How to construct Breather Solutions using Nonlinear Helmholtz Systems

Dominic Scheider
Outline

1 Breather solutions

2 Breather solutions for the cubic Klein-Gordon Equation

\[ \partial_t^2 U - \Delta U + m^2 U = U^3 \text{ on } \mathbb{R} \times \mathbb{R}^3 \]

3 Breather solutions for the nonlinear Wave Equation

\[ \partial_t^2 U - \Delta U = |U|^{p-2} U \text{ on } \mathbb{R} \times \mathbb{R}^N \]
Breather solutions

The Sine-Gordon Breather

\[ \frac{\partial^2}{\partial t^2} U - \frac{\partial^2}{\partial x^2} U + \sin(U) = 0 \quad \text{on} \quad \mathbb{R} \times \mathbb{R} \]

▷ Ablowitz et al. 1973:

\[ U(t, x) = 4 \arctan \left( \frac{\sqrt{1 - \omega^2}}{\omega} \frac{\cos(\omega(t - t_0))}{\cosh(\sqrt{1 - \omega^2}(x - x_0))} \right). \]

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“breathing takes place only for isolated nonlinearities”
Breathers in periodic structures (1/3)

\[ V(x) \partial_t^2 U - \partial_x^2 U + q(x) U = U^3 \quad \text{on} \quad \mathbb{R} \times \mathbb{R} \]
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Assumptions:
\[ V(x), q(x) = (q_0 - \varepsilon^2) V(x) \] explicit periodic step potentials.

Methods:
Spatial dynamics, center manifold reduction, bifurcation.

Result: For \( 0 < \varepsilon < \varepsilon_0 \) existence of a solution \( U(t, x) \) with
▷ explicit period in \( t \),  ▷ exponential decay in \( x \).
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Breather solutions

Breathers in periodic structures (2/3)

\[ V(x) \partial_t^2 U - \partial_x^2 U + q(x) U = \Gamma(x)|U|^{p-2}U \quad \text{on} \quad \mathbb{R} \times \mathbb{R} \]

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▷ Hirsch, Reichel 2019:
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Assumptions:
\[ V(x) \sim q(x) \text{ specific periodic delta / step / } H^r \text{ potentials; } \]
\[ 2 < p < p^*(V). \]

Methods:
Variational approach (Nehari manifold).

Result:
Existence of (possibly large) time-periodic ground state solutions.
Breather solutions

Breathers in periodic structures (3/3)

\[ V(x) \frac{\partial^2}{\partial t^2} U - \frac{\partial^2}{\partial x^2} U + m^2 V(x) U = f(x, U) \quad \text{on} \quad \mathbb{R} \times \mathbb{R} \]

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Guiding principles:

\[ U(t, x) = \sum_k e^{ik\omega t} u_k(x) \quad \leadsto \quad -u''_k + (m^2 - k^2 \omega^2) V(x) u_k = f_k(x, U). \]

Aim for \(0 \notin \sigma\left(-\frac{d^2}{dx^2} + (m^2 - k^2 \omega^2) V(x)\right)\). Problem: \(|k| \to \infty \ldots\)
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Periodicity & roughness of \(V(x)\) yield uniformly open spectral gaps.
\(\leadsto\) “breathing takes place only for carefully designed potentials”
Breather solutions

Why not allow 0 in the spectra?
Breather solutions

Why not allow 0 in the spectra?

Then $V(x) \equiv 1$ is fine. Klein-Gordon equation:

$$\partial_t^2 U - \Delta U + m^2 U = f(x, U) \text{ on } \mathbb{R} \times \mathbb{R}^N$$
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**Cost:** $N = 1$ not accessible, $V(x) \not\equiv \text{const.}$ hard, breathers decay slowly.

**Gain:** $N > 1$ accessible, $V(x) \equiv \text{const.}$ accessible, many breathers.
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Here, breathing is not a rare phenomenon.
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(i) \( U(t, x) = \sum_k e^{ik\omega t} u_k(x) \) yields

\[ -\Delta u_k - (k^2 \omega^2 - m^2) u_k = \sum_{l+m+n=k} u_l u_m u_n \quad \text{on} \quad \mathbb{R}^3. \]
Breathers for the Klein-Gordon Equation

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Nonlinear Helmholtz System: \[ 0 \in \sigma(-\Delta - (k^2 \omega^2 - m^2)) \text{ for } k \neq 0 \]
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Breathers for the Klein-Gordon Equation

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**Bifurcation from simple eigenvalues:**

Need 1-dim. kernel of linearization

\[ -\Delta v_k - (k^2 \omega^2 - m^2) v_k = 3w_0^2 v_k \quad \text{on} \quad \mathbb{R}^3. \]
Breathers for the Klein-Gordon Equation

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Key ideas: Radial symmetry, asymptotic phase condition.
Let $X := \{ u \in C^{rad}(\mathbb{R}^3, \mathbb{R}) | \sup (1+|x|) |u(x)| < \infty \}$. Let $w_0 \in X$ with $-\Delta w_0 + m^2 w_0 = w_0^3$ on $\mathbb{R}^3$; choose $\omega > m$ and $s \in \mathbb{N}$.

