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Time-periodic solutions of semilinear wave equations

Andreas Hirsch, Simon Kohler and Wolfgang Reichel

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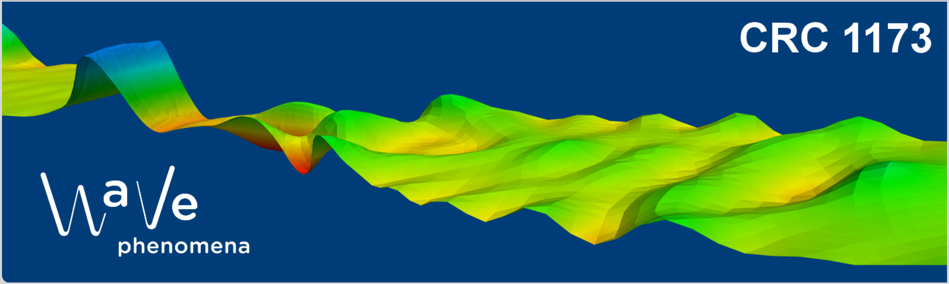
Institute for Analysis



Semilinear wave equations

Find time-periodic solutions $U : \mathbb{R}^{N+1} \times \mathbb{R} \to \mathbb{R}$ solving

$$(*) \qquad w(x)U_{tt} - \Delta_{N+1} U = \Gamma(x)|U|^{p-1}U$$

traveling wave $U(x, x_{N+1}, t) = u(x, t - c^{-1}x_{N+1})$ with $x = (x_1, \dots, x_N)$

$$(**) \qquad \underbrace{\left(w(x) - c^{-2}\right)}_{=:V(x)} u_{tt} - \Delta_N u = \Gamma(x)|u|^{p-1}u$$

Properties of the profile u

- time-periodicity: $u(x, t + T) = u(x, t)$
- localization: $\lim_{|x|\to\infty} u(x, t) = 0$

under suitable conditions on $V, \Gamma : \mathbb{R}^N \to \mathbb{R}$ and $1 < p < \infty$.
Typically $N = 1, 2$.

Outline of the talk

(A) motivation

(B) our results

(C) methods

(A) Motivation

1973 Ablowitz, Kaup, Newell, Segur: Sine-Gordon breather

$$u_{tt} - u_{xx} + \sin u = 0$$

$$u(x,t) = 4\arctan\left(\frac{m\sin(\omega t)}{\omega\cosh(mx)}\right), m^2 + \omega^2 = 1$$

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1993 Denzler, 1994 Birnir, McKean, Weinstein: non-persistence

$$u_{tt} - u_{xx} + f(u) = 0, \qquad f(0) = 0, f'(0) = 1$$

has no breather-solution except for $f(u) = \sin u$.

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Why is it so delicate?
Why is it so difficult?

(A) Motivation

Sine-Gordon equation: $Lu + \sin u = 0$ with $L = \partial_t^2 - \partial_x^2$
Try standard approach for the modified equation:

$$(\#) \qquad Lu + u = u^3$$

Ansatz $u(x,t) = \sum_{k\in\mathbb{Z}} u_k(x)\mathrm{e}^{\mathrm{i}k\omega t}, \quad u_k = u_{-k}$

$$Lu(x,t) = \sum_{k\in\mathbb{Z}} (L_k u_k)(x)\mathrm{e}^{\mathrm{i}k\omega t}, \qquad L_k = -\frac{d^2}{dx^2} - k^2\omega^2$$

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Then the nonlinear wave equation $(\#)$ becomes an infinitely coupled elliptic system

$$(\#\#) \qquad (L_k + 1)u_k = (\hat{u} * \hat{u} * \hat{u})_k$$

where $\hat{u} = (u_k)_{k\in\mathbb{Z}}$ and $(\hat{u} * \hat{u} * \hat{u})_k = \sum_{l,m\in\mathbb{Z}} u_{k-l}u_{l-m}u_m$.

