Institute for Analysis
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– Nonlinear Schrödinger Equations (final Update 4 on Feb. 27th) –

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Goals of this lecture-series:

- Discuss existence of standing waves for the nonlinear Schrödinger equation (NLS)

(1) $$\mathrm{i}\frac{\partial w}{\partial t} = -\Delta w + \tilde{V}(x)w - \Gamma(x)|w|^{p-1}w, \qquad x \in \mathbb{R}^n, t \in \mathbb{R}$$

  for functions $w : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C}$ and for $p > 1$.
- Discuss aspects of symmetry breaking and bifurcation.

1. Standing waves and their variational formulation

A solution $w(x,t) = u(x)\mathrm{e}^{-\mathrm{i}\omega t}$ with u decaying to 0 at infinity is called *standing wave*. It has to satisfy

(2) $$-\Delta u + \underbrace{(\tilde{V}(x) - \omega)}_{=:V(x)} u = \Gamma(x)|u|^{p-1}u, \qquad x \in \mathbb{R}^n \text{ with } u(x) \to 0 \text{ as } |x| \to \infty.$$

Exercise 1. *Let $n = 1$, $V(x) = 1$, $\Gamma(x) = 1$, $p = 3$. Show that the only solutions of*

$$-u'' + u = u^3 \text{ in } \mathbb{R} \text{ with } u(x) \to 0 \text{ as } |x| \to \infty$$

are given by the one-parameter families

$$u(x) = \frac{\pm\sqrt{2}}{\cosh(x - x_0)}, \quad x_0 \in \mathbb{R}.$$

1.1. **Weak solutions for NLS** (2).

(A1) $V : \mathbb{R}^n \to [0,\infty)$ and $\Gamma : \mathbb{R}^n \to \mathbb{R}$ are bounded

Recall the definition of the Sobolev space

$$H^1 = \{v : \mathbb{R}^n \to \mathbb{R} | v, \frac{\partial v}{\partial x_i} \in L^2 \text{ for } i = 1, \dots, n\}$$

where the partial derivatives are understood in the distributional sense. H^1 is a Hilbert with inner product

$$\langle v, w\rangle = \int_{\mathbb{R}^n} \nabla v \cdot \nabla w + vw \, dx, \quad v, w \in H^1$$

and we have the Sobolev embedding

$$H^1 \subset L^q \text{ for } \begin{cases} 2 \le q \le \infty & \text{for } n = 1, \\ 2 \le q < \infty & \text{for } n = 2, \\ 2 \le q \le \frac{2n}{n-2} & \text{for } n \ge 3. \end{cases}$$

A *weak solution* $u \in H^1$ of NLS (2) is defined as a solution of

$$\int_{\mathbb{R}^n} \nabla u \cdot \nabla \phi + V(x) u \phi \, dx = \int_{\mathbb{R}^n} \Gamma(x) |u|^{p-1} u \phi \, dx \text{ for all } \phi \in C_c^\infty(\mathbb{R}^n) \tag{3}$$

$$\boxed{\text{(A2) } 1 < p < \frac{n+2}{(n-2)_+} = \begin{cases} \infty & \text{for } n = 1, 2 \\ \frac{n+2}{n-2} & \text{for } n \ge 3. \end{cases}}$$

Exercise 2. *With the assumptions* (A1), (A2) *show that*

$$\Gamma(x)|u|^{p-1}u\phi \in L^1 \textit{ for all } u, \phi \in H^1.$$

Show moreover that in the definition of a weak solution the condition $\phi \in C_c^\infty(\mathbb{R}^n)$ *in* (3) *can be replaced by* $\phi \in H^1$.

1.2. Variational setup for NLS (2).

There are many ways to find a setup in which the NLS (2) can be solved by variational methods. The following setup is suitable for right-hand sides of the form $\Gamma(x)|u|^{p-1}u$. For more general right-hand sides $f(x, u)$ other setups have been developed.

Let us define the following minimization problem

$$\text{Minimize } J[u] := \int_{\mathbb{R}^n} |\nabla u|^2 + V(x) u^2 \, dx \text{ for } u \text{ in } S = \Big\{ v \in H^1 : \underbrace{\int_{\mathbb{R}^n} \Gamma(x) |v|^{p+1} \, dx}_{=:K[v]} = 1 \Big\}.$$

$$\boxed{\text{(A3) } \Gamma^+ \not\equiv 0}$$

Exercise 3. *With the assumptions* $(A1), (A3)$ *check that the set* S *is non-empty.*

1.3. An analogy with quantum mechanics.

There is an analogy between the previous minimization problem and the ground state of the hydrogen atom. For the hydrogen atom (1 electron whizzing around one fixed proton located at $x_0 \in \mathbb{R}^3$) the quantum mechanical ground state is obtained as follows:

$$\text{Minimize } J[u] := \int_{\mathbb{R}^3} |\nabla u|^2 - \frac{u^2}{|x - x_0|} \, dx \text{ for probability densities } u \text{ s.t. } \int_{\mathbb{R}^3} |u|^2 \, dx = 1.$$

A minimizer u^* satisfies (in the weak sense)

$$-\Delta u^* - \frac{u^*}{|x - x_0|} = c u^* \text{ in } \mathbb{R}^3,$$

where c is (on one hand) a Lagrange multiplier and (on the other hand) it is actually

$$c = J[u^*] = \inf_{\|u\|_2=1} J[u] = \text{ least energy level}$$

and thus u^* is called a *ground state.*

1.4. **Back to the variational setup for NLS** (2)**.**

Provided a minimizer $u^* \in S$ exists. Then it satisfies

$$-\Delta u^* + V(x)u^* = c\Gamma(x)|u^*|^{p-1}u^* \text{ in } \mathbb{R}^n$$

where (as before) c is on one hand a Lagrange multiplier and at the same time

$$c = J[u^*] = \inf_{u\in S} J[u] = \text{ least energy level.}$$

Note that (A1) implies that $c \geq 0$. If there is a minimizer, then we even have $c > 0$. Hence, if we now define

$$\tilde{u} = c^{\frac{1}{p-1}}u^*$$

then one can check that $\tilde{u}$ solves

$$-\Delta\tilde{u} + V(x)\tilde{u} = \Gamma(x)|\tilde{u}|^{p-1}\tilde{u} \text{ in } \mathbb{R}^n,$$

and thus $\tilde{u}$ is called a *ground state* for (2).

Exercise 4. *If $\tilde{u}$ is a ground state for* (2) *then the least energy level (as defined above) is obtained from $\tilde{u}$ as follows:*

$$c = \left(\int_{\mathbb{R}^n} |\nabla\tilde{u}|^2 + V(x)\tilde{u}^2\,dx\right)^{\frac{p-1}{p+1}}.$$

Next we make an assumption on the operator $-\Delta + V(x)$. Provided $V \in L^\infty$ the following is equivalent to saying that the infimum of the spectrum of $-\Delta + V(x)$ is positive:

(A4) $\left(\int_{\mathbb{R}^n} |\nabla u|^2 + V(x)u^2\,dx\right)^{1/2}$ is equivalent to the H^1-norm.

