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Notes on Bifurcation Theory – Version of January 28, 2020

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Notation:

- For functions $F(u)$ we use the notation F_u to denote the derivative ($n \times n$ Jacobi matrix) if $F : \mathbb{R}^n \to \mathbb{R}^n$).
- For functions $F(\lambda, u)$ with $F : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$ we use the notation $F_u, F_\lambda, F_{uu}, F_{\lambda u}, F_{\lambda\lambda}$ etc. to denote (partial) derivatives of first, second order (and analogously of higher order).

1. Solving nonlinear equations

Let us consider a nonlinear equation

$$F(u) = 0 \text{ with } F : \mathbb{R}^n \to \mathbb{R}^n.$$

Suppose u_0 is an approximate solution, i.e., $F(u_0) \approx 0$. Let us find a better approximation $u_1 = u_0 + \delta_u$. Use the Taylor expansion

$$0 \overset{!}{=} F(u_1) = F(u_0) + F_u(u_0) \underbrace{\delta_u}_{u_1 - u_0} + O(\delta_u^2).$$

Dropping the $O(\delta_u^2)$-term and solving for u_1 leads to

$$u_1 = u_0 - (F_u(u_0))^{-1} F(u_0).$$

One can iterate this process and this leads to Newton's scheme:

$$u_{n+1} = u_n - (F_u(u_n))^{-1} F(u_n), \qquad n \in \mathbb{N}_0.$$

Newton's scheme converges, provided

$$\boxed{u_0 \text{ is a "good" first approximation and the matrix } L := F_u(u_0) \text{ is invertible}}$$

2. Solving nonlinear equations with parameters

Now we consider the nonlinear equation

$$F(\lambda, u) = 0 \text{ with } F : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$$

so that λ is a k-dimensional parameter. Suppose the pair (λ_0, u_0) is a known solution, and suppose $\lambda_1 = \lambda_0 + \delta_\lambda$ is given. Let us find the corresponding nearby solution $u_1 = u_0 + \delta_u$.

Again, we use the Taylor expansion – this time in two variables.

$$0 \overset{!}{=} F(\lambda_1, u_1) = F(\lambda_0, u_0) + \underbrace{F_u(\lambda_0, u_0)}_{=:L} \underbrace{\delta_u}_{u_1 - u_0} + F_\lambda(\lambda_0, u_0)\delta_\lambda + \underbrace{\text{h.o.t.}}_{\text{higher order terms}}.$$

Dropping the higher order terms this leads to

$$u_1 = u_0 - L^{-1} F_\lambda(\lambda_0, u_0).$$

The iteration of this process (as before in Newton's scheme) will converge provided the $n \times n$-matrix $L = F_u(\lambda_0, u_0) \in \mathbb{R}^{n \times n}$ is invertible. It leads for λ near λ_0 to a differentiable map

$$\boxed{\lambda \mapsto u(\lambda) \text{ such that } F(\lambda, u(\lambda)) = 0}$$

Moroever, the pair $(\lambda, u(\lambda))$ is the **only** solution to the nonlinear problem in a (small) neighbourhood of (λ_0, u_0).

This results is called: the implicit function theorem. It generalizes to functions $F : \mathbb{R}^k \times X \to Y$ provided X, Y are (normed) vector spaces and the operator $L = F_u(\lambda_0, u_0) : X \to Y$ is invertible, i.e., $L^{-1} : Y \to X$ exists (and is bounded).

3. Solving bifurcation problems

Reference: M. G. Crandall and P. H. Rabinowitz, J. Functional Analysis 8, 321 (1971).

Now we consider again the nonlinear equation with a parameter

$$F(\lambda, u) = 0 \text{ with } F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n.$$

More generally we can consider $F : \mathbb{R} \times X \to Y$ for (normed) vectorspaces X, Y. Now we assume that a **trivial solution** exists, i.e.,

$$F(\lambda, 0) = 0 \text{ for all } \lambda.$$

The task is to identity conditions such that the point $(\lambda_0, 0)$ is a **bifurcation point**. Again we denote by

$$L := F_u(\lambda_0, 0) : \mathbb{R}^n \to \mathbb{R}^n \quad \text{or in the more general context } L := F_u(\lambda_0, 0) : X \to Y.$$

We distinguish several cases:

<u>1^{st} case:</u> L is invertible.
Then we can apply the result (implicit function theorem) from Section 2 and get that $u \equiv 0$ is the only small solution for $\lambda \approx \lambda_0$ and the point (λ_0, u_0) is not a bifurcation point.

<u>2^{nd} case:</u> Kernel $L = [\phi]$ is one-dimensional.
This is the interesting case, for which we will show that it leads to **bifurcation** if we additionally suppose

$$\boxed{\text{transversality: } F_{u\lambda}(\lambda_0, 0) \notin \text{Range}\, L}$$

<u>3^{rd} case:</u> Kernel L has a dimension at least 2.
This case will not be considered here.