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Note the difficulty arising from the spectrum

$$\sigma(L_k + 1) = [-k^2\omega^2 + 1, \infty)$$

(A) Motivation

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Idea: replace $L = \partial_t^2 - \partial_x^2$ by $L = V(x)\partial_t^2 - \partial_x^2$ and consider the modified equation

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in particular if $V(x), q(x)$ are periodic in x

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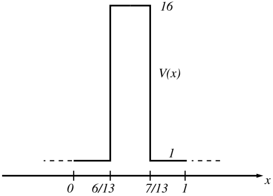
2011 Comm.M.Phys.: Blank, Chirilus-Bruckner, Lescarret, Schneider

$$V(x)u_{tt} - u_{xx} + q(x)u = \pm u^3 \text{ in } \mathbb{R} \times \mathbb{R}$$

V 1-periodic, $q(x) = (q_0 - \epsilon^2)V(x)$, $q_0 \approx 3.7703$



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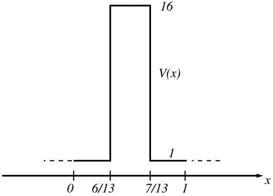

$\exists$ small breathers for $0 < \epsilon < \epsilon_0$

$u(x,t) = O(\epsilon)$

$T = \frac{32}{13} = 2\int_0^1 \sqrt{V(x)}\,dx$

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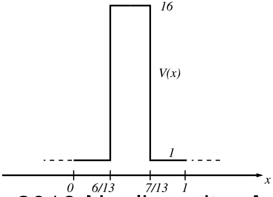

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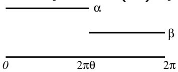
2019 Nonlinearity: A. Hirsch & W.R. Variational proof for large breathers

$$V(x)u_{tt} - u_{xx} = \pm|u|^{p-1}u \text{ on } \mathbb{R} \times \mathbb{R}$$

(1) $V(x) = \alpha + \beta\delta^{per}(x)$, $\beta > 32\alpha$, $\quad 1 < p < p^* = 2$

(2) $V(x) =$ 

, $0 < \theta \ll \frac{1}{2}$, $\frac{\alpha}{\beta} = \frac{(1-\theta)^2}{\theta^2}$ $\quad 1 < p < p^* = 3$

(3) $\exists$ suitable $V \in H^r_{per}(\mathbb{R})$ near $V_0 \equiv 1$, $r \in [1, 3/2)$, $1 < p < p^* = \frac{7-2r}{1+2r}$

(B) Our results

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general understanding so far:

strong spatial structures allow for breathers

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In this talk: replace periodic structure by a waveguide structure

$$(*) \qquad W(x)U_{tt} - \Delta_{N+1} U = \Gamma(x)|U|^{p-1} U$$

where for $x = (x_1, \ldots, x_N)$ we have a cylindrically symmetric coefficient

$$W(x) = \begin{cases} W_0, & |x| < R, \\ W_1, & |x| > R. \end{cases} \qquad 0 < W_1 < W_0.$$

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Traveling wave $U(x, x_{N+1}, t) = u(x, t - c^{-1}x_{N+1})$ leads to

$$(**) \qquad V(x)u_{tt} - \Delta_N u = \Gamma(x)|u|^{p-1}u$$

where

$$V(x) = \begin{cases} V_0 = W_0 - c^{-2} & |x| < R, \\ V_1 = W_1 - c^{-2} & |x| > R. \end{cases}$$

choose c s.t. $0 < W_1 < c^{-2} < W_0$.

(B) Our results

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$$(**) \qquad V(x)u_{tt} - \Delta_N u = \Gamma(x)|u|^{p-1}u$$

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$$V(x) = \begin{cases} V_0 > 0 & |x| < R, \\ V_1 < 0 & |x| > R. \end{cases}$$

write $V(x) = -\alpha + \beta 1_{B_R(0)}$ with $\beta > \alpha > 0$.

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Theorem 1 (S. Kohler & W.R., part of PhD thesis)

For V as above, breathers with frequency $\omega = \frac{\pi}{2R\sqrt{\beta-\alpha}}$ exist in the following three cases:

(1) $n = 1$, $\Gamma > 0$ & $\lim_{|x|\to\infty} \Gamma(x) = 0$, $1 < p < 3$.

(2) $n = 2$, $\Gamma = \Gamma(r)$ *bounded* & $\Gamma > 0$, $1 < p < 2$.

(3) $n = 1$, $\Gamma =$*periodic, bounded* & $\inf \Gamma > 0$, $1 < p < 3$

(B) Comparison with the linear case in $N = 1$

$$V(x)u_{tt} - u_{xx} = 0$$

where

$$V(x) = \begin{cases} \beta - \alpha > 0, & |x| < R, \\ -\alpha < 0, & |x| > R. \end{cases}$$

With $u(x,t) = \mathrm{e}^{\mathrm{i}k\omega t}\phi(x)$ we get

$$L_k\phi = -\phi'' - k^2\omega^2 V(x)\phi = 0 \text{ on } \mathbb{R}$$

$\Rightarrow$ exponential decay to 0 outside, oszillations inside the waveguide.