Theorem 1 [S. 2019]

There exist an interval $I \subseteq \mathbb{R}$, $0 \in I$ and a family $(U_{\eta})_{\eta \in I \subseteq C^2_{per}(\mathbb{R}, X)}$ of real-valued, classical breather solutions $U_{\eta}(t, x) = \sum_k e^{ik\omega t} u_{\eta k}(x)$ of the Klein-Gordon equation $\partial^2_t U - \Delta U + m^2 U = U^3$ on $\mathbb{R} \times \mathbb{R}^3$ which is a continuous curve in $C(\mathbb{R}, X)$ with $\Delta U_0(t, x) = w_0(x)$, $\Delta U_{\eta}$ is $2\pi \omega$-periodic in time with $\infty$ many nonzero modes ($\eta \neq 0$), $\Delta d_{\eta | \eta = 0} u_{\eta k} \neq 0$ iff $k = s$ (excitation of $s$-th mode).
Let $X := \{ u \in C_{rad}(\mathbb{R}^3, \mathbb{R}) | \sup (1 + |x|) |u(x)| < \infty \}$. 
Breathers for the Klein-Gordon Equation

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which is a continuous curve in $C(\mathbb{R}, X)$ with

- $U^0(t, x) = w_0(x)$,
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Breathers for the Klein-Gordon Equation

Remarks

\[ \partial^2_t U - \Delta U + m^2 U = \Gamma(x) U^3 \text{ on } \mathbb{R} \times \mathbb{R}^3 \] with $\Gamma$ bounded, radial, continuously differentiable.

Open: Other space dimensions / powers (easy?); non-constant potentials (hard!).

Aspects of the Proof

- Bifurcation from simple eigenvalues (in a nutshell),
- Linear Helmholtz equations in $X$ (likewise),
- How to prove the Theorem.
Remarks

▷ Extension to

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Breathers for the Klein-Gordon Equation

Aspects of the Proof 1/3: Bifurcation from simple eigenvalues.

**Situation:** Banach space $E$, $u_0 \in E$ and $f \in C^1(E \times \mathbb{R}, E)$ with
\[ f(u_0, \lambda) = 0 \quad \text{for all } \lambda \in \mathbb{R}. \]
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$$\dim \ker D_u f(u_0, \lambda_0) = 0$$

Implicit Function Theorem
Breathers for the Klein-Gordon Equation

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*Implicit Function Theorem*
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(and more)

**Crandall-Rabinowitz Theorem:**

Bifurcation

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Breathers for the Klein-Gordon Equation

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**Crandall-Rabinowitz Theorem:**
Bifurcation
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\[T = \{(u_0, \lambda) \mid \lambda \in \mathbb{R}\}\]
Breathers for the Klein-Gordon Equation

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\[ \partial_t^2 U - \Delta U + m^2 U = U^3 \quad \text{on} \quad \mathbb{R} \times \mathbb{R}^3 \]

yields via \( U(t, x) = \sum_k e^{ik\omega t} u_k(x) \) the infinite system

\[ -\Delta u_k - (k^2 \omega^2 - m^2) u_k = \sum_{l+m+n=k} u_l u_m u_n \quad \text{on} \quad \mathbb{R}^3. \]
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▷ Reformulate as \( f((u_k)_k, \lambda) = 0. \)
▷ Introduce a bifurcation parameter \( \lambda. \)
▷ Ensure 1-dim. kernel of linearized problem

\[ -\Delta v_k - (k^2 \omega^2 - m^2) v_k = 3w_0^2(x) v_k \quad \text{on} \quad \mathbb{R}^3. \]

▷ Verify remaining conditions of the CR Bifurcation Theorem (transversality).
Breathers for the Klein-Gordon Equation

Aspects of the Proof 2/3: Linear Helmholtz equations.
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\[-\Delta u - \mu u = f \quad \text{on } \mathbb{R}^3, \quad \mu > 0\]  \hspace{1cm} (H)

▷ “Helmholtz” case: \(0 \in \sigma(-\Delta - \mu)\)

\[
\begin{aligned}
-\Delta u - \mu u &= f \quad \text{on } \mathbb{R}^3, \quad \mu > 0 \\
\sigma(-\Delta - \mu) &= \{0\}
\end{aligned}
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▷ Particular solution of (H):

Limiting Absorption Principle,

\[
u_1 = \Re \left[ \lim_{\varepsilon \to 0} (-\Delta - \mu - i\varepsilon)^{-1} f \right] = \frac{\cos(|\cdot| \sqrt{\mu})}{4\pi|\cdot|} \ast f.
\]
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\]

- General solution of (H): \(u = u_1 + u_2\)
  with any Herglotz wave \(-\Delta u_2 - \mu u_2 = 0\), e.g. \(u_2 = \frac{\sin(|\cdot|\sqrt{\mu})}{4\pi|\cdot|} * f\).
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- Particular solution of (H):
  Limiting Absorption Principle,
  
  \[
  u_1 = \Re \left[ \lim_{\varepsilon \to 0} (-\Delta - \mu - i\varepsilon)^{-1} f \right] = \frac{\cos(|\cdot| \sqrt{\mu})}{4\pi|\cdot|} \ast f.
  \]

- General solution of (H): $u = u_1 + u_2$
  
  with any Herglotz wave $-\Delta u_2 - \mu u_2 = 0$, e.g. $u_2 = \frac{\sin(|\cdot| \sqrt{\mu})}{4\pi|\cdot|} \ast f$.

