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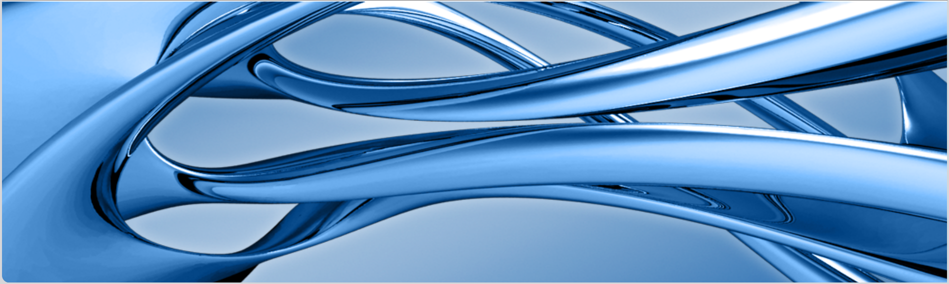
Surface Gap Solitons for the NLS

Dedicated to the memory of Wolfgang Walter (May 2, 1927 – June 26, 2010)

Tomas Dohnal, Michael Plum, Wolfgang Reichel

Research Training Group Seminar • June 28, 2010

Institute for Analysis



Ground states for the periodic NLS

Purely periodic problems: $1 < p < \frac{n+2}{(n-2)_+}$

$$\underbrace{(-\Delta + V_i(x))}_{=:L_i} u = \Gamma_i(x)|u|^{p-1}u \text{ in } \mathbb{R}^n, \qquad i = 1, 2$$

$V_1, V_2, \Gamma_1, \Gamma_2$ 1-periodic, $\operatorname{ess\,inf} \Gamma_i > 0$ (focusing)

Pankov: $0 \notin \sigma(L_i) \Rightarrow \exists$ ground state w_i

$$c_i = \inf_{u \in \mathcal{M}_i} \int_{\mathbb{R}^n} \frac{|\nabla u|^2 + V_i(x)u^2}{2} - \frac{\Gamma_i(x)|u|^{p+1}}{p+1}\, dx$$

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Interface problems for the NLS

$$(-\Delta + V(x))u = \Gamma(x)|u|^{p-1}u \text{ in } \mathbb{R}^n$$

$$V(x) = \begin{cases} V_1(x), & x_1 > 0, \\ V_2(x), & x_1 < 0, \end{cases}$$

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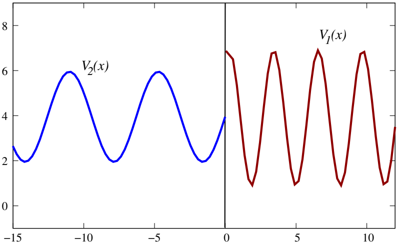

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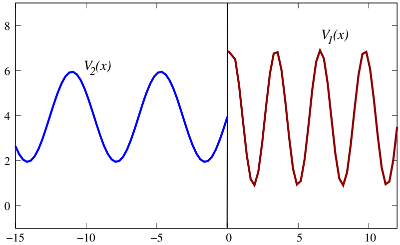

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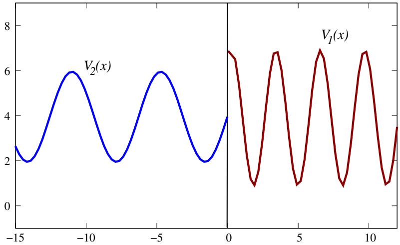

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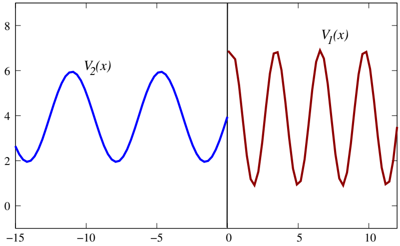

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Question:

- Is c attained, i.e., are there ground states for the interface problem?

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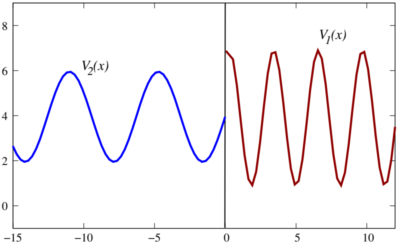

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T. Dohnal and D. Pelinovsky: *Surface gap solitons at a nonlinearity interface.* SIAM J. Appl. Dyn. Syst. **7** (2008), 249–264.

