# On the Long-term Behavior of a Nonlinear Stochastic Wave Equation

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# 1 Introduction

## 1.1 Context

Consider a particle at position $u(t) \in \mathbb{R}$ in a quadratic potential $V(u) = \frac{1}{2}|u|^2$. The force acting on the particle is given by

$$u_{tt} = -\partial_u V(u) = -u.$$

This system is a harmonic oscillator. Since one could say that all smooth potentials are in approximation just quadratic potentials up to second order, locally around an equilibrium point, the harmonic oscillator is one of the most fundamental systems in physics. Consider now the case of a quartic potential $V(u) = \frac{1}{2}|u|^2 + \frac{1}{4}|u|^4$. Then the force is given by

$$u_{tt} = -u - u^3,$$

which is now a nonlienar equation. Visually, the difference to the harmonic oscillator is that the frequency of oscillation increases with the amplitude. Suppose now that this system experiences a friction or drag force proportional to the velocity of the particle, but acting against it. Up to constants, we then have the equation

$$u_{tt} = -u_t - u - u^3.$$

Here the amplitude of the oscillation will decay exponentially due to the damping term $-u_t$. Suppose now in addition that for every $x \in \mathbb{R}^d$ we have such a damped nonlinear harmonic oscillator $u(t,x)$, and that these are coupled, meaning that in each point $x$ the particle $u(t,x)$ experiences forces proportional to the height difference to its surrounding particles. This coupling force can be modeled by the laplacian $\Delta u$, so the equation becomes

$$u_{tt} = \Delta u - u_t - u - u^3.$$

Lastly, we add white noise. Informally, let for each $t$ and $x$ let $\xi(t,x)$ be independent standard normal random variables. Then, with a certain constant $\sqrt{2}$, we arrive at the system

$$\begin{aligned} u_{tt} &= \Delta u - u_t - u - u^3 + \sqrt{2}\xi \\ u(0) &= u_0, u_t(0) = u_{t,0}. \end{aligned}$$

The actual definition of $\xi$ is more involved as one wants to avoid working with an uncountable number of independent random variables (in fact, one can not construct the family of random variales $\{\xi(t,x)\}$ on the probability space $([0,1], \mathcal{B}([0,1]), dx)$).

We restrict ourselves on the two-dimensional case

$$\begin{aligned} \partial_t^2 u + \partial_t u + (1-\Delta)u + u^3 &= \sqrt{2}\xi \\ u(0) = u_0, u_t(0) &= u_{t,0} \end{aligned} \qquad \text{on } B = \{x \in \mathbb{R}^2 : |x| \leq 1\}, \tag{1.1}$$

where $u$ is a radially symmetric function with zero boundary values and $\xi$ is radially symmetric space-time white noise. This is now a stochastic nonlinear partial differential equation and the system that we will study in this work. Let us informally present three questions that one may ask when faced with such an SPDE.

(1) **Global Well-posedness**. Do solutions exist globally in time and with continuous dependence on the initial data? What is a natural function space for the initial data so that the problem is globally well-posed but little regularity was assumed?

(2) **Existence of Invariant Measure** Does an invariant measure exist? Let $\Phi(t, \mathbf{u}_0)$ be the solution to the equation with initial data $\mathbf{u}_0 = (u_0, u_{t,0})$. Then, neglecting the presence of random forces for now, a measure $\rho(d\mathbf{u}_0)$ on the space of initial data is invariant if $\Phi(t,\cdot)_{\#}\rho = \rho$ for all $t \geq 0$.

    We can phrase this in a practical manner. Suppose we are observing an experiment which consists of a random initial data $\mathbf{u}_0$ evolving according to the equation. Suppose that at any time $t$ we can perform a number of measurements on the state of the system, each represented by the function $\mathbb{1}_A(u, u_t)$ with $A$ being a (measurable) set of pairs of functions. Then an invariant measure is a distribution for the random initial data such that for any time $t$ there is no difference in the statistics we get when applying any of our possible measurements.

(3) **Ergodicity** There are various ways to state ergodicity. In this setting we say that the evolution induced by the equation is ergodic if

$$\lim_{T\to\infty} \int_0^T \mathbb{E}\left[\mathbb{1}_A(\Phi(t, \mathbf{u}_0))\right] dt = \int \mathbb{1}_A(\mathbf{u})\, d\rho(\mathbf{u})$$

for all initial data $\mathbf{u}_0 = (u_0, u_{t,0})$ and measurable sets of initial data $A$.

We also can interpret this in the practical setting. What this means for us, the experimenter, is that for any measurement $\mathbb{1}_A$ and deterministic initial data $\mathbf{u}_0$ the expected value of the measurement $\mathbb{E}\left[\mathbb{1}_A(\Phi(t,\mathbf{u}_0))\right]$, when time-averaged over a long time span, will converge to the expected result of the measurement if we had sampled a random initial data according to the invariant distribution.

In this work we offer the reader the following:

1. An introduction to stochastic PDEs: How white noise is defined, what it means, and how we can handle it in the equation using the stochastic convolution.
2. A classical local well-posedness argument for the deterministic system, reducing the nonlinear problem to finding a pertubation of the linear solution.
3. A global well-posedness result for system (1.1) using a non-trivial energy estimate.
4. A construction of the Gibbs measure and a proof that it is invariant for the system (1.1), via a reduction to the finite dimensional (SDE) setting.

These results are closely related to the work of L. Tolomeo in [19]. There he shows global well-posedness, existence of invariant measure and ergodicity for the class of equations

$$\begin{aligned} \partial_t^2 u + \partial_t u + (1(-\Delta)^{\frac{s}{2}})u + u^3 &= \sqrt{2}\xi \\ u(0) &= u_0 \end{aligned} \qquad \text{on the torus } \mathcal{T}$$

where $d \in \mathbb{N}$ and $s > d$. Our case is $s = d = 2$ restricted to the ball $B$ and assuming radial symmetry. These assumptions improve the regularity of the white noise in such a way that a procedure called renormalization is not necessary. Further work could aim to transfer the ergodicity argument in [19] over to this setting.

In [17] N. Burq and N. Tzvetkov studied the local existence of strong solutions to the cubic nonlinear wave equation

$$\begin{aligned} u_{tt} - \Delta u + u^3 &= 0 \\ (u(0), \partial_t u(0)) &= (u_0, u_{t,0}) \end{aligned} \tag{1.2}$$

in Sobolev spaces $H^s(M)$ with $s < \frac{1}{2}$, where $M$ is a compact three-dimensional manifold. As the homogeneous sobolev norm $\|\cdot\|_{\dot{H}^s(\mathbb{R}^d))}$ is invariant under the scaling symmetry $u_\lambda(t,x) = \lambda u(\lambda t, \lambda x)$ of (1.2) precisely when $s_{\text{crit}} = \frac{d}{2} - 1$, we call $s_{\text{crit}} = 0$ for $d = 2$ and $s_{\text{crit}} = \frac{1}{2}$ for $d = 3$ the **critical index** of the system. While the local well-posedness works often works well for the subcritical case $s > s_{\text{crit}}$, in the supercritical case $s < s_{\text{crit}}$ the usual methods such as Strichartz estimates fail. In some cases even ill-posedness can be shown ([6]). Burq and Tzvetkov nevertheless manage to show local well-posedness in three dimensions for a "large" set of initial data in the supercritical case. The idea is that they consider the problem for a random initial data $(u_0, u_{t_0})$. This regularizes the problem in a certain sense: Consider the inequality

$$\mathbb{E}\left[|X|^p\right] \lesssim p^{\frac{p}{2}}\, \mathbb{E}\left[|X|^2\right]^{\frac{p}{2}}$$

for a Gaussian random variable $X$ (it is extended to more general probability distributions in [17]). This allows one to improve estimates for the solutions to the random initial data problem which are for $L^2$- or $L^4$-norms to the case of $L^p$-norms, a strategy which Burq uses for his existence result and which we will also in our local well-posedness theory. In our case we show the largest space for which we show well-posedness is a space $\mathcal{X}^0$, which is only slightly smaller than the critical case $\mathcal{H}^0$.

Regarding the global well-posedness, the problem with white noise has recently been studied in [12]. There the authors show global well-posedness of the renormalized cubic stochastic nonlinear wave equation

$$v_{tt} + (1-\Delta)v + v^3 + 3v^2\psi + 3v :\psi^2: + :\psi^3: = 0 \tag{1.3}$$

in $H^s(\mathbb{T}^2)$, where $s > \frac{4}{5}$ and $\mathbb{T}^2$ is the two-dimensional torus. Here $\psi$ is the stochastic convolution, which represents the solution to the wave equation forced by white noise. The regularity of $\psi$ plays a crucial role in these arguments, as it is a priori only a distribution in space. If one can not show that $\psi$ is in fact represented by a measurable function, then one can not make sense of powers of $\psi$ and has to resort to a procedure called "renormalization", which involves additional terms. This is the case in (1.3), where the colons $: - :$ denote the renormalization. In our case, the two-dimensional and radially symmetric one, we will find that $\psi$ is indeed a function and hence we do not have to worry about the process of renormalization. This poses the question if we can achieve a better global well-posedness result than in [12]. Specifically, for the energy

$$E(v, v_t) = \int \frac{1}{2}|v_t|^2 + \frac{1}{2}|v|^2 + \frac{1}{2}|\nabla v|^2 + \frac{1}{4}|v|^4 \, dx,$$

the authors find only a double-exponential energy estimate. The reason here is that when attempting a Grönwall-type argument, they encounter a term which looks something like $\int |v_t||v|^2|\psi| \, dx$. Since we want to arrive at a differential inequality of the type $\frac{d}{dt}E \lesssim E$, we need to estimate this by the energy. Since

$$\int |v_t||v|^2|\psi| \, dx \lesssim E\|\psi\|_{L^\infty}$$

is not good enough due to alack of regularity of $\psi$, one has make an estimate that results in too large a power of the energy being present on the right hand side. The same issue occurs in our case as well, giving evidence that it was not the lack of regularity of $\psi$ that caused the problem, but it is instead related to the dimensionality of the space. In [12] the authors resolve this in a fashion that in spirit is very similar to the expression below:

$$\int |v_t||v|^2|\psi| \, dx \lesssim E^{1+\frac{1}{p}}\|\psi\|_{L^p}.$$

The above is in fact precisely how we will resolve this problem. After a smart choice of $p$ one obtains the differential inequality $\frac{d}{dt}E \lesssim E \ln E$, which then leads to the double-exponential energy estimate.

Besides the global well-posedness, we show the invariance of the Gibbs measure

$$\mu = \exp\left(-E(u, u_t)\right) \text{“} \, du \, du_t \text{”},$$

which of course has to be rigorously defined on an infinite dimensional function space. Important and early contributions in this area were made by J. Bourgain in [3] and [23], where he showed the invariance of the respective Gibbs measures for nonlinear Schrödinger equations. Notably he introduced what is now known as "Bourgain's invariant measure argument", a method that exploits the invariant measure as a replacement for conservation of the energy to gain global existence of solutions. For a great overview of the literature regarding the questions of global existence and invariant measures for cubic nolinear wave and Schrödinger equations, we recommend figure 1 in [5].

In the cases where the invariance of the Gibbs measure fails, one may ask the question if the weaker condition of quasi-invariance holds. Quasi-invariance means that as the initial distributions transforms under the flow induced by the equation, the measure at any positive time remains absolutely continuous with respect to the initial measure. We refer the reader to [10] for a recent result.

## 1.2 Function Spaces and Notation

We will study this equation in Bessel potential spaces of radially symmetric functions. In this section we construct these spaces as subspaces of the space of distributions and define the specific notations used in this text.

Let $D \subseteq \mathbb{R}^d$ be the closure of a non-empty open set. For $1 \leq p \leq \infty$ we define

$$L^p_{\mathrm{r}}(D) = \{f \in L^p(D) : |x| = |y| \implies f(x) = f(y) \text{ for almost all } x, y \in D\}$$

We may write just $L^p_{\mathrm{r}}$ or $L^p$ for $L^p_{\mathrm{r}}(D)$. Generally for any function space $X$ the space notation $X_{\mathrm{r}}$ refers to the corresponding subspace of radially symmetric functions or distributions. If we are considering functions in time and space, we may write $L^p_{\mathrm{r}}([0,T] \times D)$ or $L^p_{t,x}$, in which case we mean $L^p([0,T], L^p_{\mathrm{r}}(D))$. The same holds for the case of distributions below. When writing the respective norms we will usually write $\|\cdot\|_{L^p}$ instead of $\|\cdot\|_{L^p_{\mathrm{r}}}$.

We denote the spaces of **test functions** and **radially symmetric test functions** on $D$ by $\mathcal{D}(D)$ and

$$\mathcal{D}_{\mathrm{r}}(D) \coloneqq \{f \in \mathcal{D}(D) : |x| = |y| \implies f(x) = f(y)\}$$

respectively. We denote the spaces of **distributions** and **radially symmetric distributions** on $D$ by $\mathcal{D}'(D)$ and $\mathcal{D}'{}_{\mathrm{r}}(D)$. They are the spaces of continuous linear functionals on the corresponding space of test functions respectively, equipped with the usual topology. We denote the **Schwartz space** by $\mathcal{S}_{\mathrm{r}}(\mathbb{R}^d)$ and its subspace of radially symmetric functions by $\mathcal{S}_{\mathrm{r}}(\mathbb{R}^d)$. Correspondingly we write $\mathcal{S}(\mathbb{R}^d)'$ and $\mathcal{S}'_{\mathrm{r}}(\mathbb{R}^d)$ for the spaces of tempered distributions and radially symmetric tempered distributions.

On a domain $D$ we define $\mathcal{S}(D) = \mathcal{D}(D) \cap \mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}_{\mathrm{r}}(D) = \mathcal{D}_{\mathrm{r}}(D) \cap \mathcal{S}_{\mathrm{r}}(\mathbb{R}^d)$. We define $\mathcal{S}'(D)$ and $\mathcal{S}'_{\mathrm{r}}(D)$ as the topological dual spaces. Later we will only deal with the case where $D = B$ is a bounded domain and so will be able to use $\mathcal{D}$ and $\mathcal{S}$ interchangeably, choosing $\mathcal{D}$ by default.

For some $1 < p, p' < \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$ we can consider the subspace of $\mathcal{S}'(D)$ given by the norm

$$\|f\|_{L^p(D)} = \sup_{\substack{g \in \mathcal{D}(D) \\ \|g\|_{L^{p'}(D)} \leq 1}} |\langle f, g\rangle|.$$

What we are doing here is simply identifying the dual space $(L^{p'})^*(D)$ with $L^p(D)$. We use the same approach to define the Bessel potential spaces $H^{\alpha,p}(\mathbb{R}^d)$ for $\alpha \in \mathbb{R}$. For this we need a definition of the fractional operator $(1-\Delta)^{\frac{\alpha}{2}}$ on test functions. Let $\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \longrightarrow \mathcal{S}(\mathbb{R}^d)$ be the Fourier transform given by

$$\mathcal{F}(f)(\xi) \coloneqq (4\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(x) e^{-i\xi x}\, dx.$$

It has the important property that

$$-\Delta f = \mathcal{F}^{-1}(|\xi|^2 \mathcal{F}(f)).$$

We therefore define for $\alpha \in \mathbb{R}$ the **Bessel potential on $\mathbb{R}^d$ of order $-\alpha$** by

$$(1-\Delta)^{\frac{\alpha}{2}}_{\mathbb{R}^d} f = \mathcal{F}^{-1}((1+|\xi|^2)^{\frac{\alpha}{2}} \mathcal{F}(f))$$

This definition makes sense not only for $f \in \mathcal{S}(\mathbb{R}^d)$, but also $f \in \mathcal{S}'(\mathbb{R}^d)$. In that case we have

$$\langle \langle \nabla \rangle^{\alpha}_{\mathbb{R}^d} f, g \rangle = \langle f, \langle \nabla \rangle^{\alpha} g \rangle$$

for all $g \in \mathcal{S}(\mathbb{R}^d)$. For $\lambda \in \mathbb{R}$ we use the shorthand notations

$$\langle \lambda \rangle \coloneqq \sqrt{1+|\lambda|^2}$$
$$[\lambda] \coloneqq \sqrt{\frac{3}{4}+|\lambda|^2},$$

which allow us to define $\langle \nabla \rangle^{\alpha}_{\mathbb{R}^d} \coloneqq (1-\Delta)^{\frac{\alpha}{2}}_{\mathbb{R}^d}$. Now let $\alpha \in \mathbb{R}$ and $1 < p < \infty$. we define the **Bessel potential space** $H^{\alpha,p}(\mathbb{R}^d)$ as the subspace of $\mathcal{S}(\mathbb{R}^d)$ given by the norm

$$\|f\|_{H^{\alpha,p}(\mathbb{R}^d)} \coloneqq \|\langle \nabla \rangle^{\alpha}_{\mathbb{R}^d} f\|_{L^p(\mathbb{R}^d)} = \sup_{\substack{g \in \mathcal{S}(\mathbb{R}^d) \\ \|g\|_{L^{p'}(\mathbb{R}^d)} \le 1}} |\langle f, \langle \nabla \rangle^{\alpha}_{\mathbb{R}^d} g \rangle|.$$

We define $H^{\alpha,p}_{\mathrm{r}} \coloneqq H^{\alpha,p}(\mathbb{R}^d) \cap \mathcal{S}'_{\mathrm{r}}(\mathbb{R}^d)$ and note that this is a closed subspace.