Summary: Multitude of (weakly) localized solutions.
Breathers for the Klein-Gordon Equation

Aspects of the Proof 2/3: Linear Helmholtz equations.

\[-\Delta u - \mu u = f(r) \text{ on } \mathbb{R}^3, \quad \mu > 0 \quad (H)\]

\[\triangleright \text{“Helmholtz” case: } 0 \in \sigma(-\Delta - \mu)\]
Breathers for the Klein-Gordon Equation

Aspects of the Proof 2/3: Linear Helmholtz equations.

\[-\Delta u - \mu u = f(r) \quad \text{on } \mathbb{R}^3, \quad \mu > 0 \quad \tag{H}\]

▷ "Helmholtz" case: $0 \in \sigma(-\Delta - \mu)$

▷ Particular solution of (H):

\[-(ru_1)'' - \mu (ru_1) = rf(r), \quad u_1(0) = 1, \quad u_1'(0) = 0.\]
Breathers for the Klein-Gordon Equation

Aspects of the Proof 2/3: Linear Helmholtz equations.

\[-\Delta u - \mu u = f(r) \quad \text{on } \mathbb{R}^3, \quad \mu > 0 \quad (H)\]

- "Helmholtz" case: \(0 \in \sigma(-\Delta - \mu)\)
- Particular solution of (H):
  \[-(ru_1)'' - \mu (ru_1) = rf(r), \quad u_1(0) = 1, \quad u_1'(0) = 0.\]

- General solution of (H):
  \[u = u_1 + c \cdot \frac{\sin(| \cdot | \sqrt{\mu})}{4\pi | \cdot |} \quad \text{often} \quad \cos\left(\frac{| \cdot | \sqrt{\mu}}{4\pi | \cdot |}\right) \ast f + \tilde{c} \cdot \frac{\sin(| \cdot | \sqrt{\mu})}{4\pi | \cdot |} \ast f, \quad c, \tilde{c} \in \mathbb{R}.\]
Breathers for the Klein-Gordon Equation

Aspects of the Proof 2/3: Linear Helmholtz equations.

\[ -\Delta u - \mu u = f(r) \quad \text{on } \mathbb{R}^3, \quad \mu > 0 \]  

\( \triangleright \) “Helmholtz” case: \( 0 \in \sigma(-\Delta - \mu) \)

\( \triangleright \) Particular solution of (H):

\[ -(ru_1)'' - \mu(ru_1) = rf(r), \quad u_1(0) = 1, \quad u_1'(0) = 0. \]

\( \triangleright \) General solution of (H):

\[ u = u_1 + c \cdot \frac{\sin(|\cdot|\sqrt{\mu})}{4\pi|\cdot|} \]

\[ \text{often} \quad \cos(|\cdot|\sqrt{\mu}) \frac{1}{4\pi|\cdot|} \ast f + \tilde{c} \cdot \frac{\sin(|\cdot|\sqrt{\mu})}{4\pi|\cdot|} \ast f, \quad c, \tilde{c} \in \mathbb{R}. \]

Radial symmetry \( \leadsto \) 1-dim. solution spaces.
Aspects of the Proof 2/3: Linear Helmholtz equations.

\[-\Delta u - \mu u = g(r) \cdot u \quad \text{on } \mathbb{R}^3, \quad \mu > 0 \quad (H^*)\]
Breathers for the Klein-Gordon Equation

Aspects of the Proof 2/3: Linear Helmholtz equations.

\[-\Delta u - \mu u = g(r) \cdot u \quad \text{on } \mathbb{R}^3, \quad \mu > 0 \quad (H^*)\]

▷ Asymptotically, if $g$ is localized:

\[-(ru)'' - \mu (ru) \approx 0 \quad \implies u(r) \approx q_\infty \frac{\sin(r\sqrt{\mu} + \tau_\infty)}{r}\]
Breathers for the Klein-Gordon Equation

Aspects of the Proof 2/3: Linear Helmholtz equations.

\[-\Delta u - \mu u = g(r) \cdot u \quad \text{on } \mathbb{R}^3, \quad \mu > 0 \quad \text{(H\(^*\))}\]

- Asymptotically, if \( g \) is localized:

\[-(ru)'' - \mu (ru) \approx 0 \quad \implies \quad u(r) \approx \varrho_\infty \frac{\sin(r \sqrt{\mu} + \tau_\infty)}{r}\]

- Lemma:

(H\(^*\)) has a unique normalized solution in \( X \). It satisfies

\[u(r) = \frac{\sin(r \sqrt{\mu} + \tau_\infty)}{r} + O \left( \frac{1}{r^2} \right)\]

for some unique \( \tau_\infty \in [0, \pi) \).
Aspects of the Proof 2/3: Linear Helmholtz equations.

\[-\Delta u - \mu u = g(r) \cdot u \quad \text{on } \mathbb{R}^3, \quad \mu > 0 \quad (H^*)\]

▷ Asymptotically, if \( g \) is localized:

\[-(ru)'' - \mu (ru) \approx 0 \quad \rightsquigarrow \quad u(r) \approx \varrho_\infty \frac{\sin (r \sqrt{\mu} + \tau_\infty)}{r} \]

▷ Lemma:

\((H^*)\) together with an asymptotic phase condition (far field condition)

\[u(r) \sim \frac{\sin (r \sqrt{\mu} + \tau)}{r} + O \left( \frac{1}{r^2} \right) \quad (A_\tau)\]

has a nontrivial solution in \( X \) iff \( \tau = \tau_\infty \). (Unique up to constant multiple.)
Aspects of the Proof 2/3: Linear Helmholtz equations.

\[
\Delta u - \mu u = g(r) \cdot u \quad \text{on} \quad \mathbb{R}^3, \quad \mu > 0 \quad (H^* u(r) \sim \sin(r \sqrt{\mu} + \tau r) + O(\frac{1}{r^2})),
\]

Lemma:

\( (H^*) \) together with the asymptotic phase condition \( (A \tau) \) has a nontrivial solution in \( X \) iff \( \tau = \tau_\infty \). (Unique up to constant multiple.)

