$$

- vector case: radial symmetry, $E(x,t) = \psi(|x|,t)\frac{x}{|x|}$, ODE in time
- scalar case: $p=3$, $V(x) =$ cst.+periodic delta, $\Gamma(x) = \gamma$ delta at 0
  - $0 \notin \sigma(-\partial_x^2 + V(x)\partial_t^2)$
  - use Fourier-Bloch decomposition $\sum_{k \text{odd}} a_k \phi_k(x) e^{ik\omega t}$
  - solve indefinite variational problem for $(a_k)_{k \text{odd}}$