Exercise 5. *$V \in L^\infty$, $V \geq 0$ and $V(x) \geq \alpha > 0$ for $|x| \geq R > 0$ is sufficient for* (A4).

1.5. **General strategy for existence of minimizers.**

(1) Let $(u_k)_{k\in\mathbb{N}}$ be a minimizing sequence in S so that $J[u_k] \to c$ as $k \to \infty$.
(2) By (A4): the sequence $(u_k)_{k\in\mathbb{N}}$ is bounded in H^1
(3) Take a subsequence converging weakly in H^1 (denote the subsequence again by $(u_k)_{k\in\mathbb{N}}$): $u_k \rightharpoonup u^*$ as $k \to \infty$.
(4) J is convex, continuous and hence weakly lower-semicontinuous so that
$$J[u^*] \leq \liminf_{k\in\mathbb{N}} J[u_k] = c.$$
(5) If we can show that $K[u^*] = \lim_{k\to\infty} K[u_k] = 1$ then $u^* \in S$ and hence it is the minimizer.
(6) In general, we do not know if $K[u^*] = 1$. Even in the case $\Gamma \geq 0$ we know in general only $K[u^*] \leq 1$.

Exercise 6. *Show that if*

$$\lim_{|x|\to\infty} V(x) = \inf_{x\in\mathbb{R}^n} V(x) > 0 \text{ and } \lim_{|x|\to\infty} \Gamma(x) = \sup_{x\in\mathbb{R}^n} \Gamma > 0$$

and at least one of the functions V, Γ is not constant, then the infimum $c = \inf_{u\in S} J[u]$ is not attained and every minimizing sequence $(u_k)_{k\in\mathbb{N}}$ converges weakly to 0.

In the following cases the above strategy for proving existence of minimizers can be successfully completed (under some extra conditions):

• $\Gamma(x) \to 0$ as $\lvert x\rvert \to \infty$ • V, Γ radially symmetric and $n \geq 2$	• $V, \Gamma > 0$ constant • $V(x) \to V_\infty > 0$, $\Gamma(x) \to \Gamma_\infty > 0$ as $\lvert x\rvert \to \infty$	• V, Γ periodic • V, Γ of interface type
easy	*harder*	*advanced*

2. Existence of ground states - Part I

In this section we begin with working on the easy parts of the above list.

2.1. Existence when Γ vanishes at infinity.

Theorem 1. *Assume* (A1)–(A4) *and* $\lim_{|x|\to\infty} \Gamma(x) = 0$. *Then* (2) *has a ground state.*

Proof. We show that $u_k \rightharpoonup u$ as $k \to \infty$ implies $K[u_k] \to K[u]$, i.e., the functional K is weakly continuous in H^1.

$$\left| \int_{\mathbb{R}^n} \Gamma(x)(|u_k|^{p+1} - |u|^{p+1})\,dx \right|$$
$$\leq \|\Gamma\|_\infty \underbrace{\int_{B_R(0)} \Big| |u_k|^{p+1} - |u|^{p+1} \Big|\,dx}_{\to 0 \text{ by compact embedding}} + \int_{\mathbb{R}^n\setminus B_R(0)} \underbrace{\Gamma(x)}_{\to 0 \text{ as } |x|\to\infty} \Big| |u_k|^{p+1} + |u|^{p+1} \Big|\,dx$$

Here we use that H^1 embeds compactly in $L^{p+1}(B_R(0))$ and that $\Gamma(x)$ tends to 0 at infinity. So for given $\epsilon > 0$ we choose $R > 0$ so large that $\Gamma \leq \epsilon$ on $\mathbb{R}^n \setminus B_R(0)$. Then we choose k so large that the first integral becomes small by the compact Sobolev embedding so that

$$\left| \int_{\mathbb{R}^n} \Gamma(x)(|u_k|^{p+1} - |u|^{p+1})\,dx \right| \leq \epsilon + \epsilon \left(\|u_k\|_{p+1}^{p+1} + \|u\|_{p+1}^{p+1} \right).$$

Since $(u_k)_{k\in\mathbb{N}}$ is bounded in H^1, the L^{p+1}-norms of u_k are bounded and the last expression can be estimated by a multiple of ϵ. This shows that $K[u_k] \to K[u]$ as $k \to \infty$.

□

2.2. Existence when V, Γ are radially symmetric.

With a similar trick as before we can handle the case of radially symmetric coefficients, i.e, $V(x) = V(Ax)$, $\Gamma(x) = \Gamma(Ax)$ for all orthogonal matrices $A \in O(n)$ and for almost all $x \in \mathbb{R}^n$. In fact, we can also look for radially symmetric solutions if we replace H^1 by

$$H^1_{rad} = \{u \in H^1 : u(x) = u(Ax) \quad \forall A \in O(n), \quad \forall_a x \in \mathbb{R}^n\}.$$

Analogously, we replace the Lebesgue spaces L^q by L^q_{rad}. Note: $\forall_a$ means *for almost all.*

Theorem 2. *Assume $n \geq 2$, (A1)–(A4) and V, Γ radially symmetric. Then (2) has a solution.*

Remarks.

- Here the minimizer of our variational problem is obtained in H^1_{rad} and not in H^1.
- Theorem 2 makes no statement about the existence of a minimizer in H^1.
- There are examples where $c = \inf_{u \in S} J[u] < c_{rad} = \inf_{u \in S \cap H^1_{rad}} J[u]$.

Before we give the proof here is the key-result.

Lemma 3. *There exists a constant $K = K(n)$ such that*

$$|u(x)| \leq K \|u\|_{H^1} |x|^{\frac{1-n}{2}} \text{ for } |x| \geq 1 \text{ and all } u \in H^1_{rad}. \tag{4}$$

Proof. Do the proof as an Exercise with some hints. It is enough to show the inequality for $u \in H^1_{rad} \cap C^\infty_c$. Proceed as follows:

- Define a function $v(r) = u(x)$ for $|x| = r$ and set $w(r) = r^{\frac{n-1}{2}} v(r)$.
- Check that $w, w' \in L^2(1, \infty)$.
- Use the fundamental theorem of calculus to show that for all $r \geq 1$:

$$w^2(r) = -2 \int_r^\infty w(s) w'(s)\, ds.$$

- Deduce $w^2(r) \leq K(n) \|u\|^2_{H^1}$ for all $r \geq 1$.

□

Remarks.