Assumptions and observations

Spectrum of L:

$$\sigma(L) = \sigma(L_1) \cup \sigma(L_2) \cup \{\text{ possibly eigenvalues}\}$$

Assume in the following: $0 < \inf \sigma(L)$

Energy-levels:

$$c \leq \min\{c_1, c_2\}$$

There are sequences $\{u_k\}_{k\in\mathbb{N}}$ in $\mathcal{M}$ with

- $J[u_k] \to c_1$ (likewise for c_2)
- $J'[u_k] \to 0$
- $u_k \rightharpoonup 0$ as $k \to \infty$

Non-existence:

- if $V_1 \leq V_2$, $\Gamma_1 \geq \Gamma_2$ (strict somewhere)
- minimizing sequence wants to look like $w_1(x - te_1)$, t large
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Abstract result

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Theorem (Dohnal, Plum, R., 2010)

Assume $0 < \inf \sigma(L)$ *and* $1 < p < \frac{n+2}{(n-2)_+}$. *If* $c < \min\{c_1, c_2\}$ *then* c *is attained, i.e., there exists a ground state for the interface problem*

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If $\liminf \|u_k\|_{H^1(S_\delta)} = 0$, $S_\delta = [-\delta, \delta] \times \mathbb{R}^{n-1} \Rightarrow c \geq \min\{c_1, c_2\}$. Impossible!

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Reason:

$$u_k = \underbrace{\chi_{[\delta,\infty)\times\mathbb{R}^{n-1}} u_k}_{=:v_k} + \underbrace{\chi_{(-\infty,-\delta]\times\mathbb{R}^{n-1}} u_k}_{=:w_k} + \underbrace{\text{rest}}_{\to 0}$$

Hence

$$J[u_k] = J_1[v_k] + J_2[w_k] + o(1) \ge \min\{c_1, c_2\} + o(1)$$

Consequence: a practical criterion

Find one function $u \in \mathcal{M}$ such that

$$J[u] < \min\{c_1, c_2\}$$

Candidate: If $c_1 \leq c_2$ take

$$u_t(x) := w_1(x - te_1), \quad t \to \infty$$

Scale: Choose $s > 0$ such that $su_t \in \mathcal{M}$ ($\mathcal{M}$ is a top. sphere)

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Compute:

$$\begin{aligned} J[su_t] &= c_1 \frac{c_1 + \frac{1}{2}\int_{\mathbb{R}^n_-}(V_2 - V_1)u_t^2\,dx(1+o(1))}{c_1 + \frac{1}{p+1}\int_{\mathbb{R}^n_-}(\Gamma_2 - \Gamma_1)|u_t|^{p+1}\,dx(1+o(1))} \\ &< c_1 \text{ for large } t >> 1 \text{ provided} \end{aligned}$$

$$(p+1)\int_{\mathbb{R}^n_-}(V_2 - V_1)u_t^2\,dx < 2\int_{\mathbb{R}^n_-}(\Gamma_2 - \Gamma_1)|u_t|^{p+1}\,dx$$

Strategy

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How to verify the criterion (assuming $c_1 \leq c_2$):

$$(p+1)\int_{\mathbb{R}^n_-}(V_2 - V_1)u_t^2\,dx < 2\int_{\mathbb{R}^n_-}(\Gamma_2 - \Gamma_1)|u_t|^{p+1}\,dx \quad ?$$

- Make sure $c_1 \leq c_2$
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- Use the candidate $u_t(x) = w_1(x - te_1) = O(e^{-\sqrt{|\lambda|}|x - te_1|})$

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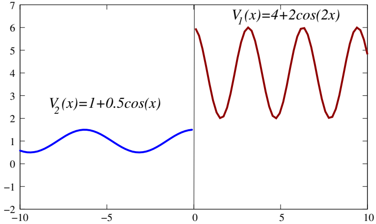
$$\begin{aligned} V_1(x) &= k^2 V_2(x), \quad k \in \mathbb{N}, \\ \Gamma_1(x) &= \gamma \Gamma_2(kx), \quad \gamma > 0 \end{aligned}$$



$$\text{ground states: } w_1(x) = \left(\frac{k}{\gamma}\right)^{\frac{2}{p-1}} w_2(kx)$$

$$\text{energies: } c_1 = \left(\frac{k}{\gamma}\right)^{\frac{4}{p-1}} k^{2-n} c_2$$

Verify criterion:

- $c_1 \le c_2$: take γ so large that $\gamma^{\frac{4}{p-1}} > k^{\frac{n-2-p(n-2)}{p-1}}$
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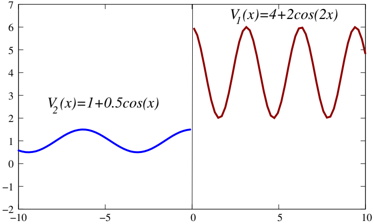

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Example 2 (large contrast in Γ_1/Γ_2)

Let $V_2 < V_1$. How to ensure $c_1 \leq c_2$?

Assume: $\Gamma_1(x) \geq \beta_0 > 0, \quad \beta_0$ large

$$\begin{aligned} c_1 &= c(V_1, \Gamma_1) \leq c(V_1, \beta_0) \qquad \text{(monotonicity)} \\ &= \beta_0^{\frac{-2}{p-1}} c(V_1, 1) \\ &\leq c(V_2, \Gamma_2) = c_2 \qquad \text{(provided } \beta_0 \text{ large)} \end{aligned}$$

Theorem (Dohnal, Plum, R., 2010)

Assume $0 < \inf \sigma(L)$ and let $V_1(x) > V_2(x)$. There is a value $\beta_0 > 0$ such that if $\Gamma_1(x) \geq \beta_0$ then there exists a ground state for the interface problem.