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In the nonlinear case: $\sqrt{\beta-\alpha}\omega R = \frac{\pi}{2}$
which is designed s.t. $0 \notin \sigma_p(L_k)$. However: $\sigma_{ess}(L_k) = [k^2\omega^2\alpha, \infty)$.

(B) Our results - extensions

$$(**) \qquad V(x)u_{tt} - \Delta_N u = \Gamma(x)|u|^{p-1}u$$

Theorem 1 (S. Kohler & W.R., part of PhD thesis)

For $V = -\alpha + \beta 1_{B_R(0)}$ *with* $\beta > \alpha > 0$, *breathers with* $\omega = \frac{\pi}{2R\sqrt{\beta-\alpha}}$ *exist if*

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Generalizations for (1) and (2): rhs. in $(**) = f(x,t,u)$

(H1) f continuous, $T = \frac{2\pi}{\omega}$-periodic in t, $|f(x,t,s)| \leq c(1+|s|^p)$

(H2) $f(x,t,s) = o(s)$ as $s \to 0$ uniformly in $x, t \in \mathbb{R}$

(H3) $f(x,t,s)$ odd in $s \in \mathbb{R}$, $s \mapsto f(x,t,s)/|s| \nearrow\!\!\nearrow_s$ on $(-\infty, 0)$ and $(0,\infty)$

(H4) $\frac{F(x,t,s)}{s^2} \to \infty$ as $s \to \infty$ unif. in $x, t \in \mathbb{R}$, $F(x,t,s) := \int_0^s f(x,t,\sigma)d\sigma$

Inspired by Szulkin, Weth: The method of Nehari manifold, 2010.

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Generalizations for (3):

(A1) $\Gamma = \Gamma_\infty + \tilde{\Gamma}$

(A2) $\Gamma_\infty =$periodic, bounded & $\inf \Gamma_\infty > 0$

(A3) $\lim_{|x|\to\infty} \tilde{\Gamma} = 0$, $\tilde{\Gamma} \geq -\text{const.}\, e^{-\delta|x|}$ with $\delta > 2\sqrt{\alpha}\omega$.

(C) Methods

$$(**) \qquad V(x)u_{tt} - \Delta_N u = \Gamma(x)|u|^{p-1}u$$

Ansatz

$u(x,t) = (S\hat{u})(x,t) \sum_{k\in 2\mathbb{Z}+1} u_k(x)\mathrm{e}^{\mathrm{i}k\omega t}, \quad u_k = u_{-k}, \quad \hat{u} = (u_k)_{k\in 2\mathbb{Z}+1}.$

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Then the nonlinear wave equation $(**)$ becomes an infinitely coupled elliptic system:

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Note: $k \in 2\mathbb{Z}+1 \Leftrightarrow u = \frac{T}{2}$ antiperiodic.

- avoids $k = 0$ problem with $0 \in \sigma(L_0)$
- compatible with nonlinearity

(C) Methods

$(***)\ \ L_k u_k = (\Gamma(x)|u|^{p-1}u)_k$ with $L_k = -\Delta_N - k^2\omega^2 V(x), \quad k \in 2\mathbb{Z}+1$

Variational formulation:

$$J(\hat{u}) = \sum_{k\in 2\mathbb{Z}+1} \frac{1}{2} b_{L_k}(u_k, u_k) - \frac{1}{p+1}\int_{\mathbb{R}^N}\int_0^T \Gamma(x)|S\hat{u}|^{p+1}\, dt\, dx.$$

with

$$b_{L_k}(u_k, u_k) = \int_{\mathbb{R}^N} |\nabla u_k|^2 - k^2\omega^2 V(x) u_k^2\, dx$$

$\mathrm{dom}(b_{L_k}) = H^1(\mathbb{R}^N)$. But what is $\mathrm{dom}(J) = \mathcal{H}$?