- For $n \geq 2$ this gives a uniform decay rate for functions belonging to H^1_{rad}.
- For $n = 1$ the Lemma does not give any decay. But infact, in the one-dimensional case there cannot be any such decay rate. To see this, consider $u \in H^1_{rad}$ which means $u(x) = u(-x)$. Suppose $u(0) = 0$ and set

$$u_t(x) := \begin{cases} u(x-t) & \text{for } x \geq t, \\ 0 & \text{for } -t \leq x \leq t, \\ u(x+t) & \text{for } x \leq -t \end{cases}$$

  In other words: cut u into two pieces, shift the right piece by t, the left by $-t$ and complete in the middle by 0. The resulting function u_t belongs to H^1_{rad} and satisfies $\|u\|_{H^1} = \|u_t\|_{H^1}$. Suppose a decay rate of the type (4) did hold, e.g. $|u(x)| \leq K \|u\|_{H^1} |x|^{-\alpha}$ with some $\alpha > 1$. Then it has to bold both for u and u_t (since the have the same norm) which is impossible.

- For $n \geq 2$ one could think of making a similar construction. Take $u \in H^1_{rad} \setminus \{0\}$ with $u(0) = 0$ and define a function $v : [0, \infty) \to \mathbb{R}$ by $u(x) = v(r)$ for $|x| = r$. Then
$$u_t(x) := \begin{cases} v(|x| - t) & \text{for } |x| \geq t, \\ 0 & \text{for } 0 \leq |x| \leq t \end{cases}$$
has the property that $\|u_t\|_{H^1} < \|u\|_{H^1}$ and hence (unlike in the case $n = 1$) there is no contradiction to (4).

Proof of Theorem 2. We can follow the proof of Theorem 1 and show that the functional $K : H^1_{rad} \to \mathbb{R}$ is weakly continuous. Taking a weakly convergent subsequence $u_k \rightharpoonup u$ as $k \to \infty$ we can split the integral $\int_{\mathbb{R}^n} \Gamma(x)(|u_k|^{p+1} - |u|^{p+1})\, dx$ into two parts: $\int_{B_R(0)} \ldots\, dx + \int_{\mathbb{R}^n \setminus B_R(0)} \ldots\, dx$. For the first integral we use as before the compact embedding H^1_{rad} into $L^{p+1}_{rad}(B_R(0))$. Hence it remains to treat the integral over $\mathbb{R}^n \setminus B_R(0)$. Let $\delta > 0$ be so small that $p + 1 - \delta \geq 2$. Then

$$\begin{aligned} & \int_{\mathbb{R}^n \setminus B_R(0)} \Gamma(x) \left| |u_k|^{p+1} - |u|^{p+1} \right| dx \\ \leq \; & \|\Gamma\|_\infty \int_{\mathbb{R}^n \setminus B_R(0)} |u_k|^\delta |u_k|^{p+1-\delta} + |u|^\delta |u|^{p+1-\delta}\, dx \\ \leq \; & \|\Gamma\|_\infty K^\delta (\|u_k\|^\delta_{H^1} + \|u\|^\delta_{H^1}) \int_{\mathbb{R}^n \setminus B_R(0)} |x|^{\frac{\delta(1-n)}{2}} \left(|u_k|^{p+1-\delta} + |u|^{p+1-\delta} \right) dx \\ \leq \; & \tilde{K} R^{\frac{\delta(1-n)}{2}} (\underbrace{\|u_k\|^{p+1}_{H^1}}_{\text{bounded}} + \|u\|^{p+1}_{H^1}) \leq \epsilon \text{ for } R \text{ sufficiently large.} \end{aligned}$$

□

3. Aspects of bifurcation theory for NLS (2)

Let us consider the NLS (2) with a parameter $\mu \in \mathbb{R}$:

$$(2)_\mu \qquad -\Delta u + (V(x) - \mu) u = \Gamma(x)|u|^{p-1}u, \qquad x \in \mathbb{R}^n \text{ with } u(x) \to 0 \text{ as } |x| \to \infty.$$

Definition 4. *$\mu_0 \in \mathbb{R}$ is called a bifurcation point for $(2)_\mu$ if there exists a sequence $(\mu_k, u_k) \in \mathbb{R} \times H^1$ such that $u_k \neq 0$ solves $(2)_{\mu_k}$ and $\mu_k \to \mu_0$, $u_k \to 0$ as $k \to \infty$.*

The point μ_0 is called *bifurcation point* since at μ_0 non-trivial solutions split off (bifurcate) from the trivial solution.

Exercise 7. *Assume $V \in L^\infty$. If μ_0 is a bifurcation point for $(2)_\mu$ then $\mu_0 \in \sigma(-\Delta + V(x))$, i.e., μ_0 belongs to the spectrum of $-\Delta + V(x)$.* Hint: *A proof by contradiction could be done by using the following result from regularity theory: whenever $\mu \notin \sigma(-\Delta + V(x))$ then $(-\Delta + V(x) - \mu)^{-1} : L^s \to W^{2,s}$ is bounded for all $1 < s < \infty$ where $W^{2,s} = \{v : \mathbb{R}^n \to \mathbb{R} | v, \frac{\partial v}{\partial x_i}, \frac{\partial^2 v}{\partial x_i \partial x_j} \in L^s$ for all $i, j = 1, \ldots, n\}$.*

Theorem 5. *Let $V \equiv 0$, $\Gamma \in L^\infty$ with $\Gamma \geq 0$, $\Gamma \not\equiv 0$. Suppose* (A3) *and*

(i) *there exists $A, R > 0$, $t \in [0,2)$ such that $\Gamma(x) \geq \frac{A}{|x|^t}$ for all $|x| \geq R$,*

(ii) $1 < p < 1 + \frac{2(2-t)}{n}$.

If either $\Gamma(x) \to 0$ as $|x| \to \infty$ or Γ is radially symmetric and $n \geq 2$ then $\mu_0 = 0$ is a bifurcation point for $(2)_\mu$.

Remark: Conditions (i) and (ii) are in some sense necessary to obtain bifurcation results. If (i) is satisfied but (ii) is violated then there may not be bifurcation, cf. Appendix.