Note that the map

$$\langle \nabla \rangle^{\beta}_{\mathbb{R}^d} : H^{\alpha,p}(\mathbb{R}^d) \longrightarrow H^{\alpha-\beta,p}(\mathbb{R}^d)$$

is an isometry for all $\alpha, \beta \in \mathbb{R}$.

Let us now compare this to some other function spaces. It is well-known that if $\alpha \geq 0$ then $H^{\alpha,p}(\mathbb{R}^d) \subseteq L^p(\mathbb{R}^d)$. In fact, the Bessel potential spaces can be seen as a definition of fractional Sobolev spaces: if $\alpha = k$ is an integer then $H^{k,p}(\mathbb{R}^d) = W^{k,p}(\mathbb{R}^d)$ ([18, Thm. 3]), where $W^{k,p}(\mathbb{R}^d)$ are the classical Sobolev spaces.

There is an alternative way to define fractional sobolev spaces. It is the case that $H^{\alpha,p}(\mathbb{R}^d) = F^{p,2}_{\alpha}(\mathbb{R}^d)$, where $F^{p,q}_{\alpha}$ is the Triebel-Lizorkin scale of function spaces which the reader may read up on in [21]. We will not need these function spaces but want to make it clear that the Bessel potential spaces $H^{\alpha,p}(\mathbb{R}^d)$ should not be confused with an alternative scale of fractional sobolev spaces which we denote by $\overline{W}^{\alpha,p}(\mathbb{R}^d)$, where for non-integer $\alpha$ one chooses the so called

Sobolev-Slobodeckij space. Here we have $\overline{W}^{\alpha,p}(\mathbb{R}^d) = F^{p,p}_{\alpha}(\mathbb{R}^d)$. In the case $p = 2$ the two definitions agree, i.e. $H^{\alpha,2}(\mathbb{R}^d) = \overline{W}^{\alpha,2}(\mathbb{R}^d)$. We write $H^{\alpha}(\mathbb{R}^d) := H^{\alpha,2}(\mathbb{R}^d)$ for this special case.

A further way to describe this difference is that the Bessel potential spaces are the **complex** interpolation spaces and the Sobolev-Slobodeckij spaces are the **real** interpolation spaces between the integer sobolev spaces $W^{k,p}(\mathbb{R}^d)$. More on this can be found in [14], specifically theorem 6.4.5.

We now restrict ourselves to the case of radially symmetric functions on the two-dimensional ball $B = \{x \in \mathbb{R}^2 : |x| \leq 1\}$. We define

$$H^{\alpha,p}_{\mathrm{r},0}(B) = \overline{\mathcal{D}_{\mathrm{r}}(B)}^{\|\cdot\|_{H^{\alpha,p}_{\mathrm{r}}(\mathbb{R}^2)}}.$$

Note that we have an isomorphism

$$H^{\alpha,p}_{\mathrm{r},0}(B) \cong (H^{-\alpha,p'}_{\mathrm{r},0}(B))^* \tag{1.4}$$

by defining for $f \in H^{\alpha,p}_{\mathrm{r},0}(B)$ and $g \in H^{-\alpha,p'}_{\mathrm{r},0}(B)$ the dual pairing

$$\langle f, g\rangle = \langle \langle\nabla\rangle^{\alpha}_{\mathbb{R}^2} f, \langle\nabla\rangle^{-\alpha}_{\mathbb{R}^2} g\rangle,$$

where now $\langle\nabla\rangle^{\alpha}_{\mathbb{R}^2} f \in L^p_{\mathrm{r}}(B)$ and $\langle\nabla\rangle^{-\alpha}_{\mathbb{R}^2} g \in L^{p'}_{\mathrm{r}}(B)$. Then by Hölder's inequality

$$|\langle f, g\rangle| \leq \|\langle\nabla\rangle^{\alpha}_{\mathbb{R}^2} f\|_{L^p(B)} \|\langle\nabla\rangle^{-\alpha}_{\mathbb{R}^2} g\|_{L^{p'}(B)} = \|f\|_{H^{\alpha,p}_{\mathrm{r},0}(B)} \|g\|_{H^{-\alpha,p'}_{\mathrm{r},0}(B)},$$

implying that we can identify $H^{\alpha,p}_{\mathrm{r},0}(B)$ with $(H^{-\alpha,p'}_{\mathrm{r},0}(B))^*$. This means that if $\alpha \geq 0$ the expression $\langle f, g\rangle$ is well-defined for any $g \in L^p_{\mathrm{r}}(B)$.

We will now define an alternative scale of fractional Sobolev spaces $W^{\alpha,p}_{\mathrm{r}}$ specifically on the ball $B$ and then show that $W^{\alpha,p}_{\mathrm{r}} \cong H^{\alpha,p}_{\mathrm{r},0}(B)$. For this we need a special basis of $L^2_{\mathrm{r}}(B)$. Here we follow the same approach as Tzvetkov does in [24, Section 1], using a basis of rescalings of the zero order Bessel function of first kind:

$$J_0(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{z}{2}\right)^{2m} = \frac{2}{\pi} \frac{\cos(x - \pi/4)}{\sqrt{x}} + \mathcal{O}(x^{-\frac{3}{2}}). \tag{1.5}$$

The corresponding transform is sometimes also called the Hankel transform. Skipping the details, we are given an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $L^2_{\mathrm{r}}(B)$ consisting of smooth functions

$$e_n(x) = \frac{J_0(\lambda_n|x|)}{\|J_0(\lambda_n\cdot)\|_{L^2(B)}}. \tag{1.6}$$

Crucially, the $e_n$ are eigenfunctions of $-\Delta$ with corresponding eigenvalues

$$0 < |\lambda_1|^2 < |\lambda_2|^2 < \dots$$

and zero boundary values. We say that $|\lambda_n|^2$ is the $n$-th eigenvalue because this corresponds to $\lambda_n$ being the $n$-th positive root of $z \mapsto J_0(z)$, the zero order Bessel function. It also keeps our notation consistent with the usage of a Fourier basis on the torus, as instances of $|n|$ are now replaced with $|\lambda_n|$.

For two sequences $a_n, b_n \in \mathbb{R}_+$ we write $a_n \sim b_n$ if both $\frac{a_n}{b_n}$ and $\frac{b_n}{a_n}$ are bounded, and $a_n \approx b_n$ if $\lim \frac{b_n}{a_n} = \lim \frac{b_n}{a_n} = 1$.

Note that $|\lambda_n| \sim n$ for large $n$ ([24, Section 1.2]). We have

$$\big[\lambda_n\big] \approx \langle \lambda_n \rangle \sim n. \tag{1.7}$$

We call this basis $(e_n)_{n\in\mathbb{N}}$ the **Bessel function basis**. We define the **Bessel potential on** $B$ **of order** $-\alpha$ by

$$\langle \nabla \rangle_B^\alpha f := (1-\Delta)_B^{\frac{\alpha}{2}} f := \sum_{n=1}^{\infty} \langle \lambda_n \rangle^\alpha \langle f, e_n \rangle e_n.$$

where $f \in \mathcal{D}_{\mathrm{r}}(B)$. This definition can be extended to $f \in \mathcal{D}'_{\mathrm{r}}(B)$, in which case

$$\langle \langle \nabla \rangle_B^\alpha f, g \rangle = \langle f, \langle \nabla \rangle_B^\alpha g \rangle$$

for all $g \in \mathcal{D}_{\mathrm{r}}(B)$. Now we define $W_{\mathrm{r}}^{\alpha,p}$ to be the subspace of $\mathcal{D}_{\mathrm{r}}(B)$ corresponding to the norm

$$\|f\|_{W_{\mathrm{r}}^{\alpha,p}} = \|\langle \nabla \rangle_B^\alpha f\|_{L_{\mathrm{r}}^p} = \sup_{\substack{g\in\mathcal{S}_{\mathrm{r}}(B) \\ \|g\|_{L^{p'}(B)} \le 1}} |\langle f, \langle \nabla \rangle_B^\alpha g \rangle|.$$

As the radially symmetric test functions on $B$ are dense in this space, we can write

$$W_{\mathrm{r}}^{\alpha,p} = \overline{\mathcal{D}_{\mathrm{r}}(B)}^{\|\cdot\|_{W_{\mathrm{r}}^{\alpha,p}}}.$$

Recall that in contrast to the above, we defined

$$H_{\mathrm{r},0}^{\alpha,p}(B) = \overline{\mathcal{D}_{\mathrm{r}}(B)}^{\|\cdot\|_{H_{\mathrm{r}}^{\alpha,p}(\mathbb{R}^2)}},$$

where for $f \in \mathcal{D}'_{\mathrm{r}}(B)$ we have

$$\|f\|_{H_{\mathrm{r}}^{\alpha,p}(\mathbb{R}^2)} = \|\langle \nabla \rangle_{\mathbb{R}^2}^\alpha f\|_{L_{\mathrm{r}}^p} = \sup_{\substack{g\in\mathcal{D}(B) \\ \|g\|_{L^{p'}(B)} \le 1}} |\langle f, \langle \nabla \rangle_{\mathbb{R}^2}^\alpha g \rangle|.$$

We would therefore know that $W_{\mathrm{r}}^{\alpha,p} = H_{\mathrm{r},0}^{\alpha,p}(B)$ if we knew that

$$\langle \nabla \rangle_B g = \langle \nabla \rangle_{\mathbb{R}^2} g$$

any test function $g \in \mathcal{D}_{\mathrm{r}}(B)$. This is in general not the case!. However, if $\alpha \ge N$ is an even integer, then both $(1-\Delta)_B^{\frac{\alpha}{2}}$ and $(1-\Delta)_{\mathbb{R}^2}^{\frac{\alpha}{2}}$ become ordinary powers of $(1-\Delta)$ and hence agree. Therefore $W_{\mathrm{r}}^{\alpha,p} = H_{\mathrm{r},0}^{\alpha,p}(B)$ holds for all $\alpha \in 2\,\mathbb{N}$.

For $\alpha \ge 0$ we can now recover the spaces $H_{\mathrm{r},0}^{\alpha,p}(B)$ and $W_{\mathrm{r}}^{\alpha,p}$ as complex interpolation spaces of those spaces on their respective scales with even integer index, hence $W_{\mathrm{r}}^{\alpha,p} = H_{\mathrm{r},0}^{\alpha,p}(B)$ for all $\alpha \ge 0$. We finally extend this to $\alpha < 0$ by the duality in (1.4) and the analogous duality result for $W_{\mathrm{r}}^{\alpha,p}$.

We are mostly interested in the space $H^{\alpha}_{\mathrm{r},0} \coloneqq H^{\alpha,2}_{\mathrm{r},0}(B)$, whose norm has the formula

$$\|f\|_{H^{\alpha}_{\mathrm{r},0}(B)} = \sum_{n=1}^{\infty} \langle \lambda_n \rangle^{2\alpha} |\langle f, e_n \rangle|^2.$$

If $f : L^2_{\mathrm{r}(B)} \longrightarrow \mathbb{R}$ is linear and the right hand side above is finite, then $f \in H^{\alpha,2}_{\mathrm{r},0}$.

We now define some new and simplified notations for the objects we really care about. We set

$$H^{\alpha}_{\mathrm{r}} \coloneqq W^{\alpha,2}_{\mathrm{r}} \cong H^{\alpha}_{\mathrm{r},0}(B) = H^{\alpha,2}_{\mathrm{r},0}(B).$$

It is a Hilbert space with the inner product

$$\langle f, g \rangle_{H^{\alpha}_{\mathrm{r}}} = \sum_{n=1}^{\infty} \langle \lambda_n \rangle^{2\alpha} \langle f, e_n \rangle \langle g, e_n \rangle.$$

We also define the notation $\hat{f}(n) \coloneqq \langle f, e_n \rangle$ and sometimes write $\mathcal{F}(f)$ for $\hat{f}$, using the symbol for the Fourier transform suggestively and intentionally.

We define

$$H^{-\infty}_{\mathrm{r}} = \bigcup_{\alpha \in \mathbb{R}} H^{\alpha}_{\mathrm{r}} \qquad \text{and} \qquad H^{\infty}_{\mathrm{r}} = \bigcap_{\alpha \in \mathbb{R}} H^{\alpha}_{\mathrm{r}}.$$

We consider these merely as sets and not spaces, unless otherwise specified.

Furthermore, we define

$$\mathcal{W}^{\alpha,p}_{\mathrm{r}} = W^{\alpha,p}_{\mathrm{r}} \times W^{\alpha-1,p}_{\mathrm{r}} \text{ and } \mathcal{H}^{\alpha}_{\mathrm{r}} = H^{\alpha}_{\mathrm{r}} \times H^{\alpha-1}_{\mathrm{r}}.$$

We will use bold letters to refer to pairs of distributions $\mathbf{u} \in \mathcal{D}^2_{\mathrm{r}}(B)$ and write the individual components in any of the following ways:

$$\mathbf{u} = (u, u_t) = (\pi \mathbf{u}, \pi_t \mathbf{u}) = (u_1, u_2) = (\pi_1 \mathbf{u}, \pi_2 \mathbf{u}).$$

It should be said that $u_t$ **is not necessarily the time derivative** of $u$ here. Time derivatives will often but not always be denoted by $\partial_t$ or some similar notation. We only use the suggestive notation $u_t$ since the second component of a pair of functions will often be precisely the time derivative of the first, i.e. $\partial_t u = u_t$ holds.

We will remember to write $W^{\alpha,p}_{\mathrm{r}}$, $L^p_{\mathrm{r}}$ etc. but when writing their respective norms we will most often just write $\|\cdot\|_{W^{\alpha,p}}$, $\|\cdot\|_{L^p}$ etc.

We also define for an arbitrary Banach space $(E, \|\cdot\|)$ the space $L^{\exp}([0,\infty), E)$ of measurable functions $f : [0,\infty) \longrightarrow E$ with exponential decay by the norm

$$\|f\|_{L^{\exp}([0,\infty),E)} = \sup_{t \geq 0} e^{\frac{t}{2}} \|f(t)\|_E.$$

This will be a convenient substitute for $L^{\infty}([0,\infty), E)$ because exponential decay is often present in our damped setting. Note that

$$\|f\|_{L^p([0,\infty),E)} \leq \|f\|_{L^{\exp}([0,\infty),E)} \|e^{-\frac{t}{2}}\|_{L^p([0,\infty))} = \left(\frac{2}{p}\right)^{\frac{1}{p}} \|f\|_{L^{\exp}([0,\infty),E)}. \tag{1.8}$$

### 1.3 Space-time White Noise

The term $\xi$ in (1.1) refers to space-time white noise. We now define and construct this object. We fix some measure space $(\Omega, \mathcal{F}, \mathbb{P})$.