Remark:

For \( \tau \neq 0 \) and \( u \in X \),

\[ (H^*), (A \tau) \iff u = \sin(|\cdot| \sqrt{\mu} + \tau_{\infty} |\cdot| \sin(\tau)) \ast [g u] =: \Psi_{\tau \mu} \ast [g u]. \]

Radial symmetry \( \oplus \) “good” phase cond. \( \Rightarrow \) 1-dim. solution spaces.

Radial symmetry \( \oplus \) “bad” phase cond. \( \Rightarrow \) 0-dim. solution spaces.
Aspects of the Proof 2/3: Linear Helmholtz equations.

\[- \Delta u - \mu u = g(r) \cdot u \quad \text{on } \mathbb{R}^3, \quad \mu > 0 \quad \text{ (H*)}\]

\[u(r) \sim \frac{\sin(r \sqrt{\mu + \tau})}{r} + O \left( \frac{1}{r^2} \right) \quad \text{ (A}_\tau \text{)}\]

Lemma:

(H*) together with the asymptotic phase condition (A_\tau) has a nontrivial solution in X iff \(\tau = \tau_\infty\). (Unique up to constant multiple.)
Aspects of the Proof 2/3: Linear Helmholtz equations.

\[- \Delta u - \mu u = g(r) \cdot u \quad \text{on } \mathbb{R}^3, \quad \mu > 0 \quad (H^*)
\]
\[u(r) \sim \frac{\sin(r \sqrt{\mu} + \tau)}{r} + O \left( \frac{1}{r^2} \right) \quad (A_\tau)\]

\[\updownarrow \text{Lemma:} \quad \updownarrow\]

(H*) together with the asymptotic phase condition (A_\tau) has a nontrivial solution in X iff \(\tau = \tau_\infty\). (Unique up to constant multiple.)

\[\updownarrow \text{Remark: For } \tau \neq 0 \text{ and } u \in X, \updownarrow\]

\[(H^*), (A_\tau) \quad \Leftrightarrow \quad u = \frac{\sin(|\cdot| \sqrt{\mu} + \tau)}{4\pi|\cdot| \sin(\tau)} * [g u] =: \Psi^\tau_{\mu} * [g u].\]
Breathers for the Klein-Gordon Equation

Aspects of the Proof 2/3: Linear Helmholtz equations.

\[-\Delta u - \mu u = g(r) \cdot u \quad \text{on } \mathbb{R}^3, \quad \mu > 0 \quad \text{(H*)}\]

\[u(r) \sim \frac{\sin(r \sqrt{\mu} + \tau)}{r} + O\left(\frac{1}{r^2}\right) \quad \text{(A}_\tau\text{)}\]

▷ Lemma:

(H*) together with the asymptotic phase condition (A_τ) has a nontrivial solution in X iff \(\tau = \tau_\infty\). (Unique up to constant multiple.)

▷ Remark: For \(\tau \neq 0\) and \(u \in X\),

\[(H^*), (A_\tau) \iff u = \frac{\sin(|\cdot| \sqrt{\mu} + \tau)}{4\pi|\cdot| \sin(\tau)} * [g \ u] =: \Psi_{\tau \mu} * [g \ u].\]

Radial symmetry \(\oplus\) “good” phase cond. \(\leadsto\) 1-dim. solution spaces.

Radial symmetry \(\oplus\) “bad” phase cond. \(\leadsto\) 0-dim. solution spaces.
Aspects of the Proof 3/3.

Breathers for the Klein-Gordon Equation

\[ \partial^2_t U - \Delta U + m^2 U = U^3 \text{ on } \mathbb{R} \times \mathbb{R}^3 \]
yields via

\[ U(t, x) = \sum_{k} e^{i k \omega t} u_k(x) \]
the infinite system

\[-\Delta u_k - (k^2 \omega^2 - m^2) u_k = \sum_{l+m+n=k} u_l u_m u_n \text{ on } \mathbb{R}^3 \]
Breathers for the Klein-Gordon Equation

Aspects of the Proof 3/3.

\[ \partial_t^2 U - \Delta U + m^2 U = U^3 \quad \text{on} \quad \mathbb{R} \times \mathbb{R}^3 \]

yields via \( U(t, x) = \sum_k e^{ik\omega t} u_k(x) \) the infinite system

\[ -\Delta u_k - (k^2 \omega^2 - m^2) u_k = \sum_{l+m+n=k} u_l u_m u_n \quad \text{on} \quad \mathbb{R}^3. \]
Breathers for the Klein-Gordon Equation

Aspects of the Proof 3/3.

\[-\Delta u_k - (k^2 \omega^2 - m^2) u_k = \sum_{l+m+n=k} u_l u_m u_n \quad \text{on} \quad \mathbb{R}^3\]
Breathers for the Klein-Gordon Equation

Aspects of the Proof 3/3.

\[-\Delta u_k - (k^2 \omega^2 - m^2) u_k = (u \ast u \ast u)_k \quad \text{on} \quad \mathbb{R}^3\]

where \( u = (u_k)_k \in \ell^1(\mathbb{Z}, X) \).
Breathers for the Klein-Gordon Equation

Aspects of the Proof 3/3.

\[-\Delta u_k - \mu_k u_k = (u \ast u \ast u)_k \text{ on } \mathbb{R}^3\]

where \(u = (u_k)_k \in \ell^1(\mathbb{Z}, X)\) and \(\mu_k = k^2 \omega^2 - m^2\).
Breathers for the Klein-Gordon Equation

Aspects of the Proof 3/3.

\[ -\Delta u_k - \mu_k u_k = (u * u * u)_k \quad \text{on} \quad \mathbb{R}^3 \]

where \( u = (u_k)_k \in \ell^1(\mathbb{Z}, X) \) and \( \mu_k = k^2 \omega^2 - m^2 \).