Proof. By Theorem 1/Theorem 2 for every $\mu < 0$ there exists a ground state of $(2)_\mu$. We will show the following estimate for the least energy level c_μ

$$c_\mu \leq C|\mu|^{\frac{n-t}{p+1}+\frac{2-n}{2}} \text{ for some constant } C > 0 \text{ and } |\mu| \leq 1. \tag{5}$$

If this is done then we proceed as follows: if u_μ^* is the minimizer of the variational problem then $\tilde{u}_\mu := c_\mu^{\frac{1}{p-1}} u_\mu^*$ is the ground state, which has the properties

$$\|\nabla \tilde{u}_\mu\|_{L^2}^2 = c_\mu^{\frac{2}{p-1}} \|\nabla u_\mu^*\|_{L^2}^2 \leq c_\mu^{\frac{2}{p-1}+1}$$

and

$$\|\tilde{u}_\mu\|_{L^2}^2 \leq \frac{1}{|\mu|} \int_{\mathbb{R}^n} |\nabla \tilde{u}_\mu|^2 + |\mu||\tilde{u}_\mu|^2 \, dx = \frac{1}{|\mu|} c_\mu^{\frac{2}{p-1}} \int_{\mathbb{R}^n} |\nabla u_\mu^*|^2 + |\mu||u_\mu^*|^2 \, dx$$
$$\leq \frac{1}{|\mu|} c_\mu^{\frac{2}{p-1}+1} \overset{(5)}{\leq} C|\mu|^{\frac{p+1}{p-1}(\frac{n-t}{p+1}+\frac{2-n}{2})-1} \to 0 \text{ as } \mu \to 0$$

because of assumption (ii) of the theorem. Hence $\|\tilde{u}_\mu\|_{H^1} \to 0$ as $\mu \to 0$. It remains to prove (5). Let $v(x) := \beta \mathrm{e}^{-\alpha|x|^2}$ with $\alpha \in (0,1)$ arbitrary and $\beta > 0$ to be determined such that $v \in S$. Then

$$c_\mu \leq J[v] = \beta^2 \int_{\mathbb{R}^n} (4\alpha^2|x|^2 + |\mu|)\mathrm{e}^{-2\alpha|x|^2} \, dx \qquad \text{substitute } \sqrt{\alpha}x = y$$
$$= \beta^2 \alpha^{-\frac{n}{2}} \int_{\mathbb{R}^n} (4\alpha|y|^2 + |\mu|)\mathrm{e}^{-2|y|^2} \, dy$$
$$= \beta^2 |\mu|^{\frac{2-n}{2}} C \text{ by choosing } \alpha = |\mu|. \tag{6}$$

Now we need an upper bound for β. The condition $v \in S$ means that

$$\beta^{p+1} \int_{\mathbb{R}^n} \Gamma(x) \mathrm{e}^{-\alpha(p+1)|x|^2} \, dx = 1.$$

If we insert the assumption (i) on Γ and use $0 < \alpha < 1$ to get $\mathbb{R}^n \setminus B_{\sqrt{\alpha}R(0)} \supset \mathbb{R}^n \setminus B_R(0)$ then

$$1 \geq \beta^{p+1} A \int_{\mathbb{R}^n \setminus B_R(0)} \frac{1}{|x|^t} \mathrm{e}^{-\alpha(p+1)|x|^2} \, dx \geq \beta^{p+1} \alpha^{\frac{-n+t}{2}} \underbrace{A \int_{\mathbb{R}^n \setminus B_R(0)} \frac{1}{|y|^t} \mathrm{e}^{-(p+1)|y|^2} \, dy}_{=:D},$$

i.e. $\beta \leq D^{\frac{-1}{p+1}} \alpha^{\frac{n-t}{2(p+1)}}$. Inserting this into (6) and recalling $\alpha = |\mu|$ yields the result (5). □

4. Existence of ground states - Part II

4.1. **The NLS** (2) **with periodic coefficients.**

We consider again (2). This time we make the following assumptions on the coefficients:

(A1)' $\Gamma \geq 0$, $\Gamma \not\equiv 0$ and $V, \Gamma \in L^\infty$ are periodic in $x_1, \ldots, x_n$-directions with period 1.

Notice that we no longer require that V is non-negative. Instead Γ is supposed to be non-negative. For simplicity of the presentation we assume $\Gamma \equiv 1$.

Instead of (A4) we now make an assumption on the spectrum $\sigma(L)$ of the linear operator $L := -\Delta + V(x)$:

(A4)' $0 \notin \sigma(L)$

Remark. An example is the case where $V \equiv 1$ and $\sigma(L) = [1, \infty)$. In general the spectrum of L is the union of countably many compact intervals (bands) whose endpoints converge to infinity. There may possibly be gaps between the bands, which allows for the physically interesting situation $V \leq 0$, $\inf \sigma(L) < 0$ but $0 \notin \sigma(L)$.

As a consequence of assumption (A4)', the boundedness of V and some regularity theory we have that L has a bounded inverse $L^{-1} : L^s \to W^{2,s}$ for every $s \in (1, \infty)$. Thus NLS (2) is equivalent[1] to the fixed point problem

$$u = L^{-1}(|u|^{p-1}u) \text{ for } u \in L^{ps} \text{ and some } s \in (1, \infty).$$

The form of the fixed point problem suggests to make the substitution $v = |u|^{p-1}u$ and its inverse $u = |v|^{q-1}v$ with $q = 1/p$. Hence we end up with

$$|v|^{q-1}v = L^{-1}v, \quad v \in L^s. \tag{7}$$

The equation (7) is called dNLS which stands for dual nonlinear Schrödinger equation. Notice that $(q+1)' = p + 1$. Now dNLS (7) has a simple variational structure. It is the Euler-Lagrange equation of the functional

$$M[u] = \int_{\mathbb{R}^n} \frac{1}{q+1}|v|^{q+1} - \frac{1}{2} v L^{-1} v \, dx, \quad v \in L^{q+1}.$$

Exercise 8. *With the assumptions* (A1)', (A2), (A4)' *show that*

$$vL^{-1}w \in L^1 \textit{ for all } v, w \in L^{q+1}.$$

Show moreover that $\int_{\mathbb{R}^n} vL^{-1}w \, dx = \int_{\mathbb{R}^n} wL^{-1}v \, dx$ *for all* $v, w \in L^{q+1}$.

We follow the general strategy of Subsection 1.2 to show the existence of ground states, i.e., we

$$\text{Minimize } J[v] := \int_{\mathbb{R}^n} |v|^{q+1} \, dx \text{ for } v \text{ in } S = \Big\{ w \in L^{q+1} : \underbrace{\int_{\mathbb{R}^n} wL^{-1}w \, dx}_{=:K[w]} = 1 \Big\}.$$

[1] in a sense which needs to be made more precise than needed here

Notice that $1 < p < \frac{n+2}{(n-2)_+}$, $2 < p+1 < \frac{2n}{(n-2)_+}$ leads to $\frac{(n-2)_+}{n+2} < q < 1$, $\frac{2n}{n+2} < q+1 < 2$.

Exercise 9. *v is a ground state for dNLS $\Leftrightarrow$ $u = |v|^{q-1}v$ is a ground state for NLS.*

Theorem 6. *Suppose* (A1)', (A2), (A4'). *Then NLS* (2) *has a ground state.*

Remark. The theorem applies in the constant coefficient case $V \equiv 1$, $\Gamma \equiv 1$. In this case one can say more about the ground state: it is (up to multiplication with -1) positive and (up to shifting) radially symmetric with respect to the origin and exponentially decreasing in the radial direction.

Before we give the proof, let us begin with a result on the tangent space of S.

