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Analytical and computer assisted methods
in mathematical models

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Basics on the Lebesgue integral and the divergence theorem

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Ref.: W. Rudin, Real and Complex Analysis, 3rd Ed., McGraw-Hill, 1987, Chapter 1–3.

Let X be a set, e.g. $X = \mathbb{R}^n$, and let $\mathcal{P}(X)$ be the set of all subsets of X .

Definition 1 (σ -algebra) A system of sets $\mathcal{M} \subset \mathcal{P}(X)$ is called a σ -algebra over X if

$$(i) X \in \mathcal{M}$$

$$(ii) A \in \mathcal{M} \implies X \setminus A \in \mathcal{M}$$

$$(iii) A_i \in \mathcal{M} \quad \forall i \in \mathbb{N} \implies \bigcup_{i=1}^{\infty} A_i \in \mathcal{M}$$

Definition 2 (positive measure) Let \mathcal{M} be σ -algebra over X . A map $\mu : \mathcal{M} \rightarrow [0, \infty]$ is called a positive measure, if

$$A_i \in \mathcal{M} \quad \forall i \in \mathbb{N} \text{ with } A_i \cap A_j = \emptyset \text{ for } i \neq j \implies \mu \left(\bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} \mu(A_i).$$

Let $I = I^1 \times \dots \times I^n = (a_1, b_1) \times \dots \times (a_n, b_n)$ be an open interval in \mathbb{R}^n . Then the *volume* of I is defined by $|I| = (b_1 - a_1) \cdot \dots \cdot (b_n - a_n)$. The same applies if one or more of the component-intervals (a_i, b_i) are replaced by closed, semi-closed, semi-open intervals.

Definition 3 (outer measure on \mathbb{R}^n) Let $A \subset \mathbb{R}^n$ be an arbitrary set. Then

$$\lambda(A) := \inf \left\{ \sum_{i=1}^{\infty} |I_i| : A \subset \bigcup_{i=1}^{\infty} I_i \text{ und } I_i \text{ bounded interval } \forall i \in \mathbb{N} \right\}$$

is called the outer measure of the set A .

Remark: λ is not a positive measure on $\mathcal{P}(\mathbb{R}^n)$.

Definition 4 (Lebesgue σ -algebra, Caratheodory) A set $A \subset \mathbb{R}^n$ is called Lebesgue-measurable (short: $A \in \mathcal{L}(\mathbb{R}^n)$) if

$$\lambda(E) = \lambda(A \cap E) + \lambda(A^c \cap E) \quad \forall E \subset \mathbb{R}^n.$$

If $X \subset \mathbb{R}^n$ is Lebesgue-measurable, then let $\mathcal{L}(X) = \{A \subset X : A \text{ is Lebesgue-measurable}\}$.

Theorem 5 $\mathcal{L}(\mathbb{R}^n)$ is a σ -algebra. The outer measure λ (see Definition 3) is invariant under Euclidean motions and if it is restricted to $\mathcal{L}(\mathbb{R}^n)$ then it becomes a positive, complete measure on $\mathcal{L}(\mathbb{R}^n)$.

In the following let $X \subset \mathbb{R}^n$ be a Lebesgue-measurable set. For $f : X \rightarrow \overline{\mathbb{R}} = [-\infty, \infty]$ let $f^+ = \max\{f, 0\}$, $f^- = -\min\{f, 0\}$. Hence $f = f^+ - f^-$.

Definition 6 (mesasurable functions)

- (i) A function $f : X \rightarrow \overline{\mathbb{R}}$ is called measurable, if $f^{-1}((\alpha, \infty]) \in \mathcal{L}(X)$ for all $\alpha \in \mathbb{R}$,
- (ii) A function $s : X \rightarrow \mathbb{R}$ is called an elementary function, if s possesses only finitely many values $\alpha_1, \dots, \alpha_k$. In this case

$$s = \sum_{i=1}^k \alpha_i \chi_{A_i}, \quad A_i = s^{-1}(\alpha_i).$$

Definition 7 (Lebesgue-integral for non-negative functions)

- (i) Let $s = \sum_{i=1}^k \alpha_i \chi_{A_i}$ be a measurable elementary function. Then

$$\int_X s \, dx := \sum_{i=1}^k \alpha_i \lambda(A_i)$$

is called the Lebesgue-integral of s over X .