▷ Reformulate as \( f(u, \lambda) = 0 \).
▷ Introduce a bifurcation parameter \( \lambda \).
▷ Ensure 1-dim. kernel of linearized problem

\[ -\Delta v_k - \mu_k v_k = 3w^2_0(x) v_k \quad \text{on} \quad \mathbb{R}^3. \]

▷ Verify remaining conditions of the CR Bifurcation Theorem (transversality).
Breathers for the Klein-Gordon Equation

Aspects of the Proof 3/3.

\[ u_k - \Psi_{\mu_k}^{\tau_k} \ast [(u \ast u \ast u)_k] = 0 \quad (k \neq 0 !) \]

where \( u = (u_k)_k \in \ell^1(\mathbb{Z}, X) \) and \( \mu_k = k^2 \omega^2 - m^2 \)

with asymptotic conditions given by \( \tau_k \).

▷ ✓ Reformulate as \( f(u, \lambda) = 0 \).
▷ Introduce a bifurcation parameter \( \lambda \).
▷ Ensure 1-dim. kernel of linearized problem

\[ v_k - 3 \Psi_{\mu_k}^{\tau_k} \ast [w_0^2 v_k] = 0 \quad (k \neq 0 !). \]

▷ Verify remaining conditions of the CR Bifurcation Theorem (transversality).
Breathers for the Klein-Gordon Equation

Aspects of the Proof 3/3.

\[ u_k - \Psi^{\tau_k}_{\mu_k} \ast [(u \ast u \ast u)_k] = 0 \quad (k \neq 0 !) \]

where \( u = (u_k)_k \in \ell^1(\mathbb{Z}, X) \) and \( \mu_k = k^2 \omega^2 - m^2 \)
with asymptotic conditions given by \( \tau_k \).

▷ √ Reformulate as \( f(u, \lambda) = 0 \).
▷ √ Replace \( \tau_s \) by \( \tau_s + \lambda \): “Invisible” bifurcation parameter.
▷ Ensure 1-dim. kernel of linearized problem

\[ v_k - 3 \Psi^{\tau_k}_{\mu_k} \ast [w_0^2 v_k] = 0 \quad (k \neq 0 !). \]

▷ Verify remaining conditions of the CR Bifurcation Theorem (transversality).
Breathers for the Klein-Gordon Equation

Aspects of the Proof 3/3.

\[ u_k - \Psi^{\tau_k}_{\mu_k} [(u \ast u \ast u)_k] = 0 \quad (k \neq 0 !) \]

where \( u = (u_k)_k \in \ell^1(\mathbb{Z}, X) \) and \( \mu_k = k^2 \omega^2 - m^2 \)

with asymptotic conditions given by \( \tau_k \).

▷ ✓ Reformulate as \( f(u, \lambda) = 0 \).

▷ ✓ Replace \( \tau_s \) by \( \tau_s + \lambda \): “Invisible” bifurcation parameter.

▷ ✓ Recall Lemma: Choose \( \tau_s \) “good” and all other \( \tau_k \) “bad” s.t.

\[ v_k - 3\Psi^{\tau_k}_{\mu_k} [w_0^2 v_k] = 0 \Rightarrow v_k \equiv 0 \quad \text{holds iff } k \neq s. \]

For \( k = 0 \), this is a result in the literature.

▷ Verify remaining conditions of the CR Bifurcation Theorem (transversality).
Aspects of the Proof 3/3.

\[ u_k - \Psi_{\mu_k}^\tau \ast [(u \ast u \ast u)_k] = 0 \quad (k \neq 0 !) \]

where \( u = (u_k)_k \in \ell^1(\mathbb{Z}, X) \) and \( \mu_k = k^2 \omega^2 - m^2 \)

with asymptotic conditions given by \( \tau_k \).

\( \diamond \) ✓ Reformulate as \( f(u, \lambda) = 0. \)

\( \diamond \) ✓ Replace \( \tau_s \) by \( \tau_s + \lambda \): “Invisible” bifurcation parameter.

\( \diamond \) ✓ Recall Lemma: Choose \( \tau_s \) “good” and all other \( \tau_k \) “bad” s.t.

\[ \nu_k - 3 \Psi_{\mu_k}^\tau \ast [w_0^2 \nu_k] = 0 \Rightarrow \nu_k \equiv 0 \quad \text{holds iff } k \neq s. \]

For \( k = 0 \), this is a result in the literature.

\( \diamond \) ✓ Transversality condition: Direct computation using asymptotics.
Breathers for the Klein-Gordon Equation

Aspects of the Proof 3/3.

\[ u_k - \Psi^{\tau_k}_{\mu_k} \ast [(u \ast u \ast u)_k] = 0 \quad (k \neq 0 !) \]

where \( u = (u_k)_k \in \ell^1(\mathbb{Z}, X) \) and \( \mu_k = k^2 \omega^2 - m^2 \)
with asymptotic conditions given by \( \tau_k \).