Lemma 7. *For $v \in S$ let $T_vS = \{\phi \in L^{q+1} : \int_{\mathbb{R}^n} \phi L^{-1} v\, dx = 0\}$ (it is the tangent space to S at v). Then $L^{q+1} = T_vS \oplus [v]$. Moreover, for every bounded subset $\Sigma \subset S$ there exists a constant $C > 0$ such that*

(8) $$\|v\|_{q+1} + \|\phi\|_{q+1} \leq C\|v+\phi\|_{q+1} \text{ for all } v \in \Sigma \text{ and all } \phi \in T_vS.$$

Proof. See Appendix. □

Remark. The lemma means two things:

(a) $[v]$ and T_vS cannot become "almost" colinear (provided v is in a bounded subset $\Sigma \subset S$).
(b) The projections $P_v : L^{q+1} \to [v]$, $Q_v : L^{q+1} \to T_vS$ are bounded independently of v (provided v is in a bounded subset $\Sigma \subset S$).

Lemma 8. *Let $(v_k)_{k\in\mathbb{N}}$ be a minimizing sequence of J in S such that*

(i) $J[v_k] \to c = \inf_S J$ *as* $k \to \infty$,
(ii) $v_k \rightharpoonup v^*$ *as* $k \to \infty$,
(iii) $J'[v_k]|_{T_{v_k}S} \to 0$.

Then

(9) $$|v^*|^{q-1}v^* = cL^{-1}v^*.$$

Note: This is not yet the solution of our problem since v^* could be zero.

Proof. Let us first look at the linear functionals $K'[v_k]|_{[v_k]}$ and $J'[v_k]|_{[v_k]}$ defined on the one dimensional space $[v_k]$. Both functionals are non-zero since applied to v_k they have non-zero values. As functionals on the one-dimensional space $[v_k]$ they are linearly dependent and hence there exists $\lambda_k \in \mathbb{R}$ such that

(10) $$\lambda_k K'[v_k]|_{[v_k]} = J'[v_k]|_{[v_k]}.$$

Now we gather information on λ_k. By testing (10) with v_k we get

$$\lambda_k 2 = (q+1)J[v_k] \to (q+1)c \text{ as } k \to \infty.$$

Notice also that $K'[v_k]\circ Q_{v_k}=0$, i.e., $K'[v_k]=K'[v_k]\circ P_{v_k}$ by the definition of the tangent space $T_{v_k}S$ so that

$$J'[v_k]\circ P_{v_k}=\frac{q+1}{2}cK'[v_k]+o(1) \text{ as } k\to\infty. \tag{11}$$

Now let us use the splitting

$$J'[v_k]=J'[v_k]\circ P_{v_k}+J'[v_k]\circ Q_{v_k}. \tag{12}$$

Since Q_{v_k} is bounded uniformly in k by Lemma 7 (apply the lemma with $\Sigma=\{v_k : k\in\mathbb{N}\}$) and since by assumption $J'[v_k]|_{T_{v_k}S}\to 0$ as $k\to\infty$ we obtain from (11) and (12) that

$$J'[v_k]=\frac{q+1}{2}cK'[v_k]|_{[v_k]}+o(1) \text{ as } k\to\infty. \tag{13}$$

In particular, for every $\phi\in L^{q+1}$ we have

$$\int_{\mathbb{R}^n}|v_k|^{q-1}v_k\phi\,dx=c\int_{\mathbb{R}^n}\phi L^{-1}v_k\,dx+o(1).$$

On every bounded set $D\subset\mathbb{R}^n$ the compactness of L^{-1} implies $L^{-1}v_k|_D\to L^{-1}v^*|_D$ in $L^{p+1}(D)$. Then (13) implies that $|v_k|^{q-1}v_k$ converges in $L^{p+1}(D)$ (and thus for a subsequence also pointwise almost everywhere) to $cL^{-1}v^*$. The $L^{p+1}(D)$ convergence implies the existence of a majorant $|v_k|\le h\in L^{q+1}(D)$. A dominated convergence argument together with the pointwise almost everywhere convergence of a subsequence of $(v_k)_{k\in\mathbb{N}}$ on D allows to identify the weak limit $v_k\rightharpoonup v^*=|cL^{-1}v^*|^{p-1}cL^{-1}v^*$, which exists by hypothesis (ii). This proves the claim. □

Proof of Theorem 6: Let $(v_k)_{k\in\mathbb{N}}$ be a minimizing sequence in S such that $J[v_k]\to c=\inf_S J$. We may furthermore arrange for $(v_k)_{k\in\mathbb{N}}$ such that $J'[v_k]|_{T_{v_k}S}\to 0$ as $k\to\infty$ (this is usually done with the help of Ekeland's variational principle, cf. Appendix). Setting $z_k:=L^{-1}v_k$ we get that $(z_k)_{k\in\mathbb{N}}$ is bounded in $W^{2,q+1}$ and

$$1=\int_{\mathbb{R}^n}v_kz_k\,dx\le\|v_k\|_{q+1}\|z_k\|_{p+1}\le C\|z_k\|_{p+1}.$$

Now we apply to the sequence $(z_k)_{k\in\mathbb{N}}$ in $W^{2,q+1}$ a Concentration Compactness Lemma of Pierre-Louis Lions (1984), cf. M. Willem's book in references, Lemma 1.21 for a similar variant:

Suppose $(z_k)_{k\in\mathbb{N}}$ is bounded in $W^{2,q+1}$ and let $q+1\le s<(q+1)^{**}=\frac{n(q+1)}{(n-2(q+1))_+}$. If

$$\sup_{y\in\mathbb{R}^n}\int_{B_R(y)}|z_k|^s\,dx\to 0 \text{ as } k\to\infty$$

then $z_k\to 0$ in L^r for all $r\in\big((q+1),(q+1)^{**}\big)$.

Since our sequence $(z_k)_{k\in\mathbb{N}}$ does not converge to 0 in L^{p+1} and $p+1\in\big((q+1),(q+1)^{**}\big)$ we get that for a suitable subsequence (again denoted by $(z_k)_{k\in\mathbb{N}}$) and some $R>0$

$$\sup_{y\in\mathbb{R}^n}\int_{B_R(y)}|z_k|^{p+1}\,dx\ge 2\delta>0$$

and hence there exists centers $y_k \in \mathbb{N}^n$ and $\rho > R$ such that

$$\int_{B_\rho(y_k)} |z_k|^{p+1}\,dx \geq \delta > 0. \tag{14}$$

Set $z_k'(\cdot) := z_k(\cdot + y_k)$. Then $\|z_k'\|_{p+1,B_\rho(0)}^{p+1} \geq \delta$. Moreover, if we set $v_k'(\cdot) = v_k(\cdot + y_k)$ then $z_k' = L^{-1}v_k'$ and v_k' is also a minimizing sequence with $J'[v_k']|_{T_{v_k'}S} \to 0$. By boundedness we may furthermore (by selecting a subsequence) assume $v_k' \rightharpoonup v^*$, $z_k' \rightharpoonup z^*$, $z_k' \to z^*$ in L^{p+1}_{loc} as $k \to \infty$ with $L^{-1}v^* = z^*$. Due to (14) $z^* \neq 0$ and hence also $v^* \neq 0$. Now we apply Lemma 8 to $(v_k')_{k\in\mathbb{N}}$ and find that v^* is a non-trivial solution of (9).