**Definition 1.1.** *(i) A* ***random radially symmetric space-time distribution*** *$\xi$ is a continuous linear map*

$$\xi : \mathcal{D}_{\mathrm{r}}(\mathbb{R}_+ \times B) \longrightarrow L^2(\Omega).$$

*We analogously define a* ***random radially symmetric space distribution*** *and* ***random time distribution****. We may also consider vectors of random distributions where the test functions are in $\mathcal{D}_{\mathrm{r}}\left(\bigsqcup^k(\mathbb{R}_+ \times B)\right) \cong \mathcal{D}_{\mathrm{r}}(\mathbb{R}_+ \times B)^k$ for $k \in \mathbb{N}$.*

*(ii) A random radially symmetric space-time distribution $\xi$ is called* ***radially symmetric space-time white noise*** *if $\langle \xi, f \rangle$ is a centered Gaussian random variable and*

$$\mathbb{E}\left[\langle \xi, f \rangle \langle \xi, g \rangle\right] = \langle f, g \rangle_{L^2_{t,x}}$$

*for all $f, g \in \mathcal{D}_{\mathrm{r}}(\mathbb{R}_+ \times B)$.*

*(iii) Let $W$ be a one-dimensional Brownian motion. We define a random time distribution $dW$. For $f \in \mathcal{D}(\mathbb{R})$, set*

$$\langle dW, f \rangle := \int_{\mathbb{R}_+} f(s)\, dW(s).$$

*This is a Wiener integral, an object which is not the focus of this text. A definition of this integral in the infinite dimensional case can be found in [8]. Since $f$ is smooth and has compact support, we can choose a simple pathwise definition for this object here:*

$$\int_{\mathbb{R}_+} f(s)\, dW(s) := - \int_{\mathbb{R}_+} W(s) \dot{f}(s)\, ds.$$

*In Lemma 1.3 we will use further properties of the Wiener integral that require its full definition, but we do not want to elaborate on this.*

**Lemma 1.2** (Extension onto $L^2(\mathbb{R}_+ \times B)$)**.** *There exists a unique estension of $\xi$ to a bounded linear operator $\xi : L^2_{\mathrm{r}}(\mathbb{R}_+ \times B) \longrightarrow L^2(\Omega)$.*

*Proof.* Let $\phi_n \in \mathcal{D}_{\mathrm{r}}(\mathbb{R}_+ \times B)$ be a Cauchy sequence in $L^2_{\mathrm{r}}(\mathbb{R}_+ \times B)$ with limit $\phi \in L^2_{\mathrm{r}}(\mathbb{R}_+ \times B)$. Then

$$\mathbb{E}\left[|\langle \xi, \phi_n - \phi_m \rangle|^2\right] = \|\phi_n - \phi_m\|^2_{L^2},$$

and so $\langle \xi, \phi_n \rangle$ is a Cauchy sequence in $L^2(\Omega)$. We define $\langle \xi, \phi \rangle$ as the limit. From the construction it immediately follows that this is a continuous function. Since the test functions are dense in $L^2$, the linearity is inherited and we have a bounded linear operator. Since it is uniquely defined on the dense subset of test functions, it is unique. □

We will identify $\xi$ with this extension from now on.

**Lemma 1.3** (Construction of space-time white noise)**.** *Let $e_n$ be any ONB of $L^2_{\mathrm{r}}$ and let $\xi$ be a random radially symmetric space-time distribution. Then the following are equivalent:*

*(i)* $\xi$ *is a radially symmetric space-time white noise on* $\mathbb{R}_+ \times B$.

*(ii) There exist independent, one-dimensional Brownian motions* $(W_n)_{n\in\mathbb{N}}$ *so that almost surely*

$$\langle \xi, f\rangle = \sum_{n=1}^{\infty} \langle dW_n, \langle f, e_n\rangle_{L^2_x}\rangle = \sum_{n=1}^{\infty} \int_{\mathbb{R}_+}\int_B f(t,x)e_n(x)\,dx\,dW_n(t).$$

*for all* $f \in \mathcal{D}_{\mathrm{r}}(\mathbb{R}_+ \times B)$. *The Brownian motions* $W_n(t)$ *are modifications of* $\langle \xi, \mathbb{1}_{[0,t]}(s)e_n(x)\rangle$.

*Proof.* We first show $(ii) \Longrightarrow (i)$. Let $(W_n)_{n\in\mathbb{N}}$ be a sequence of independent one-dimensional Brownian motions and let $f \in \mathcal{D}_{\mathrm{r}}(\mathbb{R}_+ \times B)$.

Note that $\int_B f(t,x)e_n(x)\,dx$ is a test function in time and hence admissible for $dW_n$. For $N \leq M \in \mathbb{N}$ and any other test function $g$, we calculate

$$\mathbb{E}\left[\sum_{n=N}^{M}\int_{\mathbb{R}_+}\int_B f(t,x)e_n(x)\,dx\,dW_n(t)\sum_{k=N}^{M}\int_{\mathbb{R}_+}\int_B g(t,x)e_k(x)\,dx\,dW_k(t)\right]$$

$$= \sum_{\substack{k,n=N\\k\neq n}}^{M} \mathbb{E}\left[\int_{\mathbb{R}_+}\int_B f(t,x)e_n(x)\,dx\,dW_n(t)\right]\mathbb{E}\left[\int_{\mathbb{R}_+}\int_B g(t,x)e_k(x)\,dx\,dW_k(t)\right]$$

$$+ \sum_{n=N}^{M}\mathbb{E}\left[\int_{\mathbb{R}_+}\int_B f(t,x)e_k(x)\,dx\,dW_k(t)\int_{\mathbb{R}_+}\int_B g(t,x)e_k(x)\,dx\,dW_k(t)\right]$$

The first sum vanishes since we are taking the expectations of local martingales starting in zero. For the second sum we use Itô isometry to get

$$= \sum_{n=N}^{M}\mathbb{E}\left[\int_{\mathbb{R}_+}\int_B f(t,x)e_n(x)\,dx\cdot\int_B g(t,x)e_n(x)\,dx\,dt\right]$$

$$= \int_{\mathbb{R}_+}\sum_{n=N}^{M}\widehat{f}(t,n)\widehat{g}(t,n)\,dt.$$

Setting $f = g$ we can follow from this that we indeed have a Cauchy sequence in $L^2(\Omega)$. As each element in the sequence depends linearly on $f$, the limit is also linear. Hence $\xi$ is a random space-time distribution. Taking $M \nearrow \infty$ and $N = 1$ on both sides of the computation, we see that

$$\mathbb{E}\left[\langle \xi, f\rangle\langle \xi, g\rangle\right] = \int_{\mathbb{R}_+}\langle f, g\rangle_{L^2_x}\,dt = \langle f, g\rangle_{L^2_{t,x}}.$$

We also know that time changes of Brownian Motion yield centered Gaussian random variables and hence the limit is also centered Gaussian. Therefore $\xi$ is space-time white noise.

Now we show $(i) \Longrightarrow (ii)$. Define $W_n(t) := \langle \xi, \mathbb{1}_{[0,t]}(s)e_n(x)\rangle_{s,x}$. We will show that $W_n$ has a modification $\tilde{W}_n$ which is a Brownian motion with the desired property.

Clearly $W_n(0) = 0$. The definition of radial space-time white noise also implies that for all $t_1 < \cdots < t_k$, all $W_n(t_{j+1}) - W_n(t_j) = \langle \xi, \mathbb{1}_{[t_{j+1},t_j]}(s)e_n(x)\rangle_{s,x}$ are a centered Gaussian random variable with variances $t_{j+1} - t_j$, and that they are pairwise uncorrelated, hence independent.

We have

$$\mathbb{E}\left[|W_n(t_1) - W_n(t_0)|^2\right] = \|\mathbb{1}_{[t_0,t_1]}\|_{L^2}^2 = |t_1 - t_0|^2,$$

so by the Kolmorogov continuity theorem there exists a continuous modification $\tilde{W}_n$ of $W_n$. This then fulfills the definition of a Brownian motion.

Now let $f \in L^2([0,\infty),\mathbb{R})$ be a left-continuous simple function in time, meaning that

$$f(t,x) = \sum_{j=1}^{k} \mathbb{1}_{(t_j,t_{j+1}]}(t) g_j(x).$$

Then

$$\begin{aligned}
\langle \xi, f\rangle &= \sum_{j=1}^{k} \left\langle \xi, \mathbb{1}_{(t_j,t_{j+1}]}(t) \sum_{n=1}^{\infty} e_n(x)\langle g_j, e_n\rangle \right\rangle_{t,x} \\
&= \sum_{j=1}^{k}\sum_{n=1}^{\infty} \left\langle \xi, \mathbb{1}_{(t_j,t_{j+1}]}(t) e_n(x)\right\rangle_{t,x} \langle g_j, e_n\rangle \\
&= \sum_{n=1}^{\infty}\sum_{j=1}^{k} (W_n(t_{j+1}) - W_n(t_j))\langle g_j, e_n\rangle \\
&\overset{\text{a.s.}}{=} \sum_{n=1}^{\infty}\sum_{j=1}^{k} (\tilde{W}_n(t_{j+1}) - \tilde{W}_n(t_j))\langle f(t_{j+1}), e_n\rangle \\
&= \sum_{n=1}^{\infty} \int_{\mathbb{R}_+} \int_B f(t,x) e_n(x)\, dx\, d\tilde{W}_n(t).
\end{aligned}$$

This identity can now be concluded for all $f \in L^2_{\mathrm{r}}(\mathbb{R}_+ \times B)$ since both the space-time white noise and the stocastic integral are continuous from $L^2_{\mathrm{r}}(\mathbb{R}_+ \times B)$ to $L^2(\Omega)$ and these simple functions in time are dense. □

## 1.4 On the Matter of Measurability

Regarding measurability we will take a lenient approach and consider various maps as random variables without explicitly proving their measurability. Let it be said that by default we always use the Borel $\sigma$-algebra of whatever Polish space we are considering, and that in the case of separable Banach spaces, which almost all of the spaces we consider are, the Borel $\sigma$-algebras induced by the strong and weak topologies coincide. This makes it easier to prove the measurability of the solutions to various problems we will find throughout this work, which is a priori not a trivial matter.

Note also that we prove in Lemma 3.5 that the embeddings between various function spaces we consider are bimeasurable, meaning that the $\sigma$-algebras of $\mathcal{H}^0_{\mathrm{r}}$ and $\mathcal{H}^1_{\mathrm{r}}$ for example are, in a sense, identical.

# 2 Local Well-posedness

From now on $(e_n)_{n\in\mathbb{N}}$ refers explicitly to the Bessel function basis unless otherwise specified. Our strategy to prove local well-posedness is to decompose the problem into three easier problems.

1. For an data $\mathbf{w}_0$ in a certain space $\mathcal{X}^\alpha$ we solve the linear problem

$$\begin{aligned} \partial_t^2 w + \partial_t w + (1-\Delta)w &= 0 \\ w(0) &= w_0 \\ \partial_t w(0) &= w_{t,0} \end{aligned} \tag{2.1}$$

   with a possibly random initial data.

2. For the space-time white noise $\xi$ we construct a mild solution to the linear problem with inhomogeneity $\sqrt{2}\xi$:

$$\begin{aligned} \partial_t^2 \psi + \partial_t \psi + (1-\Delta)\psi &= \sqrt{2}\xi \\ \psi(0) &= 0 \\ \partial_t \psi(0) &= 0. \end{aligned} \tag{2.2}$$

   This is called the stochastic convolution.

3. Given $w$ and $\psi$, we now solve the deterministic and homogeneous nonlinear problem

$$\begin{aligned} \partial_t^2 v + \partial_t v + (1-\Delta)v + (w+\psi+v)^3 &= 0 \\ v(0) &= 0 \\ \partial_t v(0) &= 0. \end{aligned} \tag{2.3}$$

   Problem (2.1) will be solved in a rather strong sense and for (2.2) and (2.3) we will find so-called **mild solutions**. Then $u = w + \psi + v$ will almost surely be a solution to (1.1) in the sense of distributions.

   **Definition 2.1** (Distributional solution)**.** *Let $T \in \mathbb{R}_+ \cup \{\infty\}$. We say that a measurable function $\mathbf{u} = (u, u_t) : [0,\infty) \longrightarrow \mathcal{H}^0_{\mathrm{r}}$ is a distributional solution to (1.1) on $[0,T]$ with initial data $\mathbf{u}_0 = (u_0, u_{t,0}) \in \mathcal{H}^0_{\mathrm{r}}$ if*

$$\langle u_0, f_t(0) - f(0)\rangle_x + \langle u_{t,0}, -f(0)\rangle + \langle u, f_{tt} - f_t + (1-\Delta)f\rangle_{t,x} + \langle u^3, f\rangle_{t,x} = \langle \sqrt{2}\xi, f\rangle_{t,x}$$

   *for all $f \in \mathcal{D}_{\mathrm{r}}([0,T) \times B)$.*

## 2.1 The Linear Problem with Random Initial Data

In this section we compute and analyze the solution operator $S$ for the linear wave equation (2.1).

We view the equation as a Hilbert space valued ODE of degree 1 by considering pairs of functions. The equation becomes

$$\begin{aligned} \partial_t \mathbf{w} &= L\mathbf{w} \\ \mathbf{w}(0) &= \mathbf{w}_0 \end{aligned}$$

where

$$L = \begin{pmatrix} 0 & 1 \\ -(1-\Delta) & -1 \end{pmatrix}.$$

Then formally the solutions should be given by $\mathbf{w}(t) = S(t)\mathbf{w}_0$, where

$$S(t) = \exp(tL).$$

is the solution operator. One could use a functional calculus to define this operator exponential but we will simply construct it and prove the desired properties.
The operator $L$ is well-behaved under conjugation with our "Fourier transform" $\mathcal{F}$ and so we can define $S(t)$ in the following fashion:

$$\widehat{S(t)\mathbf{w}_0}(n) := \exp\left(t\begin{pmatrix} 0 & 1 \\ -1-|\lambda_n|^2 & -1 \end{pmatrix}\right)\widehat{\mathbf{w}_0}(n).$$

We therefore have to only compute a matrix exponential. For $c > 0$ we make the ansatz

$$\begin{pmatrix} 0 & 1 \\ -1-c & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ a_1 & a_2 \end{pmatrix}\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ a_1 & a_2 \end{pmatrix}^{-1}.$$

As

$$\det\begin{pmatrix} -a_{1,2} & 1 \\ -1-c & -1-a_{1,2} \end{pmatrix} = 0 \iff a_{1,2}^2 + a_{1,2} + 1 + c = 0$$

we see that

$$a_{1,2} = -\frac{1}{2} \pm i\sqrt{c+\frac{3}{4}}$$

is the correct choice. We get

$$\begin{aligned}
\exp\left(t\begin{pmatrix} 0 & 1 \\ -1-c & -1 \end{pmatrix}\right) &= \begin{pmatrix} 1 & 1 \\ a_1 & a_2 \end{pmatrix}\begin{pmatrix} e^{ta_1} & 0 \\ 0 & e^{ta_2} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ a_1 & a_2 \end{pmatrix}^{-1} \\
&= \frac{1}{a_2-a_1}\begin{pmatrix} e^{ta_1} & e^{ta_2} \\ a_1e^{ta_1} & a_2e^{ta_2} \end{pmatrix}\begin{pmatrix} a_2 & -1 \\ -a_1 & 1 \end{pmatrix} \\
&= \frac{1}{a_2-a_1}\begin{pmatrix} e^{ta_1}a_2 - e^{ta_2}a_1 & e^{ta_2} - e^{ta_1} \\ e^{ta_1}a_1a_2 - e^{ta_2}a_1a_2 & e^{ta_2}a_2 - e^{ta_1}a_1 \end{pmatrix}.
\end{aligned}$$