- ✔ Reformulate as \( f(u, \lambda) = 0 \).
- ✔ Replace \( \tau_s \) by \( \tau_s + \lambda \): “Invisible” bifurcation parameter.
- ✔ Recall Lemma: Choose \( \tau_s \) “good” and all other \( \tau_k \) “bad” s.t.

\[ v_k - 3 \Psi^{\tau_k}_{\mu_k} \ast [w_0^2 v_k] = 0 \Rightarrow v_k \equiv 0 \quad \text{holds iff } k \neq s. \]

For \( k = 0 \), this is a result in the literature.

- ✔ Transversality condition: Direct computation using asymptotics.
- ✔ Regularity of breathers: Scaling property of convolution with \( \Psi^{\tau_k}_{\mu_k} \).
Let $X := \{ u \in C_{\text{rad}}(\mathbb{R}^3, \mathbb{R}) \mid \sup(1 + |x|)|u(x)| < \infty \}$.

Let $w_0 \in X$ with $-\Delta w_0 + m^2 w_0 = w_0^3$ on $\mathbb{R}^3$; choose $\omega > m$ and $s \in \mathbb{N}$.

**Theorem 1** [S. 2019]

There exist an interval $I \subseteq \mathbb{R}$, $0 \in I$ and a family $(U^\eta)_{\eta \in I} \subseteq C^2_{\text{per}}(\mathbb{R}, X)$ of real-valued, classical breather solutions $U^\eta(t, x) = \sum_k e^{ik\omega t} u^\eta_k(x)$ of the Klein-Gordon equation

$$\partial^2_t U - \Delta U + m^2 U = U^3 \quad \text{on } \mathbb{R} \times \mathbb{R}^3$$

which is a continuous curve in $C(\mathbb{R}, X)$ with

- $U^0(t, x) = w_0(x)$,
- $U^\eta$ is $\frac{2\pi}{\omega}$-periodic in time with $\infty$ many nonzero modes ($\eta \neq 0$),
- $\frac{d}{d\eta} \big|_{\eta=0} u^\eta_k \neq 0$ iff $k = s$ (excitation of $s$-th mode).
1 Breather solutions

2 Breather solutions for the cubic Klein-Gordon Equation

\[ \partial_t^2 U - \Delta U + m^2 U = U^3 \text{ on } \mathbb{R} \times \mathbb{R}^3 \]

3 Breather solutions for the nonlinear Wave Equation

\[ \partial_t^2 U - \Delta U = |U|^{p-2} U \text{ on } \mathbb{R} \times \mathbb{R}^N \]
Breathers for the Wave Equation

This is work in progress.

**Theorem 2** [sketch]
Let \( \Gamma \in S(\mathbb{R}^N) \), \( \Gamma > 0 \) and \( N \geq 2 \), \( 2 < p < 2(N + 1)/(N - 1) \). Then the nonlinear wave equation

\[
\partial_t^2 U - \Delta U = \Gamma(x)|U|^{p-2}U \quad \text{on} \ [0, 2\pi] \times \mathbb{R}^N
\]

has a nontrivial dual ground state \( U : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R} \), which is \( 2\pi \)-periodic (\( \pi \)-antiperiodic) in time.
Breathers for the Wave Equation

Remarks

“Large” breathers via variational methods: dual ground states.

Drawbacks: Need localized coefficient $\Gamma(x)$ in the nonlinearity.

$N=3, p=4$ not accessible (endpoint case).

To do: Regularity, extension to Klein-Gordon Equation.

Aspects of the Proof

A formal solution map for $\partial^2_t U - \Delta U = F$ on $[0, 2\pi] \times \mathbb{R}^N$.

Dual variational techniques.
Breathers for the Wave Equation

Remarks

▷ “Large” breathers via variational methods: dual ground states.
▷ Drawbacks: Need localized coefficient $\Gamma(x)$ in the nonlinearity.

\[ N = 3, \, p = 4 \text{ not accessible (endpoint case).} \]
▷ To do: Regularity, extension to Klein-Gordon Equation.
Breathers for the Wave Equation

Remarks

▷ “Large” breathers via variational methods: dual ground states.
▷ Drawbacks: Need localized coefficient $\Gamma(x)$ in the nonlinearity.

$$N = 3, p = 4 \text{ not accessible (endpoint case).}$$
▷ To do: Regularity, extension to Klein-Gordon Equation.

Aspects of the Proof
Remarks

▷ “Large” breathers via variational methods: dual ground states.
▷ Drawbacks: Need localized coefficient $\Gamma(x)$ in the nonlinearity.

$$N = 3, \ p = 4 \text{ not accessible (endpoint case).}$$

▷ To do: Regularity, extension to Klein-Gordon Equation.