(C) Methods

How to define $\mathrm{dom}(J)$?

$$J(\hat{u}) = \sum_{k \in 2\mathbb{Z}+1} \frac{1}{2} b_{L_k}(u_k, u_k) - \frac{1}{p+1} \int_{\mathbb{R}^N} \int_0^T \Gamma(x) |S\hat{u}|^{p+1} \, dt \, dx.$$

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To get an idea: suppose $L_k = \gamma_k \,\mathrm{Id}$, i.e., γ_k is the eigenvalue of L_k. Then

$$\sum_{k\in 2\mathbb{Z}+1} b_{L_k}(u_k, u_k) = \sum_{k\in 2\mathbb{Z}+1} \gamma_k \|u_k\|_{L^2}^2$$

and

$$\mathcal{H} = \mathrm{dom}(J) = \left\{ \hat{u} = (u_k)_{k\in 2\mathbb{Z}+1} : \sum_{k\in 2\mathbb{Z}+1} |\gamma_k| \|u_k\|_{L^2}^2 < \infty \right\}$$

Analagously, we proceed if L_k is a self-adjoint operator ...

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Use functional calculus:

$$\begin{aligned} L_k &= \int_{\mathbb{R}} \lambda dP_\lambda^k = \int_{\sigma(L_k)} \lambda dP_\lambda^k \\ |L_k| &= \int_{\mathbb{R}} |\lambda| dP_\lambda^k = \int_{\sigma(L_k)} |\lambda| dP_\lambda^k \end{aligned}$$

$$\mathrm{dom}(b_{L_k}) = \mathrm{dom}(b_{|L_k|}) = H^1(\mathbb{R}^N)$$

$$\mathcal{H} = \mathrm{dom}(J) = \Big\{ \hat{u} = (u_k)_{k \in 2\mathbb{Z}+1} : u_k \in H^1(\mathbb{R}^N), \underbrace{\sum_{k \in 2\mathbb{Z}+1} b_{|L_k|}(u_k, u_k)}_{=:|||\hat{u}|||^2} < \infty \Big\}$$

(C) Methods

How to define $\mathrm{dom}(J)$?

$$J(\hat{u}) = \sum_{k\in 2\mathbb{Z}+1} \frac{1}{2} b_{L_k}(u_k, u_k) - \frac{1}{p+1} \int_{\mathbb{R}^N} \int_0^T \Gamma(x) |S\hat{u}|^{p+1} \, dt \, dx.$$

If $0 \notin \sigma(L_k)$ then define projections

$$(\hat{u}^+)_k = \int_{0-}^{\infty} 1 dP_\lambda^k(u_k), \quad (\hat{u}^-)_k = \int_{-\infty}^{0+} 1 dP_\lambda^k(u_k)$$

so that $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ and

$$\sum_{k\in 2\mathbb{Z}+1} b_{L_k}(u_k, u_k) = |||\hat{u}^+|||^2 - |||\hat{u}^-|||^2$$

(C) Methods

How to define $\mathrm{dom}(J)$?

$$J(\hat{u}) = \sum_{k \in 2\mathbb{Z}+1} \frac{1}{2} b_{L_k}(u_k, u_k) - \frac{1}{p+1} \int_{\mathbb{R}^N} \int_0^T \Gamma(x) |S\hat{u}|^{p+1} \, dt \, dx.$$

If $0 \notin \sigma(L_k)$ then define projections

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$\Rightarrow$ we can apply saddle point methods of J (e.g., generalized Nehari manifold method: Szulkin, Weth, The method of Nehari manifold, 2010.)

(C) Methods

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The proof strategy:

(i) choice of ω & ODE analysis $\Rightarrow (-c|k|, c|k|) \in \sigma(L_k)^c = \rho(_k)$ for $|k| > K$.

(ii) for $|k| \le K$: 0 is at most an eigenvalue of finite multiplicity of L_k

(iii) $b_{|L_k|}(u_k, u_k) \ge c|k| \|u_k\|_{L^2}^2$, $b_{|L_k|}(u_k, u_k) \ge c|k|^{-1} \|\nabla u_k\|_{L^2}^2$

(iV) by Riesz-Thorin interpolation & Fourier-transform we get that $S : \mathcal{H} \to L^{r+1}(\mathbb{R}^N \times [0, T])$ locally compact for $1 \le r < p^* = 1 + \frac{2}{N}$

(v) compactness of Palais-Smale sequences achieved either by $\Gamma(x) \to 0$ as $|x| \to \infty$ or by radial symmetry for $N = 2$.