We start the analysis of the 2^{nd} case. Using orthogonal complements we split the spaces as follows:

$$\begin{array}{llllll} X & = \mathbb{R}^n & = [\phi] & \oplus & [\phi]^\perp \\ X & \ni u & = t\phi & + & v \end{array}$$

We also split the target space

$$Y = \mathbb{R}^n = \underbrace{[\phi^*]}_{=\text{Kernel}\, L^*} \oplus \underbrace{[\phi^*]^\perp}_{=\text{Range}\, L}$$

where L^* is the adjoint operator which satisfies $\langle Lu, w\rangle = \langle u, L^* w\rangle$. For a $n \times n$ matrix L the adjoint matrix L^* is given by $L^* = \overline{L}^T$ (complex conjugate and transpose).

Next we define orthogonal projections:

$$P : Y = \mathbb{R}^n \to \text{Range}\, L, \quad Q : Y = \mathbb{R}^n \to \text{Kernel}\, L^*.$$

Let us consider an **example**. Suppose

$$\phi = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \qquad \phi^* = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Then

$$L(t\phi + v) = \begin{pmatrix} 0 & \dots\dots\dots & 0 \\ & \text{invertible} & \vdots \\ & \text{part of } L & \vdots \\ & & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_{n-1} \\ t \end{pmatrix}$$

which means that the $(n-1) \times (n-1)$ submatrix of L with lines $2 \dots n$ and columns $1 \dots n-1$ is invertible.

Now let us come back to the original equation, and use the scalar product $\langle \cdot, \phi^* \rangle$ to denote the projection onto $[\phi^*]$. We have the following equivalence

$$F(\lambda, u) = 0 \Leftrightarrow \begin{cases} G(\lambda, t, v) := PF(\lambda, t\phi + v) = 0 & (1) \\ QF(\lambda, t\phi + v) = \langle F(\lambda, t\phi + v), \phi^* \rangle \phi^* = 0. & (2) \end{cases}$$

This means that the original problem is solved provided (1) and (2) are solved.

<u>Solve (1):</u> Note that $G(\lambda_0, 0, 0,) = 0$. Moroever

$$G_v(\lambda_0, 0, 0) = PF_u(\lambda_0, 0)|_{[\phi]^\perp} = \text{ invertible part of } L.$$

Therefore, Section 2 applies (implicit function theorem) and we get a differentiable map

$$(\lambda, t) \mapsto v(\lambda, t) \text{ for } \lambda \approx \lambda_0, t \approx 0$$

such that

$$v(\lambda, 0) = 0 \text{ and } G(\lambda, t, v(\lambda, t)) = 0.$$

Solve (2): To solve (2) means that

$$f(\lambda,t) := \langle F(\lambda, t\phi + v(\lambda,t)), \phi^* \rangle = 0.$$

To see how to solve this equation, let us write down the Taylor expansion of the function f:

$$\begin{aligned} f(\lambda,t) =& f(\lambda_0,0) + f_\lambda(\lambda_0,0)(\lambda-\lambda_0) + f_t(\lambda_0,0)t \\ &+ \frac{1}{2} f_{\lambda\lambda}(\lambda_0,0)(\lambda-\lambda_0)^2 + f_{\lambda t}(\lambda_0,0)(\lambda-\lambda_0)t + \frac{1}{2} f_{tt}(\lambda_0,0)t^2 \\ &+ \frac{1}{6} f_{\lambda\lambda\lambda}(\lambda_0,0)(\lambda-\lambda_0)^2 + O\big((\lambda-\lambda_0)^2 t + (\lambda-\lambda_0)t^2 + t^3\big). \end{aligned}$$

Many of the terms in the Taylor expansion vanish.

- $f(\lambda, 0) = 0$. Therfore all pure λ-derivatives of f vanish.
- $f_t(\lambda_0, 0) = 0$. To see this first compute

(3)
$$f_t(\lambda,t) = \langle F_u\big(\lambda, t\phi + v(\lambda,t)\big)(\phi + v_t(\lambda,t)), \phi^* \rangle.$$

  In particular

$$f_t(\lambda_0,0) = \langle L(\phi + v_t(\lambda_0,0)), \phi^* \rangle = \langle L v_t(\lambda_0,0), \phi^* \rangle = \langle v_t(\lambda_0,0), \underbrace{L^*\phi^*}_{=0} \rangle = 0$$

- $v_t(\lambda_0, 0) = 0$. To see this, differentiate (1) with respect to t at $t = 0$ to get

$$0 = PL(\phi + v_t(\lambda_0,0)) = PLv_t(\lambda_0,0) = 0.$$

  Since PL applied to an element in $[\phi]^\perp$ is invertible (invertible part of L) this means $v_t(\lambda_0, 0) = 0$.