Aspects of the Proof

▷ A formal solution map for

$$\partial_t^2 U - \Delta U = F \quad \text{on } [0, 2\pi] \times \mathbb{R}^N,$$

▷ Dual variational techniques.
Breathers for the Wave Equation

Aspects of the Proof 1/2: Formal solution map.

\[ -\Delta - k^2 u_k = f_k \text{ on } \mathbb{R}^N. \]

Restriction to odd \( k \in \mathbb{Z} \) yields a nonlinear Helmholtz system. Idea:

\[ u_k = \Psi_k^2 \ast f_k, \]

\[ U(t, x) = \sum_{k \text{ odd}} e^{i k t} (\Psi_k^2 \ast f_k)(x) \]

with \( f_k(x) = \int_0^{2\pi} F(t, x) e^{-i k t} dt \).
Breathers for the Wave Equation

Aspects of the Proof 1/2: Formal solution map.

\[ \partial_t^2 U - \Delta U = F \quad \text{on} \quad [0, 2\pi] \times \mathbb{R}^N \]
Breathers for the Wave Equation

Aspects of the Proof 1/2: Formal solution map.

\[ \partial_t^2 U - \Delta U = F \quad \text{on } [0, 2\pi] \times \mathbb{R}^N \]

with \( U(t, x) = \sum_k e^{ikt} u_k(x) \) and \( F(t, x) = \sum_k e^{ikt} f_k(x) \), this leads to

\[ (-\Delta - k^2) u_k = f_k \quad \text{on } \mathbb{R}^N. \]
Breathers for the Wave Equation

Aspects of the Proof 1/2: Formal solution map.

\[
\partial_t^2 U - \Delta U = F \quad \text{on } [0, 2\pi] \times \mathbb{R}^N
\]

with \( U(t, x) = \sum_k e^{ikt} u_k(x) \) and \( F(t, x) = \sum_k e^{ikt} f_k(x) \), this leads to

\[
(-\Delta - k^2) u_k = f_k \quad \text{on } \mathbb{R}^N.
\]

Restriction to odd \( k \in \mathbb{Z} \) yields a nonlinear Helmholtz system. Idea:

\[
u_k = \Psi_{k^2} \ast f_k,
\]
Breathers for the Wave Equation

Aspects of the Proof 1/2: Formal solution map.

\[ \partial_t^2 U - \Delta U = F \quad \text{on} \quad [0, 2\pi] \times \mathbb{R}^N \]

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Restriction to odd \( k \in \mathbb{Z} \) yields a nonlinear Helmholtz system. Idea:

\[ u_k = \Psi_{k^2} \ast f_k, \]

\[ U(t, x) = \sum_{k \text{ odd}} e^{ikt}(\Psi_{k^2} \ast f_k)(x) \]
Breathers for the Wave Equation

Aspects of the Proof 1/2: Formal solution map.

\[ \partial_t^2 U - \Delta U = F \quad \text{on} \quad [0, 2\pi] \times \mathbb{R}^N \]

with \( U(t, x) = \sum_k e^{ikt} u_k(x) \) and \( F(t, x) = \sum_k e^{ikt} f_k(x) \), this leads to

\[ (-\Delta - k^2) u_k = f_k \quad \text{on} \quad \mathbb{R}^N. \]

Restriction to odd \( k \in \mathbb{Z} \) yields a nonlinear Helmholtz system. Idea:

\[ u_k = \Psi_{k2} \ast f_k, \]

\[ U(t, x) = \sum_{k \text{ odd}} e^{ikt} (\Psi_{k2} \ast f_k)(x) \quad \text{with} \quad f_k(x) = \int_0^{2\pi} F(t, x) e^{-ikt} \frac{dt}{2\pi}. \]
Aspects of the Proof 2/2: Dual variational techniques.

\[
\partial_t^2 U - \Delta U = \Gamma(x)|U|^{p-2}U \quad \text{on} \quad [0, 2\pi] \times \mathbb{R}^N
\]

\[\text{Substitution } V : \Gamma(x) = \frac{1}{p'}|U|^{p-2}U \text{ yields } \left(\partial_t^2 - \Delta\right) \left[\Gamma(x) - \frac{1}{p}V\right] = \Gamma(x) \frac{1}{p}V \text{ on } [0, 2\pi] \times \mathbb{R}^N.\]

\[\text{Then, formally, solve } \frac{1}{p'}V^{p'-2}V = L_{\Gamma[V]} \text{ on } [0, 2\pi] \times \mathbb{R}^N \]

where \(L_{\Gamma[V]}(t, x) = \sum_{k \text{ odd}} e^{ikt} \Gamma(x) \frac{1}{p}(\Psi_k^2 \ast [\frac{1}{p}v_k])(x).\)
Breathers for the Wave Equation

Aspects of the Proof 2/2: Dual variational techniques.

\[ \partial_t^2 U - \Delta U = \Gamma(x)|U|^{p-2} U \quad \text{on} \quad [0, 2\pi] \times \mathbb{R}^N \]
Aspects of the Proof 2/2: Dual variational techniques.

$$\partial_t^2 U - \Delta U = \Gamma(x)|U|^{p-2}U \quad \text{on } [0, 2\pi] \times \mathbb{R}^N$$

▷ Substitution $V := \Gamma(x)^{1/p'}|U|^{p-2}U$ yields

$$(\partial_t^2 - \Delta) \left[ \Gamma(x)^{-1/p} |V|^{p'-2}V \right] = \Gamma(x)^{1/p}V \quad \text{on } [0, 2\pi] \times \mathbb{R}^N.$$
Breathers for the Wave Equation

Aspects of the Proof 2/2: Dual variational techniques.

\[ \partial_t^2 U - \Delta U = \Gamma(x)|U|^{p-2}U \quad \text{on} \quad [0, 2\pi] \times \mathbb{R}^N \]

\(\triangleright\) Substitution \( V := \Gamma(x)^{1/p'}|U|^{p-2}U \) yields

\[ \left( \partial_t^2 - \Delta \right) \left[ \Gamma(x)^{-1/p} V \right]^{p'-2} V = \Gamma(x)^{1/p} V \quad \text{on} \quad [0, 2\pi] \times \mathbb{R}^N. \]