(C) Methods

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(C) Methods

The proof strategy when we don't have compactness of P.S.-sequences:

- dual variational method with $q = 1/p$:

$$Lu = \underbrace{\Gamma(x)|u|^{p-1}u}_{=:v(x)} \quad \Leftrightarrow \quad \Gamma(x)^{-q}|v|^{q-1}v = Kv \text{ with } K = L^{-1}$$

- comparison with problem at infinity

$$L_\infty u = \Gamma(x)|u|^{p-1}u \quad \Leftrightarrow \quad \Gamma(x)^{-q}|v|^{q-1}v = K_\infty v \text{ with } K_\infty = L_\infty^{-1}$$

where

$$L = (-\alpha + \beta 1_{[-r,r]})\partial_t^2 - \partial_x^2, \qquad L_\infty = -\alpha\partial_t^2 - \partial_x^2 \text{ (elliptic)}$$

(C) Methods

The proof strategy when we don't have compactness of P.S.-sequences:

- dual ground state level m, dual ground state level m_∞ at infinity

$$m = \inf_M \int_{\mathbb{R}} \int_0^T \Gamma(x)^{-q} |v|^{q+1} \, dtdx,$$

$$M = \left\{ v \in L^{q+1}(\mathbb{R} \times [0,T]) : \int_{\mathbb{R}} \int_0^T vKv \, dtdx = 1 \right\},$$

$$m_\infty = \inf_{M_\infty} \int_{\mathbb{R}} \int_0^T \Gamma(x)^{-q} |v|^{q+1} \, dtdx,$$

$$M_\infty = \left\{ v \in L^{q+1}(\mathbb{R} \times [0,T]) : \int_{\mathbb{R}} \int_0^T vK_\infty v \, dtdx = 1 \right\}$$

- $m < m_\infty \Rightarrow m$ is attained
- $b_{K_\infty}(v_\infty, v_\infty) < b_K(v_\infty, v_\infty) \Rightarrow m < m_\infty$ for a d.g.s. v_∞ of the problem at infinity

(C) Methods

The proof strategy when we don't have compactness of P.S.-sequences:

Proof of $b_{K_\infty}(v_\infty, v_\infty) < b_K(v_\infty, v_\infty)$:

- Take v_∞ as the dual ground state and set $u_\infty = L_\infty^{-1} v$.
- u_∞ solves $L_\infty u_\infty = \Gamma(x)|u_\infty|^{p-1} u_\infty$
- asymptotic estimates for $l \in [0,3)$

$$\partial_t^l u_\infty(x,t) = U^l \cos(\omega t) \mathrm{e}^{-\sqrt{\alpha}\omega|x|} + O(\mathrm{e}^{-(\sqrt{\alpha}\omega+\epsilon)|x|})$$

(C) Methods

The proof strategy when we don't have compactness of P.S.-sequences:
For large n consider $v_\infty^n(x) = v_\infty(x-n)$ and set $\Psi^n = -L^{-1}(L - L_\infty)u_\infty^n$.
Then compute

$$b_K(v_\infty^n, v_\infty^n) - b_{K_\infty}(v_\infty^n, v_\infty^n) = \sum_k \int_{\mathbb{R}} L_k \psi_k^n \psi_k^n + \frac{1}{\beta k^2 \omega^2} L_k \psi_k L_k \psi_k \, dx$$

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since all eigenvalues of L_1 are positive.

Summary

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Using

- saddle point methods, when P.S.-sequences are compact
- dual variational methods, when compactness does not hold
- spectral gaps of size $O(|k|)$ around zero in the spectrum of L_k

we find a variational setup for proving existence of time-periodic, spatially localited solutions of

$$(**) \qquad V(x)u_{tt} - \Delta_n u = \Gamma(x)|u|^{p-1}u$$

Theorem 1 (S. Kohler & W.R., part of PhD thesis)

For $V = -\alpha + \beta 1_{B_R(0)}$ *with* $\beta > \alpha > 0$*, breathers with* $\omega = \frac{\pi}{2R\sqrt{\beta-\alpha}}$ *exist if*

(1) $n = 1$, $\Gamma > 0$ & $\lim_{|x|\to\infty} \Gamma(x) = 0$, $1 < p < 3$.

(2) $n = 2$, $\Gamma = \Gamma(r)$ *bounded* & $\Gamma > 0$, $1 < p < 2$.

(3) $n = 1$, $\Gamma =$*periodic, bounded* & $\inf \Gamma > 0$, $1 < p < 3$

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Thank you for your attention