Karlsruhe Institute of Technology

The Paneitz equation in hyperbolic space

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Institute for Analysis

Conformally covariant 2^{nd} order problems

(M, g): n-dim. ($n > 2$), Riemannian manifold, complete, $\partial M = \emptyset$

scalar curvature: R_g

Laplace Beltrami: $\Delta_g = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j)$

conformal Laplacian: $L_g = -\Delta_g + \frac{n-2}{4(n-1)} R_g$

Yamabe (1962), conformal change of metric: $h = u^{\frac{4}{n-2}} g$

Conformal covariance: $\phi \in C^2(M)$

$$L_g(u\phi) = u^{\frac{n+2}{n-2}} L_h(\phi)$$

Yamabe equation (take $\phi \equiv 1$):

$$\boxed{L_g u = \frac{n-2}{4(n-1)} R_h u^{\frac{n+2}{n-2}} \text{ in } M.}$$

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Conformally covariant 4^{th} order problems

$$Q_g = Q\text{-curvature}$$
$$= \frac{-2}{(n-2)^2}|\,\mathrm{Ric}_g\,|^2 + \frac{n^3-4n^2+16(n-1)}{8(n-1)^2(n-2)^2}R_g^2 - \frac{1}{2(n-1)}\Delta_g R_g$$

$$P_g = \text{Paneitz operator}$$
$$= \Delta_g^2 + \mathrm{div}\Big((a_n R_g\,\mathrm{Id} + b_n \mathcal{R}ic_g)\nabla_g\Big) + \frac{n-4}{2}Q_g$$

where $a_n = \frac{(n-2)^2+4}{2(n-1)(n-2)}$, $b_n = \frac{4}{n-2}$, $\mathcal{R}ic_i^j = g^{jk}Ric_{ki}$

Paneitz (1983), $n > 4$, conformal change of metric: $h = u^{\frac{4}{n-4}}g$
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Uniqueness questions on $\mathbb{S}^n$ and $\mathbb{H}^n$

$$\left.\begin{aligned} \mathbb{S}^n &= \left(\mathbb{R}^n, \left(\tfrac{2}{1+r^2}\right)^2 \delta_{ij}\right), & R_g &= +n(n-1) \\ \mathbb{H}^n &= \left(B_1(0) \subset \mathbb{R}^n, \left(\tfrac{2}{1-r^2}\right)^2 \delta_{ij}\right), & R_g &= -n(n-1) \end{aligned}\right\} Q_g = \frac{n}{8}(n^2-4)$$

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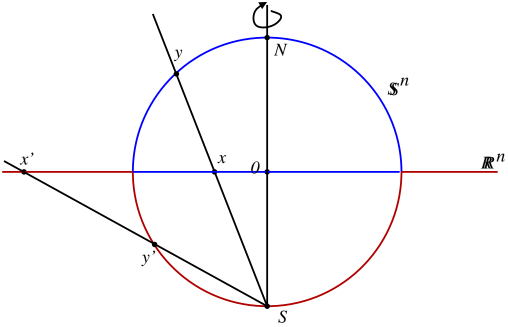

Stereographic projection of $\mathbb{S}^n \setminus \{S\}$ onto $\mathbb{R}^n$

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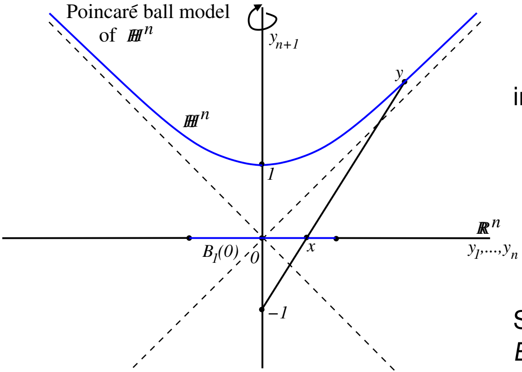

$$\mathbb{H}^n = \left\{(y_1,\dots,y_{n+1}) : y_{n+1} > 0, \; y_1^2 + \dots + y_n^2 = y_{n+1}^2 - 1\right\}$$

in Lorenz-Minkowski space with

$$(g_{ij})_{ij=1}^{n+1} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix}$$