Setting $b = \sqrt{c+\frac{3}{4}} = [\sqrt{c}]$, the above is equal to

$$\begin{aligned}
&\frac{e^{-t\frac{1}{2}}}{-2ib}\begin{pmatrix} i\frac{e^{-itb}-e^{itb}}{2i} - 2ib\frac{e^{itb}+e^{-itb}}{2} & -2i\frac{e^{itb}-e^{-itb}}{2i} \\ 2i(c+1)\frac{e^{itb}-e^{-itb}}{2i} & i\frac{e^{itb}-e^{-itb}}{2i} - 2ib\frac{e^{-itb}+e^{itb}}{2} \end{pmatrix} \\
&= \frac{e^{-t\frac{1}{2}}}{-2b}\begin{pmatrix} -\sin(tb) - 2b\cos(tb) & -2\sin(tb) \\ 2(c+1)\sin(tb) & \sin(tb) - 2b\cos(tb) \end{pmatrix}
\end{aligned}$$

Using the notations $\left[\lambda_n\right] = \sqrt{|\lambda_n|^2 + \frac{3}{4}}$ and $\left[\nabla\right] = \sqrt{-\Delta + \frac{3}{4}}$, we ultimately get that $S$ conjugated with $\mathcal{F}$ in the $n$-th coordinate is given by the matrix

$$T_n(t) = e^{-t\frac{1}{2}}\begin{pmatrix} \frac{1}{2\left[\lambda_n\right]}\sin\left(t\left[\lambda_n\right]\right) + \cos\left(t\left[\lambda_n\right]\right) & \frac{1}{\left[\lambda_n\right]}\sin\left(t\left[\lambda_n\right]\right) \\ -\left(\frac{1}{4\left[\lambda_n\right]} + \left[\lambda_n\right]\right)\sin\left(t\left[\lambda_n\right]\right) & -\frac{1}{2\left[\lambda_n\right]}\sin\left(t\left[\lambda_n\right]\right) + \cos\left(t\left[\lambda_n\right]\right) \end{pmatrix}. \tag{2.4}$$

This can be written in shorthand as

$$S(t) = e^{-t\frac{1}{2}} \begin{pmatrix} \frac{1}{2[\nabla]} \sin\left(t[\nabla]\right) + \cos\left(t[\nabla]\right) & \frac{1}{[\nabla]} \sin\left(t[\nabla]\right) \\ -\left(\frac{1}{4[\nabla]} + [\nabla]\right) \sin\left(t[\nabla]\right) & -\frac{1}{2[\nabla]} \sin\left(t[\nabla]\right) + \cos\left(t[\nabla]\right) \end{pmatrix}. \tag{2.5}$$

Take note of the componentwise estimate for $T_n(t)$ by the leading order term of $\langle\lambda_n\rangle$ in each entry:

$$T_n(t) \lesssim e^{-\frac{t}{2}} \begin{pmatrix} 1 & \langle\lambda_n\rangle^{-1} \\ \langle\lambda_n\rangle & 1 \end{pmatrix}. \tag{2.6}$$

The decay factor $e^{-\frac{t}{2}}$ is caused by the damping in the equation. If we studied the equation

$$w_{tt} + \beta w_t + (1 - \Delta)w = 0$$

where $\beta \in \mathbb{R}$ is the strength of the damping term, we would be dealing with the operator

$$\tilde{L} = \begin{pmatrix} 0 & 1 \\ -(1-\Delta) & -\beta \end{pmatrix}$$

and corresponding characteristic polynomial

$$\det \begin{pmatrix} -\tilde{a}_{1,2} & 1 \\ -1-c & -\beta - \tilde{a}_{1,2} \end{pmatrix} = \tilde{a}_{1,2}^2 + \beta\tilde{a}_{1,2}1 + c = 0 \iff \tilde{a}_{1,2} = -\frac{\beta}{2} \pm i\sqrt{c + 1 - \frac{\beta^2}{4}}.$$

The real term $-\frac{\beta}{2}$ then causes a factor $e^{-\frac{\beta}{2}}$ to appear in the matrix exponential. If $\beta = 0$ then this means we have no decay and if $\beta < 0$ we would get exponential growth.

What we can also learn from (2.6) is where $T_n$ gives and takes regularity.

**Lemma 2.2.** *For all $\alpha \in \mathbb{R}$, $S \in C([0,\infty), L(\mathcal{H}_{\mathrm{r}}^\alpha, \mathcal{H}_{\mathrm{r}}^\alpha))$. For $\mathbf{u} \in \mathcal{H}^\alpha$,*

$$\|S(t)\mathbf{u}\|_{\mathcal{H}^\alpha} \lesssim e^{-\frac{t}{2}}\|\mathbf{u}\|_{\mathcal{H}^\alpha}.$$

*Proof.* We estimate

$$\begin{aligned} e^t\|S(t)\mathbf{u}\|_{\mathcal{H}^\alpha}^2 &= e^t\|(1-\Delta)^{\frac{\alpha}{2}}\left(S(t)\mathbf{u}\right)_1\|_{L^2}^2 + e^t\|(1-\Delta)^{\frac{\alpha-1}{2}}\left(S(t)\mathbf{u}\right)_2\|_{L^2}^2 \\ &= \sum_{n=1}^{\infty} |\langle(1-\Delta)^{\frac{\alpha}{2}} e^{\frac{t}{2}}\left(S(t)\mathbf{u}\right)_1, e_n\rangle|^2 + \sum_{n=1}^{\infty} |\langle(1-\Delta)^{\frac{\alpha-1}{2}} e^{\frac{t}{2}}\left(S(t)\mathbf{u}\right)_2, e_n\rangle|^2 \\ &= \sum_{n=1}^{\infty}\langle\lambda_n\rangle^{2\alpha}\left|\left(e^{\frac{t}{2}}T_n(t)\right)_{1,1}\right|^2|\widehat{u_1}(n)|^2 + \sum_{n=1}^{\infty}\langle\lambda_n\rangle^{2\alpha}\left|\left(e^{\frac{t}{2}}T_n(t)\right)_{1,2}\right|^2|\widehat{u_2}(n)|^2 \\ &+ \sum_{n=1}^{\infty}\langle\lambda_n\rangle^{2\alpha-2}\left|\left(e^{\frac{t}{2}}T_n(t)\right)_{2,1}\right|^2|\widehat{u_1}(n)|^2 + \sum_{n=1}^{\infty}\langle\lambda_n\rangle^{2\alpha-2}\left|\left(e^{\frac{t}{2}}T_n(t)\right)_{2,2}\right|^2|\widehat{u_2}(n)|^2 \end{aligned}$$

Now use (2.6):

$$\lesssim \sum_{n=1}^{\infty}\left(\langle\lambda_n\rangle^{2\alpha} + \langle\lambda_n\rangle^{2\alpha-2}\langle\lambda_n\rangle^2\right)|\widehat{u_1}(n)|^2 + \sum_{n=1}^{\infty}\left(\langle\lambda_n\rangle^{2\alpha}\langle\lambda_n\rangle^{-2} + \langle\lambda_n\rangle^{2\alpha-2}\right)|\widehat{u_2}(n)|^2 \lesssim e^t\|\mathbf{u}\|_{\mathcal{H}^\alpha}^2.$$

□

**Lemma 2.3.** *let $\alpha \in \mathbb{R}$. We have*

*(i) For any $\gamma \in (0,1]$ we have $S \in C^{0,\gamma}([0,\infty), L(\mathcal{H}_{\mathrm{r}}^{\alpha}, \mathcal{H}_{\mathrm{r}}^{\alpha-\gamma}))$ with exponential decay*

$$\sup_{t<t_0<t_1} \frac{\|S(t_1)-S(t_0)\|_{L(\mathcal{H}_{\mathrm{r}}^{\alpha},\mathcal{H}_{\mathrm{r}}^{\alpha-\gamma})}}{|t_1-t_0|^{\gamma}} \leq Ce^{-\frac{t}{2}}.$$

*(ii) For any $\mathbf{u} \in \mathcal{H}_{\mathrm{r}}^{\alpha}$ we have $S(t)\mathbf{u} \in C([0,\infty), \mathcal{H}_{\mathrm{r}}^{\alpha}) \cap C^1([0,\infty), \mathcal{H}_{\mathrm{r}}^{\alpha-1})$ with exponential decay*

$$\|S(t)\mathbf{u}\|_{\mathcal{H}_{\mathrm{r}}^{\alpha}} + \|\partial_t S(t)\mathbf{u}\|_{\mathcal{H}_{\mathrm{r}}^{\alpha-1}} \leq Ce^{-\frac{t}{2}}.$$

*The derivative is given by the formula*

$$\partial_t S(t)\mathbf{u} = LS(t)\mathbf{u}.$$

*Proof.* To prove $(i)$ we simply proceed as in Lemma 2.2 but instead of (2.6) we use the estimate

$$|T_n(t_1)-T_n(t_0)| \lesssim e^{-\frac{t_0}{2}} \begin{pmatrix} 1 & \langle\lambda_n\rangle^{-1} \\ \langle\lambda_n\rangle & 1 \end{pmatrix} \langle\lambda_n\rangle^{\gamma}|t_1-t_0|^{\gamma}$$

for $0 \leq t_0 < t_1$. This estimate is a result of the estimate $|\sin([\lambda_n]t_1)-\sin([\lambda_n]t_0)| \leq 2[\lambda_n]^{\gamma}|t_1-t_0|^{\gamma}$ and the analogous for the difference of cosines.

Now we show $(ii)$. We start with the continuity. By the semigroup property of $S$it suffices to show that $\|S(h)\mathbf{u}-\mathbf{u}\|_{\mathcal{H}_{\mathrm{r}}^{\alpha}} \longrightarrow 0$ as $h \to 0$. Looking at the Fourier multipliers, we see that for all $n \in \mathbb{N}$ and $i,j \in \{1,2\}$ we have $(T_n(h)-\mathrm{Id})_{i,j} \longrightarrow 0$ locally uniformly in $h$. Then we proceed with the same estimates as above and use dominated convergence in $\ell^2(\mathbb{N})$ to conclude.

Like before we only have to show existence and the formula of the derivative at $t=0$, as the other cases follow from the semigroup property of $S$. We first compute $\partial_t T_n(t)$:

$$\begin{aligned}
\partial_t T_n(t) &= e^{-t\frac{1}{2}} \begin{pmatrix} \frac{1}{2}\cos\big(t\big[\lambda_n\big]\big) - \big[\lambda_n\big]\sin\big(t\big[\lambda_n\big]\big) & \cos\big(t\big[\lambda_n\big]\big) \\ -\big(\frac{1}{4}+\big[\lambda_n\big]^2\big)\cos\big(t\big[\lambda_n\big]\big) & -\frac{1}{2}\cos\big(t\big[\lambda_n\big]\big) - \big[\lambda_n\big]\sin\big(t\big[\lambda_n\big]\big) \end{pmatrix} \\
&\quad -\frac{1}{2}e^{-t\frac{1}{2}} \begin{pmatrix} \frac{1}{2\big[\lambda_n\big]}\sin\big(t\big[\lambda_n\big]\big) + \cos\big(t\big[\lambda_n\big]\big) & \frac{1}{\big[\lambda_n\big]}\sin\big(t\big[\lambda_n\big]\big) \\ -\Big(\frac{1}{4\big[\lambda_n\big]} + \big[\lambda_n\big]\Big)\sin\big(t\big[\lambda_n\big]\big) & -\frac{1}{2\big[\lambda_n\big]}\sin\big(t\big[\lambda_n\big]\big) + \cos\big(t\big[\lambda_n\big]\big) \end{pmatrix} \\
&= e^{-t\frac{1}{2}} \begin{pmatrix} -\Big(\frac{1}{4\big[\lambda_n\big]} + \big[\lambda_n\big]\Big)\sin\big(t\big[\lambda_n\big]\big) & -\frac{1}{2\big[\lambda_n\big]}\sin\big(t\big[\lambda_n\big]\big) + \cos\big(t\big[\lambda_n\big]\big) \\ -\Big(\frac{1}{4}+\big[\lambda_n\big]^2\Big)\Big(-\frac{1}{2}\frac{1}{\big[\lambda_n\big]}\sin\big(t\big[\lambda_n\big]\big) + \cos\big(t\big[\lambda_n\big]\big)\Big) & -\Big(\frac{1}{4\big[\lambda_n\big]} + \big[\lambda_n\big]\Big)\sin\big(t\big[\lambda_n\big]\big) - \cos\big(t\big[\lambda_n\big]\big) \end{pmatrix}.
\end{aligned}$$

Then

$$\partial_t T_n(0) = \begin{pmatrix} 0 & 1 \\ -\frac{1}{4}-\big[\lambda_n\big]^2 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1-\langle\lambda_n\rangle^2 & -1 \end{pmatrix}$$

Furthermore we can bound the distance of the difference quotient to the derivative uniformly in time $t$. For convenience of notation we use $\leq$ and $|\cdot|$ in the following, with the meaning that this holds for each individual component of $T_n(t)$. We write $T_n(t) = e^{-\frac{1}{2}t}e^{\frac{1}{2}t}T_n(t)$ and use the product formula for difference quotients

$$\left|\frac{T_n(t)-\mathrm{Id}}{t} - \partial_t T_n(0)\right|$$

$$= \left| e^{-\frac{t}{2}} \begin{pmatrix} \frac{\frac{1}{2[\lambda_n]}\sin\left(t[\lambda_n]\right)+\cos\left(t[\lambda_n]\right)-1}{t} - \frac{1}{2} & \frac{\frac{1}{[\lambda_n]}\sin\left(t[\lambda_n]\right)}{t} - 1 \\ \frac{-\left(\frac{1}{4[\lambda_n]}+[\lambda_n]\right)\sin\left(t[\lambda_n]\right)}{t} + \frac{1}{4} + [\lambda_n]^2 & \frac{-\frac{1}{2[\lambda_n]}\sin\left(t[\lambda_n]\right)+\cos\left(t[\lambda_n]\right)-1}{t} + \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{e^{-\frac{t}{2}}-1}{t} + \frac{1}{2} & 0 \\ 0 & \frac{e^{-\frac{t}{2}}-1}{t} + \frac{1}{2} \end{pmatrix} \right|$$

We define

$$\delta(x) \equiv \max\left\{\left|\frac{\sin(x)}{x} - 1\right|, \left|\frac{\cos(x)-1}{x}\right|\right\} \qquad \gamma(t) = \left|\frac{e^{-\frac{t}{2}}-1}{t} + \frac{1}{2}\right|.$$

and use it to estimate the above as

$$\leq \left|\begin{pmatrix} \frac{1}{2}\delta\left(t[\lambda_n]\right) + [\lambda_n]\delta\left(t[\lambda_n]\right) & \delta\left(t[\lambda_n]\right) \\ \frac{1}{4}\delta\left(t[\lambda_n]\right) + [\lambda_n]^2\delta\left(t[\lambda_n]\right) & \frac{1}{2}\delta\left(t[\lambda_n]\right) + [\lambda_n]\delta\left(t[\lambda_n]\right) \end{pmatrix}\right| + \begin{pmatrix} \gamma(t) & 0 \\ 0 & \gamma(t) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} + [\lambda_n] & 1 \\ \frac{1}{4} + [\lambda_n]^2 & \frac{1}{2} + [\lambda_n] \end{pmatrix} \delta\left(t[\lambda_n]\right) + \begin{pmatrix} \gamma(t) & 0 \\ 0 & \gamma(t) \end{pmatrix}.$$