- (ii) Let $f : X \rightarrow [0, \infty]$ be measurable. Then

$$\int_X f \, dx := \sup_{s \in \mathcal{S}} \int_X s \, dx, \quad \mathcal{S} = \{s : X \rightarrow \mathbb{R} \text{ measurable elementary function}, 0 \leq s \leq f\}$$

is called the Lebesgue-integral of f over the set X .

Definition 8 (Lebesgue-integral for real- or complex-valued functions) Let $\mathbb{K} = \overline{\mathbb{R}}$ or \mathbb{C} .

$$L^1(X) := \{f : X \rightarrow \mathbb{K} \text{ measurable} : \int_X |f| \, dx < \infty\}.$$

For $f \in L^1(X)$ let $f_1 = \Re f, f_2 = \Im f$. Then

$$\int_X f \, dx := \int_X f_1^+ \, dx - \int_X f_1^- \, dx + i \left(\int_X f_2^+ \, dx - \int_X f_2^- \, dx \right)$$

is called the Lebesgue-integral of f over the set X .

Definition 9 ($f = g$ a.e.) Let $f, g : X \rightarrow \mathbb{K}$ be measurable. Then we say $f = g$ almost everywhere, if there exists a set N of measure 0 such that $f(x) = g(x) \forall x \in X \setminus N$. Equality almost everywhere is an equivalence relation.

Definition 10 (essential supremum) Let $F : X \rightarrow \overline{\mathbb{R}}$ be measurable. Then

$$\text{ess sup } F := \inf \{s \in \overline{\mathbb{R}} : F(x) \leq s \text{ a.e. in } X\}.$$

is called the essential supremum of F .

Definition 11 (The space $L^p(X)$)

(a) For $1 \leq p < \infty$ let

$$L^p(X) = \{u : X \rightarrow \overline{\mathbb{R}} \text{ measurable: } \int_X |u|^p dx < \infty\}.$$

(b) For $p = \infty$ let

$$L^\infty(X) = \{u : X \rightarrow \overline{\mathbb{R}} \text{ measurable: } \text{ess sup}_X |u| < \infty\},$$

where $\text{ess sup}_X v = \inf\{s \in \overline{\mathbb{R}} : v(x) \leq s \text{ for almost all } x \in X\}$.

Definition 12 (Norm on $L^p(X)$) For $1 \leq p < \infty$ let

$$\|u\|_p := \left(\int_X |u|^p dx \right)^{1/p}$$

and

$$\|u\|_\infty := \text{ess sup}_X |u|.$$

Then $(L^p(X), \|\cdot\|_p)$ is a Banachspace. $L^2(X)$ is a Hilbertspace with inner product $\langle f, g \rangle := \int_X fg dx$.

Theorem 13 (Minkowski and Hölder inequalities)

(i) $\|u + v\|_p \leq \|u\|_p + \|v\|_p$ for all $u, v \in L^p(X)$.

(ii) Let $1 \leq p, q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\int_X |uv| dx \leq \|u\|_p \|v\|_q$$

for all $u \in L^p(X)$ and all $v \in L^q(X)$.

Theorem 14 (Dual space of $L^p(X)$) Let $1 \leq p < \infty$ and let $\phi : L^p(X) \rightarrow \mathbb{R}$ be a continuous linear functional. Then there exists a unique $v \in L^q(X)$, $\frac{1}{p} + \frac{1}{q} = 1$, such that

$$\phi(u) = \int_X uv dx \text{ for all } u \in L^p(X).$$

For short: $(L^p(X))^* = L^q(X)$.

Note: In general the theorem fails for $p = \infty$, i.e., $(L^\infty(X))^* \supsetneq L^1(X)$.

Theorem 15 Let $1 \leq p \leq \infty$ and $u \in L^p(X)$. If $(u_k)_{k \in \mathbb{N}}$ is a sequence of functions in $L^p(X)$ such that $\lim_{k \rightarrow \infty} \|u_k - u\|_p = 0$ then there exists a subsequence $(u_{k_l})_{l \in \mathbb{N}}$ such that

$$\lim_{l \rightarrow \infty} u_{k_l}(x) = u(x) \text{ for almost all } x \in X.$$

Theorem 16 Let $1 \leq p < \infty$. The set $C_c(X)$ of continuous functions with compact support in X are dense in $L^p(X)$. Here the support of a function is defined as $\text{supp } f = \overline{\{x \in X : f(x) \neq 0\}}$.