Now it remains to show that v^* is the minimizer. Since v^* solves (9) and since $\|\cdot\|_{q+1}$ is weakly-lower semicontinuous

$$c \geq \|v^*\|_{q+1}^{q+1} = c\underbrace{\int_{\mathbb{R}^n} v^* L^{-1} v^*\,dx}_{=:t}$$

so that $t \in (0,1]$. Setting $\tilde{v} = t^{-1/2}v^*$ we have $\tilde{v} \in S$ and

$$c \leq J[\tilde{v}] = t^{\frac{-q-1}{2}} ct \leq c.$$

Hence $t = 1$, $v^* \in S$ and $J[v^*] = c$ so that v^* is indeed a minimizer. ☐

4.2. **The NLS** (2) **with asymptotically constant coefficients.**

Now we consider (2) where the coefficients satisfy:

(A5) $\lim_{\lvert x\rvert\to\infty} V(x) = V_\infty > 0$, $\lim_{\lvert x\rvert\to\infty} \Gamma(x) = \Gamma_\infty > 0$.

Consider (and compare) now the two variational problems

$$c := \inf_{u\in S} \underbrace{\int_{\mathbb{R}^n} |\nabla u|^2 + V(x)u^2\,dx}_{=J[u]}, \quad S = \Big\{ v \in H^1 : K[v] = \int_{\mathbb{R}^n} \Gamma(x)|v|^{p+1}\,dx = 1 \Big\}.$$

and

$$c_\infty := \inf_{u\in S_\infty} \underbrace{\int_{\mathbb{R}^n} |\nabla u|^2 + V_\infty u^2\,dx}_{=J_\infty[u]}, \quad S_\infty = \Big\{ v \in H^1 : K_\infty[v] = \int_{\mathbb{R}^n} \Gamma_\infty|v|^{p+1}\,dx = 1 \Big\}.$$

Theorem 9. *Suppose* (A1)-(A5) *and* $c < c_\infty$. *Then NLS* (2) *has a ground state.*

Remark. One way to verify $c < c_\infty$ is to assume in addition to (A5) that $V_\infty = \sup_{\mathbb{R}^n} V$ and $\Gamma_\infty = \inf_{\mathbb{R}^n} \Gamma$. If not both V, Γ are constant, then there exists a set of positive measure where $V(x) < V_\infty$ or a possibly different set where $\Gamma(x) > \Gamma_\infty$. In the first case we take the minimizer $u_\infty \in S_\infty$ of the constant-coefficient problem, multiply it with a positive factor $t \leq 1$ so that $tu_\infty \in S$. Then we compute $J[tu_\infty] \leq J[u_\infty] < J_\infty[u_\infty] = c_\infty$ and we obtain the desired inequality. In the second case we already know that $t < 1$ and we also obtain $J[tu_\infty] < J[u_\infty] \leq J_\infty[u_\infty] = c_\infty$.

Proof. As usual, let $(u_k)_{k\in\mathbb{N}}$ be a minimizing sequence in S such that $J[u_k] \to c$ as $k \to \infty$ and $u_k \rightharpoonup u^*$, $u_k(x) \to u(x)$ a.e. in $\mathbb{R}^n$ for $k \to \infty$.

Step 1. We prove the following two statements:

$$\begin{aligned} &\text{(i)}\ J[u_k] = J[u^*] + J_\infty[u_k - u^*] + o(1) \text{ as } k \to \infty \\ &\text{(ii)}\ \underbrace{K[u_k]}_{=1} = \underbrace{K[u^*]}_{=:A} + \underbrace{K_\infty[u_k - u^*]}_{=:B_k} + o(1) \text{ as } k \to \infty \end{aligned}$$

Let us first look at (ii) and remember the Lemma of Brezis and Lieb (1983), cf. M. Willem's book in references, Lemma 1.32:

Let $(u_k)_{k\in\mathbb{N}}$ be a bounded sequence in L^r for $1 \le r < \infty$ with $u_k(x) \to u(x)$ a.e. in $\mathbb{R}^n$. Then

Fatou:	$\|u\|_r^r \le \liminf_{k\in\mathbb{N}} \|u_k\|_r^r$
Brezis & Lieb:	$\|u\|_r^r = \lim_{k\to\infty} \|u_k\|_r^r - \|u_k - u\|_r^r$

The result is true in $L^r(D,\mu)$ for some measure space (D,μ).

Now we apply this to the weighted L^{p+1}-norm $\left(\int_{\mathbb{R}^n} \Gamma(x)|u|^{p+1}\,dx\right)^{1/(p+1)}$. Then

$$\int_{\mathbb{R}^n} \Gamma(x)|u_k - u^*|^{p+1}\,dx = \int_{\mathbb{R}^n} \Gamma(x)|u_k|^{p+1}\,dx - \int_{\mathbb{R}^n} \Gamma(x)|u^*|^{p+1}\,dx + o(1)$$

and on the other hand

$$\int_{\mathbb{R}^n} \Gamma(x)|u_k - u^*|^{p+1}\,dx = \int_{\mathbb{R}^n} \Gamma_\infty|u_k - u^*|^{p+1}\,dx + \underbrace{\int_{\mathbb{R}^n} \big(\Gamma(x) - \Gamma_\infty\big)|u_k - u^*|^{p+1}\,dx}_{=o(1)}$$

as we have seen in the proofs in Subsection 2.1. The proof of (ii) follows from considering the two (equal) right-hand sides in the previous two formulas. For the proof of (i) the term $\int_{\mathbb{R}^n} V(x)|u|^2\,dx$ can be handled in the same way. It remains to check that

$$\int_{\mathbb{R}^n} |\nabla u_k - \nabla u^*|^2\,dx = \int_{\mathbb{R}^n} |\nabla u_k|^2 + |\nabla u^*|^2 - 2\nabla u_k \cdot \nabla u\,dx = \int_{\mathbb{R}^n} |\nabla u_k|^2 - |\nabla u^*|^2\,dx + o(1)$$

which holds due to weak convergence. Adding the corresponding identity for the terms $\int_{\mathbb{R}^n} V(x)|u_k - u^*|^2\,dx$ finishes the result of the first step.