Stereographic projection of $\mathbb{H}^n$ onto $B_1(0) \subset \mathbb{R}^n$

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Yamabe case: $h = U^{\frac{4}{n-2}} g = u^{\frac{4}{n-2}} \delta_{ij}$

$$-\Delta u = \pm \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}} \text{ in } \begin{cases} \mathbb{R}^n \\ B_1(0) \ \& \ u|_{\partial B_1(0)} = \infty \quad \text{(for completeness)} \end{cases}$$

in $\mathbb{R}^n$: $u \cong \left(\frac{2}{1+r^2}\right)^{\frac{n-2}{2}}$ Obata[62], Caffarelli, Gidas & Spruck [89], Chen & Li [91]

in $B_1(0)$: $u = \left(\frac{2}{1-r^2}\right)^{\frac{n-2}{2}}$ Loewner & Nirenberg [82]

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Theorem (Grunau, Ould-Ahmedou, R.)

Let $n > 4$.

(a) *If $Q = \frac{n}{8}(n^2-4)$ then $\exists\,\infty$-many complete metrics h on $\mathbb{H}^n$ with $Q_h = Q$ and $R_h < 0$.*

(b) *if $0 < Q_0 < Q(r) < Q_1$ and $r^q Q(r) \nearrow$ for some $q \in [0,1)$ then the same result holds.*

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Completeness for radial metrics

Conformal radial metric $h = u^{\frac{4}{n-4}}\delta_{ij}$ on $B_1(0)$, $\vec{e}$ unit vector;
$t \mapsto \gamma(t) = t\vec{e}$, $t \in [0,1]$ is a distance minimizing radial geodesic

$$L[\gamma] = \int_0^1 \sqrt{h_{ij}\gamma^i\gamma^j}\,dt = \int_0^1 u(r)^{\frac{2}{n-4}}\,dr$$

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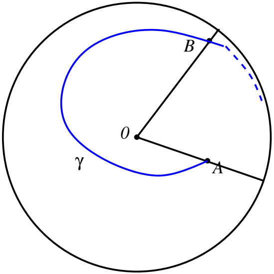
Claim: $L[\gamma] = \infty$ for radial geodesics $\Rightarrow L[\gamma] = \infty$ for every geodesic



Suppose γ is a finite length geodesic starting at A

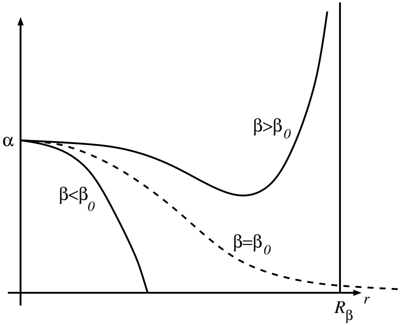
$$\text{dist}(0,B) \leq \text{dist}(0,A) + \text{dist}(A,B)$$

Elements of the proof (case $Q = \text{const.}$)

Part (I): Radial shooting

$u'(0) = u'''(0) = 0, \quad u(0) = \alpha, \quad \Delta u(0) = nu''(0) = \beta$

$\exists \beta_0 \cong \Delta$ of rescaled version of $\mathbb{S}^n$-metric $\left(\frac{2}{1+r^2}\right)^{\frac{n-4}{2}}$



The asymptote $R_\beta \searrow$ for $\beta > \beta_0$

For all $\alpha > 0 \quad \exists_1$ solution u with asymptote at $r = 1$.

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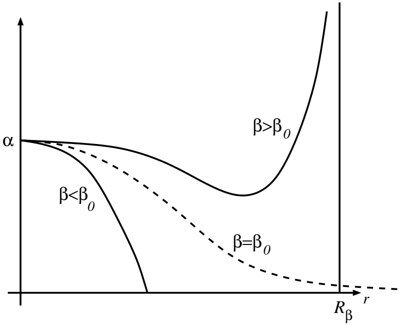

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Question: is the metric $h = u^{\frac{4}{n-2}} \delta_{ij}$ complete?