\(\triangleright\) Then, formally, solve

\[ |V|^{p'-2} V = \mathcal{L}_\Gamma[V] \quad \text{on} \quad [0, 2\pi] \times \mathbb{R}^N \]

where \( \mathcal{L}_\Gamma[V](t, x) = \sum_{k \text{ odd}} e^{ikt} \Gamma(x)^{1/p}(\Psi_k^2 * [\Gamma^{1/p} v_k])(x) \).
Breathers for the Wave Equation

Aspects of the Proof 2/2: Dual variational techniques.

\[ \partial_t^2 U - \Delta U = \Gamma(x)|U|^{p-2}U \quad \text{on } [0, 2\pi] \times \mathbb{R}^N \]

\[ \text{Substitution } V := \Gamma(x)^{1/p'}|U|^{p-2}U \text{ yields } \]

\[ (\partial_t^2 - \Delta) \left[ \Gamma(x)^{-1/p} |V|^{p'-2} V \right] = \Gamma(x)^{1/p} V \quad \text{on } [0, 2\pi] \times \mathbb{R}^N. \]

\[ \text{Then, formally, solve } \]

\[ |V|^{p'-2} V = L_\Gamma[V] \quad \text{on } [0, 2\pi] \times \mathbb{R}^N \]

where \( L_\Gamma[V](t, x) = \sum_{k \text{ odd}} e^{ikt} \Gamma(x)^{1/p} (\Psi_k^2 * [\Gamma^{1/p} V_k])(x). \)

\[ L_\Gamma : L^{p'}([0, 2\pi] \times \mathbb{R}^N) \to L^{p}([0, 2\pi] \times \mathbb{R}^N) \text{ is symmetric, compact.} \]
Breathers for the Wave Equation

Aspects of the Proof 2/2: Dual variational techniques.

\[ |V|^p' - 2V = \mathcal{L}_\Gamma[V] \quad \text{on } [0, 2\pi] \times \mathbb{R}^N \]

with \( \mathcal{L}_\Gamma : L^{p'}([0, 2\pi] \times \mathbb{R}^N) \to L^p([0, 2\pi] \times \mathbb{R}^N) \) symmetric and compact.
Breathers for the Wave Equation

Aspects of the Proof 2/2: Dual variational techniques.

\[ |V|^{p'} - 2 V = \mathcal{L}_\Gamma[V] \text{ on } [0, 2\pi] \times \mathbb{R}^N \]

with \( \mathcal{L}_\Gamma : L^{p'}([0, 2\pi] \times \mathbb{R}^N) \to L^p([0, 2\pi] \times \mathbb{R}^N) \) symmetric and compact.

\[ \triangleright \] Introduce \( J : L^{p'}([0, 2\pi] \times \mathbb{R}^N) \to \mathbb{R} \) via

\[ J(V) := \frac{1}{p'} \int |V|^{p'} \, d(t, x) - \frac{1}{2} \int V \mathcal{L}_\Gamma[V] \, d(t, x). \]
Breathers for the Wave Equation

Aspects of the Proof 2/2: Dual variational techniques.

\[ |V|^{p'-2}V = \mathcal{L}_\Gamma[V] \quad \text{on} \quad [0, 2\pi] \times \mathbb{R}^N \]

with \( \mathcal{L}_\Gamma : L^{p'}([0, 2\pi] \times \mathbb{R}^N) \rightarrow L^p([0, 2\pi] \times \mathbb{R}^N) \) symmetric and compact.

\( \triangleright \) Introduce \( J : L^{p'}([0, 2\pi] \times \mathbb{R}^N) \rightarrow \mathbb{R} \) via

\[ J(V) := \frac{1}{p'} \int |V|^{p'} \, d(t, x) - \frac{1}{2} \int V \mathcal{L}_\Gamma[V] \, d(t, x). \]

Following Evéquoz and Weth (2015, stationary case),

\( \triangleright \) \( J \) has the mountain pass geometry,

\( \triangleright \) \( J \) satisfies the Palais-Smale condition,
Breathers for the Wave Equation

Aspects of the Proof 2/2: Dual variational techniques.

\[ |V|^{p'-2} V = \mathcal{L}_\Gamma [V] \text{ on } [0, 2\pi] \times \mathbb{R}^N \]

with \( \mathcal{L}_\Gamma : L^{p'}([0, 2\pi] \times \mathbb{R}^N) \to L^p([0, 2\pi] \times \mathbb{R}^N) \) symmetric and compact.

▷ Introduce \( J : L^{p'}([0, 2\pi] \times \mathbb{R}^N) \to \mathbb{R} \) via

\[
J(V) := \frac{1}{p'} \int |V|^{p'} \, d(t,x) - \frac{1}{2} \int V \, \mathcal{L}_\Gamma [V] \, d(t,x).
\]

Following Evéquoz and Weth (2015, stationary case),

▷ \( J \) has the mountain pass geometry,

▷ \( J \) satisfies the Palais-Smale condition,

▷ hence \( J \) has a nontrivial ground state \( V_0 \in L^{p'}([0, 2\pi] \times \mathbb{R}^N) \).

\[ \rightsquigarrow \text{“Dual” ground state } U_0 = \Gamma(x)^{-1/p} |V_0|^{p'-2} V_0. \]
Thank you for your attention!
Thank you for your attention!


- The bifurcation result for the KG equation is part of my PhD thesis (KITopen, 2019).