Step 2. Recall the definition of A, B_k from Step 1: $A + B = 1$ where $B = \lim_{k\to\infty} B_k$. Assume $A \neq 0$. Then $A^{\frac{-1}{p+1}}u^* \in S$ and $J[u^*] \ge A^{\frac{2}{p+1}}c$ (which also holds in the case $A = 0$). Likewise $J_\infty[u_k - u^*] \ge B_k^{\frac{2}{p+1}} c_\infty$. Hence, using (i) of Step 1 we have

$$\begin{aligned} c = J[u_k] + o(1) &= J[u^*] + J_\infty[u_k - u^*] + o(1) \\ &\ge A^{\frac{2}{p+1}}c + B^{\frac{2}{p+1}}c_\infty + o(1) \end{aligned}$$

Taking the limit $k \to \infty$ we obtain $c \ge A^{\frac{2}{p+1}}c + B^{\frac{2}{p+1}}c_\infty \ge \left(A^{\frac{2}{p+1}} + B^{\frac{2}{p+1}}\right)c$. The strict concavity of the map $t \mapsto t^{\frac{2}{p+1}}$ forces $A = 1$, $B = 0$ or $A = 0, B = 1$. In the first case $K[u^*] = 1$ and $u^* \in S$ is the minimizer. In the second case we would get $c \ge c_\infty$ which is impossible by the assumption $c < c_\infty$. Hence the proof is complete. □

4.3. **An example of symmetry breaking for the NLS** (2)**.**

Consider the one dimensional NLS

$$-u'' + (\lambda - h(x))u = \Gamma(x)|u|^{p-1}u \quad \text{in } \mathbb{R} \tag{15}$$

for $p > 1$. The symmetry breaking result is the following:

Theorem 10. *Assume*

(i) $h, \Gamma \in L^\infty$ *are even functions and* $0 < \|h\|_\infty < \lambda$*,*
(ii) $\operatorname{supp} h \subset [-1, 1]$ *and* $h \geq 0$*,*
(iii) $\Gamma \equiv 1$ *on* $\mathbb{R} \setminus [-1, 1]$ *and* $\Gamma \leq 0$ *on* $[-1, 1]$*.*

Then (15) *has a non-symmetric ground state.*

Proof. Let c be the least energy level of (15) and let c_∞ be the least energy level of the equation

$$-u'' + \lambda u = |u|^{p-1}u \quad \text{in } \mathbb{R}. \tag{15$_\infty$}$$

The problem $(15)_\infty$ has a symmetric positive ground state u_∞ and one can show (for details cf. Arcoya, Cingolani, Gamez) that $c < c_\infty$ so that by Theorem 9 a ground state u for (15) exists. Now we show that u is not even. By contradiction, assume it is even. For $x \in [-1, 1]$ notice that $-u''(x) \leq 0$ so that u is convex on $[-1, 1]$ and thus it attains its global maximum on $\mathbb{R}$ at a point $0 < x_0 \notin [-1, 1]$. Now notice that u solves the ODE $-u'' + \lambda u = |u|^{p-1}u$ outside $[-1, 1]$, which is the same ODE that u_∞ solves. Hence (by uniqueness of the solution to this ODE with vanishing derivative at the point $\pm x_0$) we have $u(x) = u_\infty(|x| - x_0)$ for $|x| \geq x_0$. This implies the following inequality for the least energy levels c, c_∞:

$$c^{\frac{p+1}{p-1}} = \int_{\mathbb{R}} u'^2 + (\lambda - h(x))u^2 \, dx \geq \int_{\mathbb{R}\setminus[-x_0,x_0]} u'^2 + \lambda u^2 \, dx = \int_{\mathbb{R}} u_\infty^2 + \lambda u_\infty^2 \, dx = c_\infty^{\frac{p+1}{p-1}}$$

which contradicts the observation that $c < c_\infty$. Hence u must break the even symmetry. ☐

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Appendix

Remark on the bifurcation conditions in Theorem 5.

Recall the Gagliardo-Nirenberg inequality, see Th. Aubin: Nonlinear analysis on manifolds. Monge-Ampère equations p.38 (Grundlehren der Mathematischen Wissenschaften 252. Springer-Verlag, New York, 1982. xii+204 pp.):

$$\text{For } u \in H^1: \qquad \|u\|_{L^s} \le C\|\nabla u\|_{L^2}^{\alpha}\|u\|_{L^2}^{1-\alpha} \quad \text{for } s \in [2, \frac{2n}{n-2}) \text{ and } \alpha = \frac{n(s-2)}{2s}.$$

The following computation shows that the conditions (i) and (ii) of Theorem 5 are natural. Suppose

(i) there exists $A, R > 0$, $t \in [0,2)$ such that $\Gamma(x) = \frac{A}{|x|^t}$ for all $|x| \ge R$,
(ii) $1 + \frac{2(2-t)}{n} < p < \frac{n+2-2t}{n-2}$.

Then there is no bifurcation for $(2)_\mu$ at $\mu_0 = 0$ from the left (i.e. for $\mu < 0$).

Comment: Think of p being bigger than but close to $1 + \frac{2(2-t)}{n} < p$. Then bifurcation (to the left of $\mu_0 = 0$) does not occur provided Γ behaves like $|x|^{-t}$ near infinity.

Proof: Suppose $u \in H^1 \setminus \{0\}$ solves $(2)_\mu$ for $\mu < 0$. Test (2) with u and apply the Hölder- and the Gagliardo-Nirenberg inequality to get

$$\begin{aligned}
\int_{\mathbb{R}^n} |\nabla u|^2\,dx \le \int_{\mathbb{R}^n} |\nabla u|^2 + |\mu| u^2\,dx &= \int_{\mathbb{R}^n} \Gamma(x)|u|^{p+1}\,dx \\
&\le \|\Gamma\|_{L^r}\|u\|^{p+1}_{L^{(p+1)r'}} \\
&\le C\|\Gamma\|_{L^r}\|\nabla u\|^{\alpha(p+1)}_{L^2}\|u\|^{(1-\alpha)(p+1)}_{L^2}.
\end{aligned} \tag{16}$$

The L^r-norm on Γ is finite provided we choose $r > n/t$. So let us take r sufficiently close to n/t. The Gagliardo-Nirenberg inequality applies since $(p+1)r' < (p+1)\frac{n}{n-t} < \frac{2n}{n-2}$ by assumption (ii) on p. Moreover, since we may choose r arbitrarily close to n/t, we find

$$\alpha(p+1) = (p+1)\frac{n\left((p+1)r'-2\right)}{2(p+1)r'} \lessapprox \frac{(p+1)n - 2(n-t)}{2}$$

which means that $\alpha(p+1)$ is smaller than but arbitrarily close to $\frac{(p+1)n-2(n-t)}{2}$. The latter quantity is larger than 2 by the assumption (ii), i.e., by choosing r sufficiently close to n/t we may achieve that $\alpha(p+1)$ is bigger than 2. As a result we obtain from (16) that

$$1 \le C\|\Gamma\|_{L^r}\|\nabla u\|^{\alpha(p+1)-2}_{L^2}\|u\|^{(1-\alpha)(p+1)}_{L^2} \le C\|\Gamma\|_{L^r}\|u\|^{p-1}_{H^1}$$

which shows that the solutions cannot converge to 0 in H^1 as $\mu \nearrow 0$.