Now let $t \neq 0$ and $\mathbf{u} \in \mathcal{H}^\alpha_{\mathrm{r}}$. Then

$$\begin{aligned} \left\|\frac{S(t)-\mathrm{Id}}{t}\mathbf{u} - L\mathbf{u}\right\|^2_{\mathcal{H}^{\alpha-1}} \lesssim & \sum_{n=1}^{\infty} \langle\lambda_n\rangle^{\alpha 2-2}\delta\left(t[\lambda_n]\right)^2 \left(\left|[\lambda_n]\widehat{u_1}(n)\right|^2 + |\widehat{u_2}(n)|^2\right) \\ & + \sum_{n=1}^{\infty} \langle\lambda_n\rangle^{\alpha 2-4}\delta\left(t[\lambda_n]\right)^2 \left(\left|[\lambda_n]^2\widehat{u_1}(n)\right|^2 + \left|[\lambda_n]\widehat{u_2}(n)\right|^2\right) \\ & + \sum_{n=1}^{\infty} \langle\lambda_n\rangle^{\alpha 2-2}\left|\gamma(t)\widehat{u_1}(n)\right|^2 + \langle\lambda_n\rangle^{\alpha 2-4}\left|\gamma(t)\widehat{u_2}(n)\right|^2 \end{aligned}$$

Now we let $\epsilon > 0$ be arbitrarily small and in particular small enough so that $\delta$ is monotonous on $(0, \epsilon)$. We get

$$\begin{aligned} & \lesssim \sum_{n=1}^{\infty} \delta\left(|t|[\lambda_n]\right)^2 \left(\langle\lambda_n\rangle^{2\alpha}|\widehat{u_1}(n)|^2 + \langle\lambda_n\rangle^{2-2\alpha}|\widehat{u_2}(n)|^2\right) + \gamma(t) \\ & \lesssim \left(\delta(\epsilon)^2 + \gamma(t)^2\right)\|\mathbf{u}\|^2_{\mathcal{H}^\alpha_{\mathrm{r}}} \\ & + \sum_{\substack{n=1 \\ |t|[\lambda_n]>\epsilon}}^{\infty} \left(\sup_x \delta(x)\right)^2 \left(\langle\lambda_n\rangle^{2\alpha}|\widehat{u_1}(n)|^2 + \langle\lambda_n\rangle^{2-2\alpha}|\widehat{u_2}(n)|^2\right). \end{aligned}$$

We can choose $\epsilon$ so that $\delta(\epsilon)^2$ is arbitrarily small. Then in the last line we are only summing over those $n$ where $n \sim [\lambda_n] > \frac{\epsilon}{|t|}$, so the sum line vanishes as $|t| \to 0$ (note that $\sup_x \delta(x) < \infty$). Lastly $\gamma(t) \to 0$. □

We have seen that differentiating $S$ causes a loss of regularity in space. If we integrate in time, however, we can avoid some of that loss of regularity. However, we will not use this result later on as $\sqrt{h}$ is not a very strong estimate.

**Lemma 2.4.** *Let $\alpha \in \mathbb{R}$ and $T, h \geq 0$. There exists $C > 0$ so that for all $u \in L^2([0,T], \mathcal{H}^\alpha_{\mathrm{r}})$*

$$\left\| \int_0^T S(t+h)u - S(t)u \, dt \right\|_{\mathcal{H}^\alpha} \leq C\sqrt{h} \|\mathbf{u}\|_{L^2([0,T],\mathcal{H}^\alpha)}.$$

*Proof.* Let $\mathbf{u} \in \mathcal{H}^\alpha$ and $T, h \geq 0$.

$$\left\| \int_0^T (S(t+h) - S(t))\mathbf{u}(t) \, dt \right\|^2_{\mathcal{H}^\alpha}$$

$$\begin{aligned}
\leq \textstyle\sum_{n=1}^\infty \langle\lambda_n\rangle^{2\alpha} \int_0^T & \Big( \tfrac{1}{2\left[\lambda_n\right]} \Big( e^{-\frac{t+h}{2}} \sin\big((t+h)\left[\lambda_n\right]\big) - e^{-\frac{t}{2}} \sin\big(t\left[\lambda_n\right]\big)\Big) \\
& + e^{-\frac{t+h}{2}} \cos\big((t+h)\left[\lambda_n\right]\big) - e^{-\frac{t}{2}} \cos\big(t\left[\lambda_n\right]\big)\Big)^2 dt \textstyle\int_0^T \widehat{u_1}(n,t)^2 \, dt \\
+ \textstyle\sum_{n=1}^\infty \langle\lambda_n\rangle^{2\alpha} \int_0^T & \Big( \tfrac{1}{\left[\lambda_n\right]} \Big( e^{-\frac{t+h}{2}} \sin\big((t+h)\left[\lambda_n\right]\big) - e^{-\frac{t}{2}} \sin\big(t\left[\lambda_n\right]\big)\Big)\Big)^2 dt \textstyle\int_0^T \widehat{u_2}(n,t)^2 \, dt \\
+ \textstyle\sum_{n=1}^\infty \langle\lambda_n\rangle^{2\alpha-2} \int_0^T & \Big( - \Big(\tfrac{1}{4\left[\lambda_n\right]} + \left[\lambda_n\right]\Big) \Big( e^{-\frac{t+h}{2}} \sin\big((t+h)\left[\lambda_n\right]\big) - e^{-\frac{t}{2}} \sin\big(t\left[\lambda_n\right]\big)\Big)\Big)^2 dt \textstyle\int_0^T \widehat{u_1}(n,t)^2 \, dt \\
+ \textstyle\sum_{n=1}^\infty \langle\lambda_n\rangle^{2\alpha-2} \int_0^T & \Big( - \tfrac{1}{2\left[\lambda_n\right]} \Big( e^{-\frac{t+h}{2}} \sin\big((t+h)\left[\lambda_n\right]\big) - e^{-\frac{t}{2}} \sin\big(t\left[\lambda_n\right]\big)\Big) \\
& + e^{-\frac{t+h}{2}} \cos\big((t+h)\left[\lambda_n\right]\big) - e^{-\frac{t}{2}} \cos\big(t\left[\lambda_n\right]\big)\Big)^2 dt \textstyle\int_0^T \widehat{u_2}(n,t)^2 \, dt
\end{aligned}$$

We estimate the following:

$$\int_0^T \Big( e^{-\frac{t+h}{2}} \sin\big((t+h)\left[\lambda_n\right]\big) - e^{-\frac{t}{2}} \sin\big(t\left[\lambda_n\right]\big)\Big)^2 dt.$$

$$= \int_0^s e^{-(h+r)} \big( \sin\big((h+r)\left[\lambda_n\right]\big) - e^{-r} \sin\big(r\left[\lambda_n\right]\big)\big)^2 \, dr$$

$$\lesssim \int_0^s (e^{-\frac{h+r}{2}} - e^{-\frac{r}{2}})^2 \, dr + \frac{1}{\left[\lambda_n\right]} \int_0^{s[\lambda_n]} e^{-r} \big( \sin\big(h\left[\lambda_n\right] + r\big) - \sin(r)\big)^2 \, dr$$

Now we estimate one of the powers of the sines on the right hand side integrand by 2, and the other by $h\left[\lambda_n\right]$:

$$\lesssim (e^{-\frac{h}{2}} - 1)^2 \int_0^s e^{-r} \, dr + \int_0^{s\left[\lambda_n\right]} e^{-r} \, dr \frac{1}{\left[\lambda_n\right]} h\left[\lambda_n\right] \lesssim h.$$

With the same method we can also show that

$$\int_0^T \Big( e^{-\frac{t+h}{2}} \cos\big((t+h)\left[\lambda_n\right]\big) - e^{-\frac{t}{2}} \cos\big(t\left[\lambda_n\right]\big)\Big)^2 dt \lesssim h.$$

Applying those estimates to the previous yields

$$\int_0^T \Big( e^{-\frac{t+h}{2}} \sin\big((t+h)\left[\lambda_n\right]\big) - e^{-\frac{t}{2}} \sin\big(t\left[\lambda_n\right]\big)\Big)^2 dt \lesssim h \int_0^T \|\mathbf{u}\|^2_{\mathcal{H}^\alpha} \, dt.$$

In other words there exists a constant $C > 0$ (independent of $T$) so that

$$\left\| \int_0^T S(t+h) - S(t)\, dt \right\|_{(L^2([0,T],\mathcal{H}^\alpha))^*} \leq C\sqrt{h}.$$

□

With the solution operator family $S(t)$ we can now a.s. solve the linear problem for any initial data. The initial data may also be a random variable. For the case of random initial data we will ultimately consider a certain space $\mathcal{X}^\alpha$ which is built to contain just those initial data that we need. In this section we choose $\mathcal{H}_\mathrm{r}^\alpha$ as possible spaces of initial data and and perform some light, preliminary analysis.

Let $\{g_n\}_{n\in\mathbb{N}}$, $\{h_n\}_{n\in\mathbb{N}}$ be families of standard Gaussian random variables $\sim \mathcal{N}(0,1)$ so that $\{g_n\}_n$ and $\{h_m\}_m$ are an independent families of random variables. We consider random intial data of the form

$$\mathbf{w}_0 = \sum_{n=1}^\infty \begin{pmatrix} a_n g_n e_n \\ b_n h_n e_n \end{pmatrix}$$

where $a_n, b_n$ are two sequences of real numbers. This is only a preliminary investigation into the random initial data, as later on we will need more assumptions.

**Corollary 2.5.** *For all $\alpha \geq 0$ the following are equivalent:*

*(i)* $\mathbf{w}_0 \in \mathcal{H}_\mathrm{r}^\alpha$ *a.s.*

*(ii)* $\mathbb{P}(\mathbf{w}_0 \in \mathcal{H}_\mathrm{r}^\alpha) > 0$,

*(iii)* $\mathbb{E}\left[\|\mathbf{w}_0\|_{\mathcal{H}_\mathrm{r}^\alpha}^2\right] < \infty$,

*(iv)* $\left(\langle\lambda_n\rangle^\alpha a_n\right)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$ *and* $\left(\langle\lambda_n\rangle^{\alpha-1} b_n\right)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$.

This is an consequence of Lemma A.2 applied to $w$ amd $w_t$ at time $t = 1$ with $f$ being constant.

**Theorem 2.6** (Global Well-posedness for Linear Problem with Random Initial Data)**.** *Let $\alpha \geq 0$ and $\mathbf{w}_0$ be initial data of the aforementioned form so that $\mathbf{w}_0 \in \mathcal{H}^\alpha$ a.s. Then for $\mathbf{w}(t) = S(t)\mathbf{w}_0$ the following holds almost surely:*

*(i)* $\mathbf{w} \in C([0,\infty),\mathcal{H}_\mathrm{r}^\alpha) \cap C^1([0,\infty),\mathcal{H}_\mathrm{r}^{\alpha-1})$,

*(ii)* $\mathbf{w}$ *solves* $\mathbf{w}(0) = \mathbf{w}_0$ *and*

$$\partial_t \mathbf{w}(t) = Lw(t).$$

*(iii) We have the estimates*

$$\|\mathbf{w}(t)\|_{\mathcal{H}^\alpha} \lesssim e^{-\frac{t}{2}}\|\mathbf{w}_0\|_{\mathcal{H}^\alpha} \qquad \text{and} \qquad \|\partial_t \mathbf{w}(t)\|_{\mathcal{H}^{\alpha-1}} \lesssim e^{-\frac{t}{2}}\|\mathbf{w}_0\|_{\mathcal{H}^\alpha}.$$

*Proof.* By the estimate $\|S(t)\mathbf{w}_0\|_{\mathcal{H}^\alpha} \leq e^{-\frac{t}{2}}\|\mathbf{w}_0\|_{\mathcal{H}^\alpha}$ we get that $\mathbf{w} \in C([0,\infty,\mathcal{H}^\alpha)$. We see that

$$\left\| \frac{\mathbf{w}(t+h) - \mathbf{w}(t)}{h} - \begin{pmatrix} 0 & 1 \\ -(1-\Delta) & -1 \end{pmatrix} \mathbf{w}(t) \right\|_{\mathcal{H}^{\alpha-1}}$$

$$\leq \left\| \frac{S(t+h)-S(t)}{h} - \begin{pmatrix} 0 & 1 \\ -(1-\Delta) & -1 \end{pmatrix} S(t) \right\|_{L(\mathcal{H}^\alpha,\mathcal{H}^{\alpha-1})} \|\mathbf{w}_0\|_{\mathcal{H}^\alpha}.$$

$$\leq \left\| \frac{S(h)-\mathrm{Id}}{h} - L \right\|_{L(\mathcal{H}^\alpha,\mathcal{H}^{\alpha-1})} \|S(t)\|_{L(\mathcal{H}^\alpha,\mathcal{H}^\alpha)} \|\mathbf{w}_0\|_{\mathcal{H}^\alpha}.$$

We know that the first norm vanishes as $h \searrow 0$ and the second norm decays with $e^{-\frac{t}{2}}$ in $t$, so $\mathbf{w}(t)$ is differentiable in $\mathcal{H}_{\mathrm{r}}^{\alpha-1}$ with bounded, continuous derivatives and $\partial_t \mathbf{w}(t) = LS(t)\mathbf{w}_0 = L\mathbf{w}(t)$. □

Before moving on to the next section we will state and prove one more lemma, which will be crucial in section 3 when we define the space of initial data. The reason is that we need to get information about the initial data from the $L^2$-in-time behaviour of the solution.

**Lemma 2.7.** *Let $\mathbf{w}_0 \in \mathcal{H}_{\mathrm{r}}^{-\infty}$ and $\alpha \in \mathbb{R}$. For $T > 0$ there exists a decreasing function $C_1(T) > 0$ and an increasing function $C_2(T) > 0$ so that*

$$\|\mathbf{w}_0\|_{\mathcal{H}^\alpha} \leq C_1(T) \|\pi_1 S(t)\mathbf{w}_0\|_{L^2([0,T],H^\alpha)} \leq C_2(T)\|\mathbf{w}_0\|_{\mathcal{H}^\alpha}.$$

*Proof.* The second estimate is straight forward using the energy estimate from Theorem 2.6:

$$\|\pi_1 S(t)\mathbf{w}_0\|^2_{L^2([0,T],H^\alpha)} \lesssim \int_0^T e^{-t}\,dt \|\mathbf{w}_0\|^2_{\mathcal{H}^\alpha} = (1-e^{-T})\|\mathbf{w}_0\|^2_{\mathcal{H}^\alpha}.$$

Then for with some constant $\tilde{C}$ we can choose

$$C_2(T) \coloneqq \tilde{C}\sqrt{1-e^{-T}}.$$

Now we prove the first estimate. By the definitions

$$\|\pi_1 S(t)\mathbf{w}_0\|^2_{L^2([0,T],H^\alpha)} = \sum_{n=1}^\infty \langle\lambda_n\rangle^{2\alpha} \int_0^T \langle \pi_1 S(s)\mathbf{w}_0, e_n\rangle^2\,ds$$

$$= \sum_{n=1}^\infty \langle\lambda_n\rangle^{2\alpha} \langle\pi_1 \mathbf{w}_0, e_n\rangle^2 \int_0^T e^{-s}\left(\frac{1}{2\left[\lambda_n\right]}\sin\left(s\left[\lambda_n\right]\right) + \cos\left(s\left[\lambda_n\right]\right)\right)^2 ds$$

$$+ \langle\lambda_n\rangle^{2\alpha}\langle\pi_2\mathbf{w}_0, e_n\rangle^2 \int_0^T e^{-s}\left(\frac{1}{\left[\lambda_n\right]}\sin\left(s\left[\lambda_n\right]\right)\right)^2 ds$$

We denote the first integral by $(I)$ and the second by $(II)$. In the next section we will calculate $c_n^2(t)$ in (2.12) and $d_n^2(t)$ in (2.13). These integrals are very similar to our integrals $(II)$ and $(I)$ respectively and so we ask the reader to observe the computation there. Then for $(II)$ we get a lower bound

$$(II) \gtrsim \langle\lambda_n\rangle^{-2}\left(1 - e^{-T}\left(1 + \frac{1}{2}[\lambda_n]^{-1} + \frac{1}{2}[\lambda_n]^{-2}\right)\right).$$