Part (II): Estimates on the metrics

$u, u', u'', \Delta u, (\Delta u)'$ are positive and $\nearrow$ near $r = 1$

$$\underbrace{c_u \left(\frac{1}{1-r^2}\right)^{\frac{n-4}{4}}}_{\text{non-optimal}} \le u(r) \le \underbrace{C_u \left(\frac{1}{1-r^2}\right)^{\frac{n-4}{2}}}_{\text{optimal}}$$

For upper bound: Pohožaev-identity (first integral):

$$r^{n-1}(\Delta u)'\Big(ru' + \frac{n-4}{2}u\Big) + \frac{n}{2}r^{n-1}u'\Delta u = r^n\Big(\frac{1}{2}(\Delta u)^2 + \frac{n-4}{4n}Qu^{\frac{2n}{n-4}}\Big)$$

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Estimate near $r = 1$

$$u(r) = u(\rho) + \int_\rho^r u'(s)\,ds \le u'(r) + C \le Cu'(r)$$

$$\Delta u(r) \le (\Delta u)'(r) + C \le C(\Delta u)'(r)$$

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Into Pohožaev's identity:

$$u'(\Delta u)' \ge CQu^{\frac{2n}{n-4}} \quad \text{in } [\rho, 1)$$

$$n(u')^2 u'' \ge (u')^2 \Delta u \ge CQu^{\frac{3n-4}{n-4}} \quad \text{in } [\rho, 1)$$

$$(u')^4 \ge CQu^{\frac{4n-8}{n-4}} \quad \text{in } [\rho, 1) \Rightarrow \text{ (optimal) upper bound}$$

The non-optimal lower bound

$$(r^{n-1}\Delta u')' = r^{n-1}\frac{n-4}{2}Qu^{\frac{n+4}{n-4}}$$

$$(\Delta u)'(r) = C + \frac{n-4}{2}\int_{r_0}^{r}\left(\frac{s}{r}\right)^{n-1} Qu^{\frac{n+4}{n-4}}(s)\,ds \leq C + \frac{n-4}{2}Qu(r)^{\frac{n+4}{n-4}}$$

$$\Delta u(r) \leq C + \frac{n-4}{2}Qu(r)^{\frac{n+4}{n-4}} \text{ with } u(1) = \infty$$

This implies (maximum principle)

$$u \geq c\left(\frac{1}{1-r^2}\right)^{\frac{n-4}{4}}$$

Part (III): Change of variables/point of view

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$t = -\log(1 - r^2), \quad t \in [0, \infty), \quad v(t) = (1 - r^2)^{\frac{n-4}{2}} u(r)$ is bounded

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$$K_4(t)v^{(iv)} + K_3(t)v''' + K_2(t)v'' + K_1(t)v' + K_0 v = \frac{Q}{16} v^{\frac{n+4}{n-4}}$$

where $\quad K_i(t) = K_i^\infty + O(e^{-t})$ almost autonomous system

Steady states: 0 and $(16K_0/Q)^{\frac{n-4}{8}}$ ($\cong$ Poincaré-metric)

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$$K_0 = \frac{1}{16}(n^4 - 4n^3 - 4n^2 + 16n)$$

$$K_1(t) = \frac{1}{16}\Big((1-e^{-t})^2(-4n^2 + 24n - 32) + (1-e^{-t})(4n^3 - 16n^2 - 16n + 64) + 4n^3 - 4n^2 - 24n\Big)$$

$$K_2(t) = \frac{1}{16}\Big((1-e^{-t})^2(4n^2 - 40n + 80) + (1-e^{-t})(16n^2 - 16n - 96) + 4n^2 + 8n\Big)$$

$$K_3(t) = (1-e^{-t})^2(n-4) + (1-e^{-t})(n+2)$$

$$K_4(t) = (1-e^{-t})^2$$

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Autonomous limit system:

$$K_4^\infty w^{(iv)} + K_3^\infty w''' + K_2^\infty w'' + K_1^\infty w' + K_0 w = \frac{Q}{16} w^{\frac{n+4}{n-4}}$$