The proof of Lemma 7.

Proof of Lemma 7: The functional $\phi \mapsto \int_{\mathbb{R}^n} \phi L^{-1} v\,dx$ on L^{q+1} is bounded and v is not in its kernel. Therefore we have the splitting $L^{q+1} = T_v S \oplus [v]$. Let us prove the inequality (8) by contradiction, i.e., suppose that there exist sequences $v_k \in S$ with $\|v_k\|_{q+1}$ bounded and $w_k \in T_{v_k}S$ such that

$$\|v_k\|^{q+1}_{q+1} + \|w_k\|^{q+1}_{q+1} > k\|v_k + w_k\|^{q+1}_{q+1}. \tag{17}$$

First we show that $\|w_k\|_{q+1}$ is bounded. For this remember that convexity of the map $\mathbb{R} \ni s \mapsto |s|^{q+1}$ implies the inequality $|a+b|^{q+1} \ge |a|^{q+1} + (q+1)|a|^{q-1}ab$ for all $a, b \in \mathbb{R}$. Applying this to (17) together with an additional Hölder inequality yields

$$\|v_k\|^{q+1}_{q+1} + \|w_k\|^{q+1}_{q+1} > k\left(\|w_k\|^{q+1}_{q+1} - (q+1)\|w_k\|^{q}_{q+1}\|v_k\|_{q+1}\right)$$

which shows the boundedness of $\|w_k\|_{q+1}$. Thus we get from (17) that $v_k + w_k \to 0$ in L^{q+1}. Next we use $\int_{\mathbb{R}^n} w_k L^{-1} v_k\,dx = 0$ and get the contradiction

$$\begin{aligned}
1 = \int_{\mathbb{R}^n} v_k L^{-1} v_k\,dx &= \int_{\mathbb{R}^n} (v_k + w_k) L^{-1} v_k\,dx \\
&\le \|v_k + w_k\|_{q+1}\|L^{-1}\|\|v_k\|_{q+1} \to 0 \text{ as } k \to \infty,
\end{aligned}$$

where we used that $L^{-1} : L^{q+1} \to W^{2,q+1} \hookrightarrow L^{q+1}$ is a bounded operator. □

Application of Ekeland's variational principle.

We recall Ekeland's variational principle, cf. in M. Struwe's book (Theorem 5.1) in the references.

Ekeland's variational principle. *Let M be a complete metric space with metric d, and let $J : M \to \mathbb{R} \cup \{+\infty\}$ be lower semi-continuous, bounded from below, and $\not\equiv \infty$. Then, for any $\eta, \delta > 0$, and $u \in M$ with*

$$J[u] \le \inf_M J + \eta$$

there is an element $w \in M$ strictly minimizing the functional

$$J_w[z] \equiv J[z] + \frac{\eta}{\delta} d(w, z).$$

Moreover, we have $J[w] \le J[u]$ and $d(w, u) \le \delta$.

We will apply this to prove the following result that we used in the proof of Theorem 6: There exists a minimizing sequence $(v_k)_{k\in\mathbb{N}}$ of J in S with the additional property that $J'[v_k]|_{T_{v_k}S} \to 0$ as $k \to \infty$.

Choose a positive sequence $\eta_k \to 0$ as $k \to \infty$ and let $\tilde{v}_k \in S$ be such that

$$J[\tilde{v}_k] \le \inf_S J + \eta_k^2.$$

Using Ekeland's variational principle with $\eta = \eta_k^2$ and $\delta = \eta_k$ we find $v_k \in S$ such that

$$J[v_k] \le J[z] + \eta_k \|z - v_k\|_{q+1} \quad \text{for all } z \in S.$$

Consider the tangential part t_k of the derivative of J at v_k, i.e., $t_k := J'[v_k]|_{T_{v_k}S}$. Due to the following Lemma 11 we get that $\|t_k\| \le \eta_k$, which finishes the proof. □

Lemma 11. *Suppose $\eta > 0$ and $v_0 \in S$ are such that $J[v_0] - J[z] \le \eta \|v_0 - z\|_{q+1}$ for all $z \in S$. If $t := J'[v_0]|_{T_vS}$ then $\|t\| \le \eta$.*

Proof. In a first step we parameterize S near v_0 with the help of the implicit function theorem. In order to solve the equation

$$K[\tau v_0 + w] = 1$$

for τ as a function of $w \in T_{v_0}S$ near the point $v_0 \in S$ we only need to see that $\partial_{v_0} K[v_0] = 2K[v_0] = 2 \ne 0$. Hence there exists a neighborhood $W \subset T_{v_0}S$ and a C^1-function $\tau : W \to \mathbb{R}$ such that $\tau(0) = 1$ and for $w \in W$ the function $\tau(w)v_0 + w$ parameterizes S near v_0. If we differentiate $K[\tau(w)v_0 + w] = 1$ at $w = 0$ we obtain

$$0 = K'[v_0](v_0\tau'(0)\phi + \phi) \quad \forall \phi \in T_{v_0}S.$$

But since $K'[v_0]|_{T_{v_0}S} = 0$ by the definition of the tangent space $T_{v_0}S$, we obtain $0 = K'[v_0]v_0\tau'(0)\phi = 2\tau'(0)\phi$ for all $\phi \in T_{v_0}S$ and therefore we find $\tau'(0) = 0$.

Now we write for $w \in W$ the Taylor expansion for $v \in S$ near v_0:

$$v - v_0 = \big(\tau(w) - \underbrace{\tau(0)}_{=1}\big)v_0 + w = (\underbrace{\tau'(0)}_{=0} w)v_0 + w + o(w) = w + o(w) \text{ as } w \to 0$$

and the corresponding Taylor expansion for J:

$$\begin{aligned} J[v] &= J[v_0] + J'[v_0](v - v_0) + o(v - v_0) \\ &= J[v_0] + J'[v_0]w + o(w). \end{aligned}$$

Using the assumption of the lemma we deduce

$$\begin{aligned} J[v] &\leq J[v] + \eta\|v - v_0\|_{q+1} + J'[v_0]w + o(w) \\ &= J[v] + \eta\|w\|_{q+1} + J'[v_0]w + o(w). \end{aligned}$$

This implies

$$-J'[v_0]w \leq \eta\|w\|_{q+1} + o(w) \text{ as } w \to 0.$$

By choosing an arbitrary $w_0 \in T_vS$, setting $w = \pm tw_0$ and letting $t \to 0$ we obtain $\|J'[v_0]|_{T_vS}\| \leq \eta$. This finishes the proof. □