For $(I)$ we modify the computation preceding (2.13):

$$\left[\lambda_n\right]^{-1}\int_{-T[\lambda_n]}^0 e^{\frac{\tau}{[\lambda_n]}}\left(\frac{1}{2[\lambda_n]}\sin(-2\tau) + \frac{1}{2}\cos(-2\tau) + \frac{1}{2}\right) d\tau$$

$$= 2\left[\lambda_n\right]^{-1}\left[\frac{e^{\frac{\tau}{[\lambda_n]}}}{8[\lambda_n]^2+2}\left(4\left[\lambda_n\right]^3+\left[\lambda_n\right]+(2\left[\lambda_n\right]^2-1)\sin(2\tau)+\left[\lambda_n\right]\cos(2\tau)\right)\right]_{-T[\lambda_n]}^{0}$$

$$= \frac{\left[\lambda_n\right]^2}{4\langle\lambda_n\rangle^2}\left(4-e^{-T}\left(4+\left[\lambda_n\right]^{-2}-\left(2\left[\lambda_n\right]^{-1}-\left[\lambda_n\right]^{-3}\right)\sin(2T[\lambda_n])+\left[\lambda_n\right]^{-2}\cos(2T[\lambda_n])\right)\right).$$

Then

$$(I) = \frac{1}{4}(II) + \frac{\left[\lambda_n\right]^2}{\langle\lambda_n\rangle^2}\Big(1-e^{-t}\Big(1-\frac{1}{2}\left[\lambda_n\right]^{-1}\sin(2t[\lambda_n])+\frac{1}{4}\left[\lambda_n\right]^{-2}(1+\cos(2t[\lambda_n])) + \frac{1}{4}\left[\lambda_n\right]^{-3}\sin(2t[\lambda_n])\Big)\Big)$$

Finally, we can estimate

$$\begin{aligned}
(I) &\geq \frac{1}{4}(II) + \frac{\left[\lambda_n\right]^2}{\langle\lambda_n\rangle^2}\left(1-e^{-T}\left(1+\frac{1}{2}[\lambda_n]^{-1}+\frac{1}{2}[\lambda_n]^{-2}+\frac{1}{4}[\lambda_n]^{-3}\right)\right)\\
&\geq \frac{1}{4}\langle\lambda_n\rangle^{-2}\left(4[\lambda_n]^2+1-e^{-T}e^{-T}\left(1+\frac{1}{2}[\lambda_n]^{-1}+\frac{1}{2}[\lambda_n]^{-2}+\frac{1}{4}[\lambda_n]^{-3}\right)\right)\\
&= 1-\frac{1}{4}\langle\lambda_n\rangle^{-2}e^{-T}\left(1+\frac{1}{2}[\lambda_n]^{-1}+\frac{1}{2}[\lambda_n]^{-2}+\frac{1}{4}[\lambda_n]^{-3}\right)\\
&\geq 1-e^{-T}\left(1+\frac{1}{2}[\lambda_n]^{-1}+[\lambda_n]^{-2}+\frac{3}{4}[\lambda_n]^{-3}+\frac{1}{2}[\lambda_n]^{-4}\right)\\
&\gtrsim 1-e^{-T}\left(1+\langle\lambda_n\rangle^{-1}+\langle\lambda_n\rangle^{-2}+\langle\lambda_n\rangle^{-3}+\langle\lambda_n\rangle^{-4}\right)
\end{aligned}$$

Recall also that

$$(II) \gtrsim \langle\lambda_n\rangle^{-2}\left(1-e^{-T}\left(1+\langle\lambda_n\rangle^{-1}+\langle\lambda_n\rangle^{-2}\right)\right).$$

Using that $\langle\lambda_n\rangle \sim n$, this implies that there exists constants $c, C, \tilde{C} > 0$ so that for all $T > 0$ and $N \in \mathbb{N}$,

$$1 + cN^{-1} \leq e^T$$

implies

$$\|(1-P_N)\mathbf{w}_0\|_{\mathcal{H}^\alpha} \leq C\|\pi_1 S(t)\mathbf{w}_0\|_{L^2([0,T],H^\alpha)} \leq \tilde{C}\sqrt{1-e^{-T}}\|\mathbf{w}_0\|_{\mathcal{H}^\alpha}.$$

For large $N$ this holds. We deal with dimensions 1 to $N$ separately: Define

$$C_1(T) := \max\left\{C, \left(\min\left\{\inf_{1\leq n\leq N}(I)_{n,T}, \inf_{1\leq n\leq N}(II)_{n,T}\right\}\right)^{-1}\right\}.$$

This constant enusres “manually” that the estimate works not only for $n \geq N+1$, but also the cases $n = 1, ...N$. Here $(I)_{n,T}$ and $(II)_{n,T}$ refer to the corresponding integrals for the given $n$ and $T$. Both of these are positive for $T > 0$, increasing in $T$ and start in 0. Therefore $C_1(T)$ is well-defined, positive and decreasing. □

## 2.2 The Linear Problem with White Noise Inhomogeneity

In this section we construct the stochastic convolution $\Psi$, which is a almost surely solution to (1.6) in the sense of distributions. The construction is motivated by Duhamel's formula and since solutions given in the shape of Duhamel's formula are also called **mild solution**, one can consider $\Psi$ to be a mild solution to (1.6). Written in terms of pairs of functions, $\Psi$ is supposed to solve

$$\partial_t \begin{pmatrix} \psi \\ \psi_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -(1-\Delta) & -1 \end{pmatrix} \begin{pmatrix} \psi \\ \psi_t \end{pmatrix} + \begin{pmatrix} 0 \\ \sqrt{2}\xi \end{pmatrix}$$

with initial data $\Psi_0 = 0$. Since we know the solution operator $S(t)$ to the homogeneous problem, we can construct a mild solution to this problem by Duhamel's formula:

$$\Psi(t) = \int_0^t S(t-s) \cdot \begin{pmatrix} 0 \\ \sqrt{2}\xi(s) \end{pmatrix} \, ds$$

It is not entirely obvious how this has to be interpreted, so let us define it

**Definition 2.8** (Stochastic Convolution)**.** *For $t \geq 0$, define a random radially symmetric space distribution in two variables which we call the* ***stochastic convolution at time*** *$t$ by*

$$\langle \Psi(t), \mathbf{f} \rangle := \left\langle \int_0^t S(t-s) \begin{pmatrix} 0 \\ \sqrt{2}\xi(s) \end{pmatrix} \, ds, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\rangle \equiv \left\langle \xi, \mathbb{1}_{[0,t]}(s)\sqrt{2} \cdot \pi_2 S^*(t-s)\mathbf{f} \right\rangle$$

*for $\mathbf{f} = (f_1, f_2) \in \mathcal{D}_{\mathrm{r}}(B)^2$. Here $\pi_1$ and $\pi_2$ are the projections onto the first and second function. We can unfold the definition to*

$$\begin{aligned} \langle \Psi(t), \mathbf{f} \rangle = \Bigg\langle \xi, \mathbb{1}_{[0,t]}(s)\sqrt{2}e^{-\frac{t-s}{2}} \cdot \Bigg( & \left[ \tfrac{1}{[\nabla]} \sin\big((t-s)[\nabla]\big) \right] \quad f_1 \\ & + \left[ -\tfrac{1}{2[\nabla]} \sin\big((t-s)[\nabla]\big) + \cos\big((t-s)[\nabla]\big) \right] \quad f_2 \Bigg) \Bigg\rangle \end{aligned}$$

**Lemma 2.9.** *The stochastic convolution is almost surely a solution to the distributional equation*

$$\partial_t \Psi = L\Psi + \begin{pmatrix} 0 \\ \sqrt{2}\xi \end{pmatrix}.$$

*Proof.* Let $\mathbf{f} \in \mathcal{D}_{\mathrm{r}}(\mathbb{R}_+ \times B)^2$. We compute

$$\begin{aligned} \langle \partial_t \Psi - L\Psi, \mathbf{f} \rangle_{t,x} &= \int_0^\infty \langle \Psi(t), (-\partial_t - L)\mathbf{f}(t) \rangle_x \, dt \\ &= \int_0^\infty \langle \xi, \mathbb{1}_{[s,\infty]}(t)\sqrt{2}\pi_2 S^*(t-s)(-\partial_t - L)\mathbf{f}(t) \rangle_{s,x} \, dt. \end{aligned} \tag{2.7}$$

We will we use that $\partial_t S(t)^* \mathbf{f} = S(t)^* L\mathbf{f}$ by Lemma 2.3. Let $\eta_k \in \mathbb{C}^\infty(\mathbb{R}_+)$ be a sequence such that

$$\eta_k \longrightarrow \mathbb{1}_{[0,\infty)} \text{ in } L^2(\mathbb{R}_+) \text{ and } \mathcal{D}'_{\mathrm{r}}(\mathbb{R}_+) \text{ with } \operatorname{supp}\big(\eta_k - \mathbb{1}_{[0,\infty)}\big) \subset \left(0, \frac{1}{k}\right),$$

and consider the following:

$$(2.7) \xleftarrow{k\to\infty} \int_0^\infty \langle \xi, \eta_k(t-s)\sqrt{2}\pi_2 S^*(t-s)(-\partial_t - L)\mathbf{f}(t)\rangle_{s,x}\, dt \tag{2.8}$$

$$=\langle (\textstyle\int \cdot\, dt) \otimes \xi, \eta_k(t-s)\sqrt{2}\pi_2 S^*(t-s)(-\partial_t - L)\mathbf{f}(t)\rangle_{t,s,x}$$

$$=\langle \xi \otimes (\textstyle\int \cdot\, dt), \eta_k(t-s)\sqrt{2}\pi_2 S^*(t-s)(-\partial_t - L)\mathbf{f}(t)\rangle_{t,s,x}$$

$$=\left\langle \xi, \int_0^\infty \eta_k(t-s)\sqrt{2}\pi_2 S^*(t-s)(-\partial_t - L)\mathbf{f}(t)\, dt \right\rangle_{s,x}$$

$$=\left\langle \xi, \int_0^\infty \eta_k(t-s)\sqrt{2}\pi_2 (-\partial_t S^*(t-s))\mathbf{f}(t) - S^*(t-s)(\partial_t \mathbf{f}(t))\, dt \right\rangle_{s,x}$$

$$=\left\langle \xi, \int_0^\infty \partial_t \eta_k(t-s)\sqrt{2}\pi_2 S^*(t-s)\mathbf{f}(t)\, dt \right\rangle_{s,x} \tag{2.9}$$

$$\xrightarrow{k\to\infty} \left\langle \xi, \sqrt{2}\pi_2 S^*(s-s)\mathbf{f}(s)\, dt \right\rangle_{s,x} \tag{2.10}$$

$$=\left\langle \begin{pmatrix} 0 \\ \sqrt{2}\xi \end{pmatrix}, \mathbf{f}(s) \right\rangle.$$

We have to justify the limits. Let $\mathrm{supp}_t\, \mathbf{f} \subset [0,T]$. Observe that with Jensen's inequality and the definition of $\xi$ and Hölder's inequaltiy we have

$$\mathbb{E}\left[|(2.7) - (2.8)|^2\right] \leq T \int_0^T \|\eta_k(t-s) - \mathbb{1}_{[s,\infty)}(t)\|^2_{L^2_{s,x}}\, dt$$
$$\int_0^T \|\sqrt{2}\pi_2 S^*(t-s)(-\partial_t - L)\mathbf{f}(t)\|_{L^2_{s,x}}\, dt.$$

this vanishes as $k \to \infty$ as the first integrand vanishes uniformly in $t$ and the second integral is finite becasue $\mathbf{f}$ is smooth and compactly supported. Similarly with the definition of $\xi$ we get

$$\mathbb{E}\left[|(2.9) - (2.10)|^2\right] = \int_0^\infty \int_B \left| \int_0^T \partial_t \eta_k(t-s)\sqrt{2}\pi_2 S^*(t-s)\mathbf{f}(t)\, dt - \sqrt{2}\pi_2 \mathbf{f}(s) \right|^2 dx\, ds.$$

Since $\sqrt{2}\pi_2 S^*(t-s)\mathbf{f}(t)$ is smooth in $t, s$ and $x$ the convergence of the integrand to 0 is uniform. In addition we can also restrict the integral to a compact domain and so it vanishes as $k \to \infty$. □

Using that $\xi$ is a space-time white noise, we can compute for $\mathbf{f}, \mathbf{g} \in \mathcal{D}(B)^2$ that

$$\mathbb{E}\left[\langle \Psi(t), \mathbf{f}\rangle \langle \Psi(t), \mathbf{g}\rangle\right] = \left\langle \mathbb{1}_{[0,t]}(s)\sqrt{2}e^{-\frac{t-s}{2}} \cdot \pi_2 S^*(t-s) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \mathbb{1}_{[0,t]}(s)\sqrt{2}e^{-\frac{t-s}{2}} \cdot \pi_2 S^*(t-s) \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right\rangle_{L^2_{t,x}}$$

$$= \int_0^t \sum_{n=1}^\infty 2e^{s-t} \left( \frac{1}{[\lambda_n]} \sin\left((t-s)[\lambda_n]\right) \widehat{f_1}(n) + \left( -\frac{1}{2[\lambda_n]} \sin\left((t-s)[\lambda_n]\right) + \cos\left((t-s)[\lambda_n]\right) \right) \widehat{f_2}(n) \right)$$
$$\cdot \left( \frac{1}{[\lambda_n]} \sin\left((t-s)[\lambda_n]\right) \widehat{g_1}(n) + \left( -\frac{1}{2[\lambda_n]} \sin\left((t-s)[\lambda_n]\right) + \cos\left((t-s)[\lambda_n]\right) \right) \widehat{g_2}(n) \right) ds.$$

Furthermore $\langle \Psi(t), \mathbf{f} \rangle$ is a centered Gaussian random variable.
In the case that $\mathbf{f} = (e_n, 0)$ and $\mathbf{g} = (e_m, 0)$ we get that

$$\mathbb{E}\left[\langle \Psi(t), (e_n, 0)\rangle \langle \Psi(t), (e_m, 0)\rangle\right] = \begin{cases} \int_0^t \frac{2e^{s-t}}{[\lambda_n]^2} \sin\left((t-s)[\lambda_n]\right)^2 \, ds & , n = m \\ 0 & , n \neq m \end{cases}. \tag{2.11}$$

For $\mathbf{f} = (0, e_n)$ and $\mathbf{g} = (0, e_m)$ we have

$$\mathbb{E}\left[\langle \Psi(t), (0, e_n)\rangle \langle \Psi(t), (0, e_m)\rangle\right] = \begin{cases} \displaystyle\int_0^t 2e^{s-t} \left( \frac{1}{4[\lambda_n]^2} \sin\left((t-s)[\lambda_n]\right)^2 \right. \\ \qquad - \frac{1}{2}\frac{1}{[\lambda_n]} \sin\left(2(t-s)[\lambda_n]\right) & , n = m \\ \qquad \left. + \frac{1}{2}\cos\left(2(t-s)[\lambda_n]\right) + \frac{1}{2} \right) ds & \\ 0 & , n \neq m \end{cases}.$$

We can calculate the integrals:

$$\int_0^t \frac{2e^{s-t}}{\left[\lambda_n\right]^2} \sin\left((t-s)\left[\lambda_n\right]\right)^2 \, ds = 2\left[\lambda_n\right]^{-3} \int_{-t[\lambda_n]}^0 e^{\frac{\tau}{[\lambda_n]}} \sin\left(-\tau\right)^2 \, d\tau$$

$$= 2\left[\lambda_n\right]^{-3} \left[ \frac{\left[\lambda_n\right] e^{\frac{\tau}{[\lambda_n]}}}{8\left[\lambda_n\right]^2 + 2} \left( 4\left[\lambda_n\right]^2 - 2\left[\lambda_n\right]\sin(2\tau) - \cos(2\tau) + 1 \right) \right]_{-t[\lambda_n]}^0$$

$$= \frac{1}{4\langle\lambda_n\rangle^2} \left( 4 - e^{-t}\left( 4 + 2\left[\lambda_n\right]^{-1}\sin(2t[\lambda_n]) - \left[\lambda_n\right]^{-2}\cos(2t[\lambda_n]) + \left[\lambda_n\right]^{-2} \right) \right)$$

We define

$$c_n^2(t) = \frac{\Theta_n(t)}{\langle\lambda_n\rangle^2} \tag{2.12}$$

where

$$\Theta_n(t) = 1 - e^{-t}\left( 1 + \frac{1}{2}\left[\lambda_n\right]^{-1}\sin(2t[\lambda_n]) + \frac{1}{4}\left[\lambda_n\right]^{-2}\left(1 - \cos(2t[\lambda_n])\right) \right).$$

The previous results can be summarized as

$$\langle \psi(t), e_n \rangle \sim \mathcal{N}(0, c_n^2(t))$$

and they are independent in $n$. Observe that

$$\Theta_n(0) = 0 \qquad \lim_{t\to\infty} \Theta_n(t) = 1 \qquad \exists\, K > 0 : \forall\, n \in \mathbb{N}\ \forall\, t \geq 0, \quad 0 \leq \Theta_n(t) \leq K.$$

and that there exists $N \in \mathbb{N}$ so that for all $n \geq N$ and $t \geq 0$

$$1 - 2e^{-t} \leq \Theta_n(t) \leq 1 - \frac{1}{2}e^{-t}.$$

Hence for large $n$ and fixed $t > 0$ we have $c_n^2(t) \sim \langle\lambda_n\rangle^{-2}$.