Definition 17 (Locally integrable functions) Let $1 \leq p \leq \infty$. Define

$$L_{loc}^p(X) = \{u : X \rightarrow \overline{\mathbb{R}} \text{ measurable s.t. } u \in L^p(K) \text{ for every compact set } K \subset X\}.$$

$L_{loc}^p(X)$ is a vector space.

Theorem 18 (Monotone convergence) Let $(u_k)_{k \in \mathbb{N}}$ be a sequence of measurable functions on X such that

$$0 \leq u_1 \leq u_2 \leq u_3 \leq \dots$$

Then $u(x) := \lim_{k \rightarrow \infty} u_k(x)$ exists for almost all $x \in X$ and

$$\lim_{k \rightarrow \infty} \int_X u_k dx = \int_X u dx.$$

Theorem 19 (Dominated convergence) Let $(u_k)_{k \in \mathbb{N}}$ be a sequence of measurable functions on X . If there exists $w \in L^1(X)$ such that $|u_k(x)| \leq w(x)$ for almost all $x \in X$ and all $k \in \mathbb{N}$ and if $u(x) := \lim_{k \rightarrow \infty} u_k(x)$ exists almost everywhere in X then

$$\lim_{k \rightarrow \infty} \int_X u_k dx = \int_X u dx.$$

Theorem 20 (Fatou's Lemma) Let $(u_k)_{k \in \mathbb{N}}$ be a sequence of measurable functions on X such that $u_k(x) \geq 0$ almost everywhere on X . Then

$$\int_X \liminf_{k \in \mathbb{N}} u_k dx \leq \liminf_{k \in \mathbb{N}} \int_X u_k dx.$$

Definition 21 An open, connected set $\Omega \subset \mathbb{R}^n$ is called a C^1 -domain (a Lipschitz-domain) if for every $x_0 \in \partial\Omega$ there exists a radius $r > 0$ and a C^1 -function (Lipschitz-function) $\gamma : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ such that (up to rotation of the coordinate frame)

$$\Omega \cap B_r(x_0) = \{x = (x', x_n) \in B_r(x_0) : x_n > \gamma(x')\}.$$

Note that as a consequence

$$\partial\Omega \cap B_r(x_0) = \{x = (x', x_n) \in B_r(x_0) : x_n = \gamma(x')\}.$$

For $x = (x', \gamma(x')) \in \partial\Omega \cap B_r(x_0)$ the vector

$$\nu(x) = \frac{(\nabla\gamma(x'), -1)}{\sqrt{1 + |\nabla\gamma(x')|^2}}$$

is called the exterior unit-normal of $\partial\Omega$ at x . In case of a Lipschitz-domain the exterior unit-normal ν exists a.e. with respect to the surface-measure on $\partial\Omega$.

Theorem 22 (Divergence theorem, Gauss theorem) *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^1 -domain and let ν be the exterior unit-normal on $\partial\Omega$. Then*

$$\int_{\Omega} \frac{\partial f}{\partial x_i} dx = \oint_{\partial\Omega} f \nu_i d\sigma$$

for every function $f \in C^1(\bar{\Omega})$. The result also holds if Ω is a bounded Lipschitz-domain.

Often the divergence theorem appears in the following form

$$\int_{\Omega} \operatorname{div} F dx = \oint_{\partial\Omega} F \cdot \nu d\sigma$$

for a vectorfield $F : \bar{\Omega} \rightarrow \mathbb{R}^n$. The components of $F = (F_1, \dots, F_n)$ have to satisfy $F_i \in C^1(\bar{\Omega})$ for $i = 1, \dots, n$.

Lemma 23 (Green's identities) *Let $u, v \in C^2(\bar{\Omega})$. Then*

$$\begin{aligned} \int_{\Omega} \Delta u dx &= \oint_{\partial\Omega} \nabla u \cdot \nu d\sigma, \\ \int_{\Omega} \nabla u \cdot \nabla v dx &= - \int_{\Omega} u \Delta v dx + \oint_{\partial\Omega} u \nabla v \cdot \nu d\sigma, \\ \int_{\Omega} (u \Delta v - v \Delta u) dx &= \oint_{\partial\Omega} (u \nabla v - v \nabla u) \cdot \nu d\sigma. \end{aligned}$$