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Example 3 – Dislocation $(n = 1)$

$$(-\frac{d^2}{dx^2} + V(x) - \lambda)u = \Gamma(x)|u|^{p-1}u \text{ on } \mathbb{R}$$

$$V(x) = \begin{cases} V_0(x+\tau), & x_1 > 0, \\ V_0(x-\tau), & x_1 < 0, \end{cases} \qquad \Gamma(x) = \begin{cases} \Gamma_0(x+\tau), & x_1 > 0, \\ \Gamma_0(x-\tau), & x_1 < 0 \end{cases}$$

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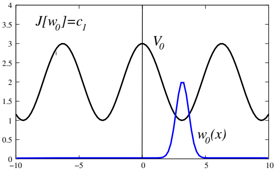

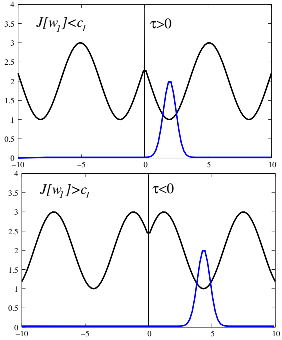

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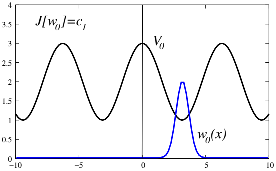

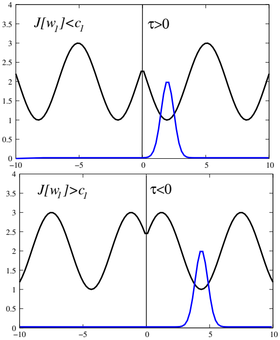

Heuristically:

$$V'(0) = 0, V''(0) < 0, \tau > 0$$

gives energetic advantage!

Example 3 – Dislocation $(n = 1)$

Check criterion for large t:

$$(p+1)\underbrace{\int_{-\infty}^{0} \delta V(x) w_0^2(x+\tau-t)\,dx}_{=:I_1} < 2\underbrace{\int_{-\infty}^{0} \delta\Gamma(x)|w_0(x+\tau-t)|^{p+1}\,dx}_{=:I_2}$$

Use asymptotics:

$$\frac{w_0(x)}{p_-(x)e^{\kappa x}} \overset{x\to-\infty}{\longrightarrow} \text{const.}, \quad p_\pm e^{\mp\kappa x} = \text{ Bloch mode for } -\frac{d^2}{dx^2} + V(x) - \lambda$$

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Example 3 – Dislocation $(n = 1)$

Check criterion for large t:

$$(p+1)\underbrace{\int_{-\infty}^{0} \delta V(x) w_0^2(x+\tau-t)\,dx}_{=:I_1} < 2\underbrace{\int_{-\infty}^{0} \delta\Gamma(x)|w_0(x+\tau-t)|^{p+1}\,dx}_{=:I_2}$$

Use asymptotics:

$$\frac{w_0(x)}{p_-(x)e^{\kappa x}} \overset{x\to-\infty}{\longrightarrow} \text{const.}, \quad p_\pm e^{\mp\kappa x} = \text{ Bloch mode for } -\frac{d^2}{dx^2} + V(x) - \lambda$$

$$I_1 = \text{const.}^2(1+o(1))e^{2\kappa(\tau-t)} \int_{-1}^{0} \delta V(x) p_-(x+\tau)^2 e^{2\kappa x}\,dx$$

$$I_2 = \text{const.}^{p+1}(1+o(1))e^{(p+1)\kappa(\tau-t)} \int_{-1}^{0} \delta\Gamma(x) p_-(x+\tau)^{p+1} e^{(p+1)\kappa x}\,dx$$

Example 3 – Result

$$\int_{-1}^{0}\Big(V_0(x-\tau)-V_0(x+\tau)\Big)p_-^2(x+\tau)e^{2\kappa x}\,dx<0 \Rightarrow \exists \text{ ground state}$$

This can be used as follows:

- check numerically
- let spectral parameter $\lambda \to -\infty$

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(same if $\int_0^1 (V_0(x+\tau)-V_0(x-\tau))p_+^2(x-\tau)e^{-2\kappa x}\,dx<0$)

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- check numerically
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Theorem (Dohnal, Plum, R., 2010)

For $\lambda << 0$ ground states for the dislocation problem exist if:

$$V_0(-\tau) \neq V_0(\tau) \quad \text{or} \quad V_0(-\tau) = V_0(\tau) \text{ and } V_0'(-\tau) > V_0'(\tau). \qquad (1)$$

For $|\tau|$ sufficiently small the above condition (1) holds if

$$V_0'(0) \neq 0 \quad \text{or} \quad V_0'(0) = 0 \text{ and } \operatorname{sign}\tau\, V_0''(0) < 0. \qquad (2)$$