For the other integral we compute

$$2\big[\lambda_n\big]^{-1}\int_{-t[\lambda_n]}^{0} e^{\frac{\tau}{[\lambda_n]}}\left(-\frac{1}{2[\lambda_n]}\sin(-2\tau)+\frac{1}{2}\cos(-2\tau)+\frac{1}{2}\right)\,d\tau$$

$$=2\big[\lambda_n\big]^{-1}\left[\frac{e^{\frac{\tau}{[\lambda_n]}}}{8[\lambda_n]^2+2}\left(4\big[\lambda_n\big]^3+\big[\lambda_n\big]+(2\big[\lambda_n\big]^2+1)\sin(2\tau)-\big[\lambda_n\big]\cos(2\tau)\right)\right]_{-t[\lambda_n]}^{0}$$

$$=\frac{\big[\lambda_n\big]^2}{4\langle\lambda_n\rangle^2}\left(4-e^{-t}\left(4+\big[\lambda_n\big]^{-2}-\left(2\big[\lambda_n\big]^{-1}+\big[\lambda_n\big]^{-3}\right)\sin(2t[\lambda_n])-\big[\lambda_n\big]^{-2}\cos(2t[\lambda_n])\right)\right).$$

We define

$$d_n^2(t)=\frac{\frac{1}{4}\Theta_n(t)+\big[\lambda_n\big]^2\Xi_n(t)}{\langle\lambda_n\rangle^2} \tag{2.13}$$

where

$$\Xi_n(t)=1-e^{-t}\left(1-\frac{1}{2}\big[\lambda_n\big]^{-1}\sin(2t[\lambda_n])+\frac{1}{4}\big[\lambda_n\big]^{-2}(1-\cos(2t[\lambda_n]))-\frac{1}{4}\big[\lambda_n\big]^{-3}\sin(2t[\lambda_n])\right)$$

We then have that

$$\langle\psi_t(t),e_n\rangle\sim\mathcal{N}(0,d_n^2(t))$$

and they are independent in $n$. Observe that

$$\Xi_n(0)=0 \qquad \lim_{t\to\infty}\Xi_n(t)=1 \qquad \exists\, K>0:\forall\, n\in\mathbb{N}\;\forall\, t\geq 0,\quad 0\leq\Theta_n(t)\leq K.$$

and that there exists $N\in\mathbb{N}$ so that for all $n\geq N$ and $t\geq 0$

$$1-\frac{1}{2}\big[\lambda_n\big]^{-2}-2e^{-t}\leq\Xi_n(t)\leq 1-\frac{1}{2}\big[\lambda_n\big]^{-2}+\frac{1}{2}e^{-t}.$$

Let us summarize the above results and more in the following lemma.

**Lemma 2.10.** *For all* $\mathbf{f}\in\mathcal{D}_{\mathrm{r}}(B)^2$ *we have*

$$\langle\Psi(t),\mathbf{f}\rangle=\sum_{n=1}^{\infty}c_n(t)g_n(t)\widehat{f}_1(n)+d_n(t)g_n(t)\widehat{f}_2(n),$$

*which means that if the stochastic convolution is actually a function in time and space (which is the case by lem. 2.11), we can write it as*

$$\Psi(t,x)=\sum_{n=1}^{\infty}\begin{pmatrix}c_n(t)g_n(t)\\ d_n(t)h_n(t)\end{pmatrix}e_n(x),$$

*where:*

*(i)* $c_n$ *and* $d_n$ *are given by (2.12) and (2.13) (the choice of sign doesn't matter). They have growth bounds* $\sup_t c_n^2(t)\lesssim\langle\lambda_n\rangle^{-2}$ *and* $\sup_t d_n^2(t)\lesssim 1$.

*(ii) For any times $0 \le t_1, ..., t_k$, the families of random vectors*

$$\left\{ \begin{pmatrix} g_n(t_1) \\ \vdots \\ g_n(t_k) \end{pmatrix} \right\}_{n\in\mathbb{N}} \quad \textit{and} \quad \left\{ \begin{pmatrix} h_n(t_1) \\ \vdots \\ h_n(t_k) \end{pmatrix} \right\}_{n\in\mathbb{N}}$$

*are each independent and each one of them follows a centered joint Gaussian distribution with 1 on the diagonal of the covariance matrix.*

*(iii) There exists $C > 0$ so that for any $s \le t$,*

$$\mathbb{E}\left[\left(c_n(t)g_n(t) - c_n(s)g_n(s)\right)^2\right] \le C\left[\lambda_n\right]^{-2}|t-s|. \tag{2.14}$$

*and*

$$\mathbb{E}\left[\left(d_n(t)h_n(t) - d_n(s)h_n(s)\right)^2\right] \le C|t-s|.$$

*Proof.* We have shown $(i)$ before and elaborated on $(ii)$. The reason that we have a joint Gaussian distribution is that $g_n(t)$ and $h_n(t)$ are given by $\xi$ tested against some function, and then by linearity of $\xi$ any linear combination of those random variable is still normally distributed.

We now calculate $(iii)$. The new term that we have to analyze here is the covariance:

$$\mathbb{E}\left[c_n(t)g_n(t)c_n(s)g_n(s)\right] = \mathbb{E}\left[\langle \psi(t), e_n\rangle\langle \psi(s), e_n\rangle\right]$$

$$= \left\langle \mathbb{1}_{[0,t]}(\tau)\sqrt{2}e^{-\frac{t-\tau}{2}} \cdot \pi_2 S^*(t-\tau)\begin{pmatrix} e_n \\ 0 \end{pmatrix}, \mathbb{1}_{[0,s]}(\tau)\sqrt{2}e^{-\frac{s-\tau}{2}} \cdot \pi_2 S^*(s-\tau)\begin{pmatrix} e_n \\ 0 \end{pmatrix} \right\rangle_{L^2_{t,x}}$$

$$= \int_0^s 2e^{\frac{\tau-t}{2}}e^{\frac{\tau-s}{2}} \frac{1}{\left[\lambda_n\right]}\sin\left((t-\tau)\left[\lambda_n\right]\right) \cdot \frac{1}{\left[\lambda_n\right]}\sin\left((t-\tau)\left[\lambda_n\right]\right)\, d\tau.$$

Then together with (2.11) we have

$$\mathbb{E}\left[c_n^2(t)g_n(t)^2 - 2c_n(t)g_n(t)c_n(s)g_n(s) + c_n^2(s)g_n(s)^2\right]$$

$$= \frac{2}{\left[\lambda_n\right]^2}\Bigg( \int_0^s \left(e^{\frac{\tau-t}{2}}\sin\left((t-\tau)\left[\lambda_n\right]\right) - e^{\frac{\tau-s}{2}}\sin\left((s-\tau)\left[\lambda_n\right]\right)\right)^2 d\tau + \int_s^t e^{\tau-t}\sin\left((t-\tau)\left[\lambda_n\right]\right)^2\, d\tau \Bigg).$$

We can easily estimate the second term by $|t-s|$. For the first term we set $h = t-s$ and substitute $r = s-\tau$ to get

$$\int_0^s \left(e^{-\frac{h+r}{2}}\sin\left((h+r)\left[\lambda_n\right]\right) - e^{-\frac{r}{2}}\sin\left(r\left[\lambda_n\right]\right)\right)^2 dr \tag{2.15}$$

$$\lesssim \int_0^s (e^{-\frac{h+r}{2}} - e^{-\frac{r}{2}})^2\, dr + \frac{1}{\left[\lambda_n\right]}\int_0^{s\left[\lambda_n\right]} e^{-\frac{r}{\left[\lambda_n\right]}}\left(\sin\left(h\left[\lambda_n\right] + r\right) - \sin\left(r\right)\right)^2 dr$$

Now we estimate one of the powers of the sines on the right hand side integrand by 2, and the other by $h\big[\lambda_n\big]$:

$$\lesssim (e^{-\frac{h}{2}}-1)^2 \int_0^s e^{-r}\,dr + \int_0^{s\big[\lambda_n\big]} e^{-\frac{r}{[\lambda_n]}}\,dr \frac{1}{[\lambda_n]} h\big[\lambda_n\big] \lesssim h.$$

For the case of $\psi_t$ we get

$$\mathbb{E}\left[d_n^2(t)h_n(t)^2 - 2d_n(t)h_n(t)d_n(s)h_n(s) + d_n^2(s)h_n(s)^2\right]$$

$$\begin{aligned}
&= 2\int_0^s \Bigg(e^{\frac{\tau-t}{2}}\left(-\frac{1}{2\big[\lambda_n\big]}\sin\big((t-\tau)\big[\lambda_n\big]\big)+\cos\big((t-\tau)\big[\lambda_n\big]\big)\right) \\
&\qquad - e^{\frac{\tau-s}{2}}\left(-\frac{1}{2\big[\lambda_n\big]}\sin\big((s-\tau)\big[\lambda_n\big]\big)+\cos\big((s-\tau)\big[\lambda_n\big]\big)\right)\Bigg)^2\,d\tau \\
&+2\int_s^t e^{\tau-t}\sin\big((t-\tau)\big[\lambda_n\big]\big)^2\,d\tau.
\end{aligned}$$

We can directly estimate the term by $4|t-s|$. For the first term we proceed with the same steps as in (2.15). It then remains to estimate

$$\frac{1}{\big[\lambda_n\big]}\int_0^{s[\lambda_n]} e^{-\frac{r}{[\lambda_n]}}\left(-\frac{1}{2\big[\lambda_n\big]}\sin\big(h\big[\lambda_n\big]+r\big)+\cos\big(h\big[\lambda_n\big]+r\big)+\frac{1}{2\big[\lambda_n\big]}\sin(r)-\cos(r)\right)^2\,dr$$

We estimate the squared part of the integrand by the square of the sine difference and the square of the cosine difference, and then again estimate one power by a constant and the other power by $h\big[\lambda_n\big]$. The remaining exponential decay integral is then finite and things work out as in the pervious case. □

**Lemma 2.11.** *Let $t>0$ and $\alpha\in\mathbb{R}$. The following are equivalent:*

*(i)* $\psi(t)\in\mathcal{H}_{\mathrm{r}}^{\alpha}$ *a.s.*

*(ii)* $\psi(t)\in L^2(\Omega,\mathcal{H}_{\mathrm{r}}^{\alpha})$,

*(iii)* $\alpha<\frac{1}{2}$.

*Proof.* We know that *(i)* $\iff$ *(ii)* by Corollary 2.5 Let $N\le M\in\mathbb{N}$. Then

$$\mathbb{E}\left[\|\psi(t)^N-\psi(t)^M\|_{H^\alpha}^2\right]=\sum_{n=N}^{M}\langle\lambda_n\rangle^{2\alpha}c_n^2(t)\underset{N,M\text{ large}}{\sim}\sum_{n=N}^{M}n^{2\alpha-2}.$$

Here we use that $c_n^2(t)\sim\langle\lambda_n\rangle\sim|\lambda_n|\sim n$. The power series at the end converges if and only if $\alpha<\frac{1}{2}$, so *(ii)* $\iff$ *(iii)*. □

This lemma tells us that $\psi(t)$ does exist not only as a random distribution, but a.s. as a function in $L_{\mathrm{r}}^2(B)$.

Let us explicitly restate again that for a fixed $t > 0$ there exist iid. random variables $g_n(t) \sim \mathcal{N}(0,1)$ so that

$$\psi(t,x) = \sum_{n=1}^{\infty} c_n(t) e_n(x) g_n(t),$$

and a fixed $n$ and any times $t_1, ..., t_k$ the random vector $(g_n(t_j))_j$ is jointly normally distributed.

The following lemma is an important inequality that we will use to get $W^{\alpha,p}$ regularity for $\psi$.

**Lemma 2.12.** *There exists $C > 0$ so that for all $0 \le \alpha < \frac{1}{2}$ and $N < M \in \mathbb{N}$ we have for almost all $x \in B$ the estimate*

$$\sum_{n=N}^{M} \langle\lambda_n\rangle^{2\alpha-2} e_n(x)^2 \tag{2.16}$$
$$\le C \begin{cases} \frac{1}{N} + 1 + \ln\left(\frac{M \wedge |x|^{-1}}{N}\right) & , |x|^{-1} > N, \alpha = 0 \\ \frac{1}{N} + \frac{|x|^{-1}}{N-1} & , |x|^{-1} \le N, \alpha = 0 \\ \frac{1}{N^{1-2\alpha}} + \frac{1}{1-2\alpha}\left(1 + |x|^{-2\alpha}\right) + \frac{1}{2\alpha}\left(\left(M \wedge |x|^{-1}\right)^{2\alpha} - N^{2\alpha}\right) & , |x|^{-1} > N, \alpha > 0 \\ \frac{1}{N^{1-2\alpha}} + \frac{1}{1-2\alpha}\frac{2|x|^{-1}}{(N-1)^{1-2\alpha}} & , |x|^{-1} \le N, \alpha > 0 \end{cases}$$

*For the case $N = 1$ and $M = \infty$ this can be simplified to*

$$\sum_{n=1}^{\infty} \langle\lambda_n\rangle^{2\alpha-2} e_n(x)^2 \le C \begin{cases} 1 + |\ln(|x|)|, & \alpha = 0 \\ 1 + \left((1-2\alpha)^{-1} + (2\alpha)^{-1}\right)|x|^{-2\alpha}, & \alpha > 0. \end{cases} \tag{2.17}$$

*Proof.* Let $N < M \in \mathbb{N}$. Recall that $\langle\lambda_n\rangle \sim n$ and that

$$e_n(x) = \|J_0(\lambda_n n|\cdot|)\|_{L^2(B)}^{-1} J_0(\lambda_n|x|) \text{ with } \|J_0(\lambda_n n|\cdot|)\|_{L^2(B)} \sim n^{-\frac{1}{2}}.$$

We split the sum into two parts:

$$\sum_{n=N}^{M} \langle\lambda_n\rangle^{2\alpha-2} e_n(x)^2 \lesssim \sum_{\substack{n=N \\ n|x|>1}}^{M} n^{2\alpha-2} n J_0(\lambda_n|x|)^2 + \sum_{\substack{n=N \\ n|x|\le 1}}^{M} n^{2\alpha-2} n J_0(\lambda_n|x|)^2.$$