Now we can go back to the Taylor expansion of f and see what remains:

$$\begin{aligned} f(\lambda,t) =& f_{\lambda t}(\lambda_0,0)(\lambda-\lambda_0)t + \frac{1}{2} f_{tt}(\lambda_0,0)t^2 \\ &+ O\big((\lambda-\lambda_0)^2 t + (\lambda-\lambda_0)t^2 + t^3\big). \end{aligned}$$

Since we want to solve (2), i.e., $f(\lambda, t) = 0$, let us divide by t. Then we get

$$\begin{aligned} 0 \overset{!}{=} \frac{1}{t} f(\lambda,t) =& f_{\lambda t}(\lambda_0,0)(\lambda-\lambda_0) + \frac{1}{2} f_{tt}(\lambda_0,0)t \\ &+ O\big((\lambda-\lambda_0)^2 + (\lambda-\lambda_0)t + t^2\big). \end{aligned}$$

From this we can get λ as a function of t provided the term $f_{\lambda t}(\lambda_0, 0)$ does not vanish, since then

$$\lambda(t) = \lambda_0 - \frac{1}{2}\frac{f_{tt}(\lambda_0,0)t}{f_{\lambda t}(\lambda_0,0)} + O(t^2)$$

and we are done.

It remains to check $\underline{f_{\lambda t}(\lambda_0, 0) \neq 0}$. For this we compute from (3)

$$\begin{aligned} f_{\lambda t}(\lambda_0,0) &= \langle F_{uu}(\lambda_0,0)(\phi+v_t(\lambda_0,0))\underbrace{v_\lambda(\lambda_0,0)}_{=0} + F_{\lambda u}(\lambda_0,0)(\phi+\underbrace{v_t(\lambda_0,0)}_{=0}) + Lv_{\lambda t}(\lambda_0,0), \phi^*\rangle \\ &= \langle F_{\lambda u}(\lambda_0,0)\phi, \phi^*\rangle + \langle Lv_{\lambda t}(\lambda_0,0), \phi^*\rangle \\ &= \langle F_{\lambda u}(\lambda_0,0)\phi, \phi^*\rangle + \langle v_{\lambda t}(\lambda_0,0), \underbrace{L^*\phi^*}_{=0}\rangle \\ &= \langle F_{\lambda u}(\lambda_0,0)\phi, \phi^*\rangle \neq 0 \end{aligned}$$

since (by the transversality assumption) $F_{\lambda u}(\lambda_0, 0)\phi \notin \operatorname{Range} L = [\phi^*]^\perp$.

What did we achieve altogether in the discussion of the 2^{nd} case? The equation

$$F(\lambda, u) = 0$$

is solved in a non-trivial way by

$$u = t\phi + v(\lambda(t), t) \text{ for } t \approx 0$$

4. Application to the Lugiato-Lefever equation

Let us consider the Lugiato-Lefever equation with Neumann boundary condition on $[0, \pi]$

(NewLLE) $$-da'' - (\mathrm{i} - \zeta)a - |a|^2 a + \mathrm{i}f = 0, \qquad a'(0) = a'(\pi) = 0$$

Let us define the function $\tilde{F}$

$$\tilde{F}(\zeta, a) := -da'' - (\mathrm{i} - \zeta)a - |a|^2 a + \mathrm{i}f$$

for functions twice (almost everywhere) differentiable functions a which satisfy $a'(0) = a'(\pi) = 0$ and where $\int_0^\pi |a''(x)|^2\, dx < \infty$. Then we set

$$F(\zeta, u) := \tilde{F}(\zeta, a_0(\zeta) + u).$$

Now $F(\zeta, 0) = 0$ and Section 3 applies. We get that

$$L = F_u(\zeta_0, 0) = \tilde{F}_a(\zeta_0, a_0(\zeta_0))$$

is the linearized operator (see slides of January 27), i.e.,

$$L\phi = -d\phi'' - (\mathrm{i} - \zeta_0)\phi - 2|a_0(\zeta_0)|^2\phi - a_0(\zeta_0)^2\bar{\phi}.$$

The transversality condition is now expressed as

$$\boxed{F_{\zeta u}(\zeta_0, 0)\phi = \phi - 2\dot{a}_0(\zeta_0)\overline{a_0(\zeta_0)}\phi - 2a_0(\zeta_0)\overline{\dot{a}_0(\zeta_0)}\phi - 2a_0(\zeta_0)\dot{a}_0(\zeta_0)\bar{\phi} \notin \operatorname{Range} L}$$

which is the condition given in the lecture of January 27.