$w \equiv 0$ is asymptotically stable in the autonomous system

$w = (16K_0/Q)^{\frac{n-4}{8}}$ has a 3-d stable and 1-d unstable manifold

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Ph. Hartman (1964, Ordinary Differential Equations)

$v \equiv 0$ is asymptotically stable in the almost autonomous system

Almost autonomous system

0 asympotically stable. If a solution $(v(t), v'(t), v''(t), v'''(t))$ ever comes close to zero then

$$|(v(t), v'(t), v''(t), v'''(t))| \leq e^{(\frac{4-n}{2}+\epsilon)t}$$

and due to the change of variables

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Our solutions (with asymptote at $r = 1$) do not do such things!
... because we have a non-optimal lower bound

$$c_u \left(\frac{1}{1-r^2} \right)^{\frac{n-4}{4}} \leq u(r)$$

Part (IV): Energy considerations

Okay: 0 is asymptotically stable. What does it help for completeness?

Suppose for contradiction: $\int_0^1 u^{\frac{2}{n-4}}\,dr < \infty$. Then: $\int_0^\infty v^{\frac{2}{n-4}}\,dt < \infty$

Since $v(t)$ is bounded also $v'(t), v''(t), v'''(t), v^{(iv)}(t)$ are bounded.

So $v(t_k) \to 0$ for a sequence $t_k \to \infty$. :-(not enough!

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Multiply once with v and once with v' and integrate $\int_0^t$:

$$K_4^\infty \int_0^t (v'')^2\,ds - K_2^\infty \int_0^t (v')^2\,ds = O(1)$$
$$-K_3^\infty \int_0^t (v'')^2\,ds + K_1^\infty \int_0^t (v')^2\,ds = O(1)$$

Part (IV): Energy considerations

Okay: 0 is asymptotically stable. What does it help for completeness?

Suppose for contradiction: $\int_0^1 u^{\frac{2}{n-4}}\,dr < \infty$. Then: $\int_0^\infty v^{\frac{2}{n-4}}\,dt < \infty$

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Karlsruhe Institute of Technology

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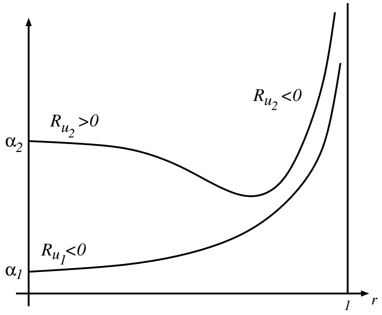
Now we can use asymptotic stability of 0 :-) Contradiction!

Negative scalar curvature

$$R_u = \frac{-4(n-1)}{n-2}\frac{\Delta u^{\frac{n-2}{n-4}}}{u^{\frac{n+2}{n-4}}}$$
$$= \frac{-4(n-1)}{n-4}\frac{1}{u^{\frac{n+2}{n-4}}}\left(u^{\frac{2}{n-4}}\Delta u + \frac{2}{n-4}u^{\frac{6-n}{n-4}}|\nabla u|^2)\right)$$

If u is such that $\Delta u > 0$ in $[0, 1]$ then $R_u < 0$



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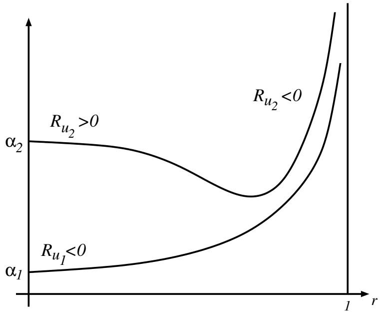

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Question 1: what makes the Poincaré-metric unique?

Question 2: what about more general (non-radial) Q?