On the first part we use the estimate $J_0(y) \in \mathcal{O}(y^{-\frac{1}{2}})$ for large $y = n|x| > 1$. On the second part we use $J_0 \in L^\infty$:

$$\lesssim |x|^{-1} \sum_{M \wedge \lceil |x|^{-1} \rceil \vee N < n \le M} n^{2\alpha-2} + \sum_{N \le n \le M \wedge \lfloor |x|^{-1} \rfloor \vee N} n^{2\alpha-1}.$$

In our notation we have the minimum $\wedge$ bind more strongly than the maximum $\vee$ so that $a \wedge b \vee c = (a \wedge b) \vee c$. Now we estimate the sums by corresponding integrals.

$$\le |x|^{-1} \int_{M \wedge \lceil |x|^{-1} \rceil \vee N-1}^{M-1} s^{2\alpha-2}\, ds + N^{2\alpha-1} + \int_{N}^{M \wedge \lfloor |x|^{-1} \rfloor \vee N} s^{2\alpha-1}\, ds = (\star).$$

In the case $\alpha = 0$ we get

$$(\star) = \frac{1}{N} + \left( \frac{|x|^{-1}}{M \wedge \lceil |x|^{-1} \rceil \vee N - 1} - \frac{|x|^{-1}}{M-1} \right) + \ln(M \wedge \lfloor |x|^{-1} \rfloor \vee N) - \ln(N).$$

If $|x|^{-1} \geq M$, then the expression in the brackets vanishes. If on the other hand $|x|^{-1} < M$, then

$$\frac{|x|^{-1}}{M \wedge \lceil |x|^{-1} \rceil \vee N - 1} - \frac{|x|^{-1}}{M-1} = \frac{|x|^{-1}}{\lceil |x|^{-1} \rceil \vee N - 1} - \frac{|x|^{-1}}{M-1} \leq \frac{\lceil |x|^{-1} \rceil}{\lceil |x|^{-1} \rceil \vee N - 1}.$$

If $\lceil |x|^{-1} \rceil \leq N$ then this is less than $2\frac{|x|^{-1}}{N-1}$. On the other hand if $\lceil |x|^{-1} \rceil > N$ then it is less than $\frac{\lceil |x|^{-1} \rceil}{\lceil |x|^{-1} \rceil - 1} \leq 2$. We therefore have

$$(\star) \leq \begin{cases} \frac{1}{N} + 2 + \ln\left(\frac{M \wedge |x|^{-1}}{N}\right) & , |x|^{-1} > N \\ \frac{1}{N} + \frac{2|x|^{-1}}{N-1} & , |x|^{-1} \leq N \end{cases}$$

for almost all $x \in B$. In the case $\alpha > 0$ we have

$$\begin{aligned} (\star) = & \frac{1}{1-2\alpha} |x|^{-1} \left( \left(M \wedge \lceil |x|^{-1} \rceil \vee N - 1\right)^{2\alpha - 1} - (M-1)^{2\alpha-1} \right) \\ & + \frac{1}{N^{1-2\alpha}} + \frac{1}{2\alpha} \left( \left(M \wedge \lfloor |x|^{-1} \rfloor \vee N\right)^{2\alpha} - N^{2\alpha} \right). \end{aligned}$$

Again if $|x|^{-1} \geq M$ then the first term vanishes. In the other case we have

$$|x|^{-1} \left( \left(M \wedge \lceil |x|^{-1} \rceil \vee N - 1\right)^{2\alpha-1} - (M-1)^{2\alpha-1} \right) \leq |x|^{-1} \left(\lceil |x|^{-1} \rceil \vee N - 1\right)^{2\alpha - 1}.$$

If $\lceil |x|^{-1} \rceil \leq N$ then this is less than $2|x|^{-1}(N-1)^{2\alpha-1}$. On the other hand if $\lceil |x|^{-1} \rceil > N$ then it is less than $\frac{\lceil |x|^{-1} \rceil}{(\lceil |x|^{-1} \rceil - 1)^{1-2\alpha}} \leq \lceil |x|^{-1} \rceil^{2\alpha} + 1 \leq 1 + 2|x|^{-2\alpha}$. We get

$$(\star) \leq \begin{cases} \frac{1}{N^{1-2\alpha}} + \frac{1}{1-2\alpha}\left(1 + 2|x|^{-2\alpha}\right) + \frac{1}{2\alpha}\left( \left(M \wedge |x|^{-1}\right)^{2\alpha} - N^{2\alpha} \right) & , |x|^{-1} > N \\ \frac{1}{N^{1-2\alpha}} + \frac{1}{1-2\alpha} \frac{2|x|^{-1}}{(N-1)^{1-2\alpha}} & , |x|^{-1} \leq N \end{cases}.$$

□

We now use Lemma 2.12 to show that certain gaussian processes, in particular $\psi$, are in Lebesgue and Sobolev spaces. We critically use the inequality

$$\mathbb{E}\left[|X|^p\right] \leq C^p p^{\frac{p}{2}} \, \mathbb{E}\left[|X|^2\right]^{\frac{p}{2}} \tag{2.18}$$

for $p \geq 1$ and a Gaussian random variable $X$. The following lemma is significantly stronger than what is necessary for the local and global well-posedness. The only case that we will use in the near future is that of $N = 1$. The general case will then be used for Theorem 3.8.

**Lemma 2.13.** *For every $\alpha < \frac{1}{2}$ there exists a constant $C = C(\alpha) > 0$ for which the following holds:*

*Let $\{X_n(t)\}_{n\in\mathbb{N}}$ be a family of independent and centered normally distributed random variables for all $t \in [0,\infty)$, i.e. a sequence of Gaussian processes. Let $\sigma_n^2(t)$ be their variances and assume that there exists a square integrable function $\beta : [0,\infty) \longrightarrow [0,\infty)$ so that*

$$\sigma_n^2(t) \leq \frac{\beta(t)^2}{\langle\lambda_n\rangle^2}$$

*for all $t \geq 0$. Now for $N < M \in \mathbb{N}\cup\{\infty\}$ define*

$$G_N^M(t,x) := \sum_{n=N}^{M} X_n(t)e_n(x).$$

*Then the following hold:*

*(i) There exists $C > 0$ so that for all $p \geq 1$ and $N < M \in \mathbb{N}\cup\{\infty\}$ we have*

$$\mathbb{E}\left[\|G_N^M\|_{L^p_{t,x}}^p\right] \leq \frac{C^p p^p \|\beta\|_{L^p_t}^p (1+\ln(N))}{N^2}.$$

*and*

$$\mathbb{E}\left[\|G_N^M(0)\|_{L^p_x}^p\right] \leq \frac{C^p p^p |\beta(0)|^p (1+\ln(N)}{N^2}.$$

*(ii) For all $0 < \alpha < \frac{1}{2}$ there exists $C(\alpha) > 0$ so that for all $p \geq 1$ with $\alpha p < 2$ and for all $N < M \in \mathbb{N}\cup\{\infty\}$ we have*

$$\mathbb{E}\left[\|G_N^M\|_{L^p_t W^{\alpha,p}_x}^p\right] \leq \frac{C^p \|\beta\|_{L^p_t}^p (1+\ln(N)}{N^{2-\alpha p}}.$$

*and*

$$\mathbb{E}\left[\|G_N^M(0)\|_{W^{\alpha,p}_x}^p\right] \leq \frac{C^p |\beta(0)|^p (1+\ln(N)}{N^{2-\alpha p}}.$$

*Proof.* We only write down the proof for the integral in time cases as the pointwise in time cases can be dealt with in the same fashion, just without the time integral. It suffices to assume $M < \infty$ as $M = \infty$ then follows by taking the limit. In this case $G_N^M$ is smooth in $x$ and so we do not have to approximate. Note that by independence of the $X_n$ we have

$$\mathbb{E}\left[\left|\sum_{n=1}^{N}\langle\lambda_n\rangle^{2\alpha}X_n(t)e_n(x)\right|^2\right] = \sum_{n=1}^{N} X_n(t)e_n(x)^2.$$

We use (2.18) to estimate

$$\begin{aligned}\mathbb{E}\left[\int_0^\infty\int_B |\langle\nabla\rangle^\alpha G_N^M(t,x)|^p\,dx\,dt\right] &\lesssim C^p p^{\frac{p}{2}}\int_0^\infty\int_B \mathbb{E}\left[|\langle\nabla\rangle^\alpha G_N^M(t,x)|^2\right]^{\frac{p}{2}}\,dx\,dt\\ &\lesssim C^p p^{\frac{p}{2}}\int_0^\infty\int_B\left(\sum_{n=N}^{M}\langle\lambda_n\rangle^{2\alpha}\sigma_n^2(t)e_n(x)^2\right)^{\frac{p}{2}}\,dx\,dt\\ &\leq C^p p^{\frac{p}{2}}\int_0^\infty|\beta(t)|^p\,dt\int_B\left(\sum_{n=N}^{M}\langle\lambda_n\rangle^{2\alpha-2}e_n(x)^2\right)^{\frac{p}{2}}\,dx.\end{aligned}$$

We now apply the estimates from the previous Lemma 2.12. For $\alpha = 0$ we get

$$\lesssim C^p p^{\frac{p}{2}} \|\beta\|_{L_t^p}^p \left( \int_{\{|x| \le N^{-1}\}} \left( \frac{1}{N} + 1 + \ln\left( \frac{M \wedge |x|^{-1}}{N} \right) \right)^{\frac{p}{2}} dx \right.$$
$$\left. + \int_{\{|x| > N^{-1}\}} \left( \frac{1}{N} + \frac{|x|^{-1}}{N-1} \right)^{\frac{p}{2}} dx \right).$$

(We also prove a bound for this in Lemma 2.14). For $\alpha > 0$ we get

$$\lesssim C^p p^{\frac{p}{2}} \|\beta\|_{L_t^p}^p \left( \int_{\{|x| \le N^{-1}\}} \left( \frac{1}{N^{1-2\alpha}} + \frac{1}{1-2\alpha} \left(1 + |x|^{-2\alpha}\right) + \frac{1}{2\alpha} \left( \left(M \wedge |x|^{-1}\right)^{2\alpha} - N^{2\alpha} \right) \right)^{\frac{p}{2}} dx \right.$$
$$\left. + \int_{\{|x| > N^{-1}\}} \left( \frac{1}{N^{1-2\alpha}} + \frac{1}{1-2\alpha} \frac{2|x|^{-1}}{(N-1)^{1-2\alpha}} \right)^{\frac{p}{2}} dx \right).$$

The estimates for those integrals that we derive in the subsequent Lemma 2.14 conclude the proof. ☐

**Lemma 2.14.** *(i) There exists $C > 0$ so that for all $p \ge 1$ and $N < M \in \mathbb{N}$,*

$$\int_{\{|x| \le N^{-1}\}} \left( \frac{1}{N} + 1 + \ln\left( \frac{M \wedge |x|^{-1}}{N} \right) \right)^p dx \le \frac{C^p p^p}{N^2}.$$

*(ii) There exists $C > 0$ so that for all $p \ge 1$ and $N < M \in \mathbb{N}$,*

$$\int_{\{1 \ge |x| > N^{-1}\}} \left( \frac{1}{N} + \frac{|x|^{-1}}{N-1} \right)^p dx \le \frac{C^p \ln(N)}{N^2}.$$

*(iii) For every $0 < \alpha < \frac{1}{2}$ there exists $C(\alpha) > 0$ so that for every $p \ge 1$ with $\alpha p < 1$,*

$$\int_{\{1 \ge |x| \le N^{-1}\}} \left( \frac{1}{N^{1-2\alpha}} + \frac{1}{1-2\alpha} \left(1 + |x|^{-2\alpha}\right) + \frac{1}{2\alpha} \left( \left(M \wedge |x|^{-1}\right)^{2\alpha} - N^{2\alpha} \right) \right)^p dx \le \frac{C(\alpha)^p}{N^{2-2\alpha p}}.$$

*(iv) For every $0 < \alpha < \frac{1}{2}$ there exists $C(\alpha) > 0$ so that for every $p \ge 1$ with $\alpha p < 1$,*

$$\int_{\{|x| > N^{-1}\}} \left( \frac{1}{N^{1-2\alpha}} + \frac{1}{1-2\alpha} \frac{2|x|^{-1}}{(N-1)^{1-2\alpha}} \right)^p dx \le \frac{C(\alpha)^p \ln(N)}{N^{2-2\alpha p}}.$$

*Proof.* Since the domain of integration $B$ is bounded it suffices to consider the case $p > 2$. We start with $(i)$:

$$\begin{aligned}
&\int_{\{|x|\le N^{-1}\}} \left(\frac{1}{N}+1+\ln\left(\frac{M\wedge|x|^{-1}}{N}\right)\right)^p dx\\
\le&\sum_{k=0}^{\infty}(k+1)^p\mathcal{L}^2\left(\left\{x\in B:|x|\le N^{-1}, k<2+\ln\left(\frac{M\wedge|x|^{-1}}{N}\right)\le k+1\right\}\right)\\
\le&\sum_{k=0}^{\infty}(k+1)^p\mathcal{L}^2\left(\left\{x\in B:|x|\le N^{-1}, Ne^{k-2}<M\wedge|x|^{-1}\le Ne^{k-1}\right\}\right)\\
\le&\mathcal{L}^2(B)N^{-2}\sum_{k=0}^{\infty}(k+1)^p e^{4-2k}\,dx\\
\lesssim&\frac{e^4}{N^2}\int_0^\infty (s+1)^p e^{-2s}\,dx\\
\lesssim&\frac{C_1^p}{N^2}2^{-(p+1)}\Gamma(p+1)\lesssim\frac{C_2^p p^p}{N^2}.
\end{aligned}$$

Now we show $(ii)$. We can trivially assume $N\ge 2$ and $p>2$. Then

$$\begin{aligned}
\int_{\{|x|>N^{-1}\}}\left(\frac{1}{N}+\frac{|x|^{-1}}{N-1}\right)^p dx &\le \frac{1}{(N-1)^p}\int_{N^{-1}}^1\left(1+\frac{1}{r}\right)^p r\,dr\\
&\le\frac{2^p}{N^p}\int_{N^{-1}}^1 r^{1-p}\,dr=\frac{2^p}{N^p}\frac{1}{2-p}\left(1-N^{p-2}\right)\\
&=\frac{2^p}{N^2}\frac{1-N^{2-p}}{p-2}\le\frac{C^p\ln(N)}{N^2}.
\end{aligned}$$

We have used that for $\epsilon>0$ and $y\ge 1$ we have $\frac{1-y^{-\epsilon}}{\epsilon}\le\ln(y)$.

Next we show $(iii)$:

$$\begin{aligned}
&\int_{\{|x|\le N^{-1}\}}\left(\frac{1}{N^{1-2\alpha}}+\frac{1}{1-2\alpha}\left(1+|x|^{-2\alpha}\right)+\frac{1}{2\alpha}\left(\left(M\wedge|x|^{-1}\right)^{2\alpha}-N^{2\alpha}\right)\right)^p dx\\
\lesssim&\, C(\alpha)^p\int_{\{|x|\le N^{-1}\}}\left(1+|x|^{-2\alpha}+\left(\left(M\wedge|x|^{-1}\right)^{2\alpha}-N^{2\alpha}\right)\right)^p ds\\
\lesssim&\, C(\alpha)^p\int_0^{N^{-1}}(1+r^{-2\alpha})^p r\,dr\le C(\alpha)^p2^p\int_0^{N^{-1}}r^{1-2\alpha p}\,dr\le C_1(\alpha)^p\frac{1}{N^{2-2\alpha p}}.
\end{aligned}$$

Finally we show $(iv)$:

$$\begin{aligned}
\int_{\{|x|>N^{-1}\}}\left(\frac{1}{N^{1-2\alpha}}+\frac{1}{1-2\alpha}\frac{2|x|^{-1}}{(N-1)^{1-2\alpha}}\right)^p dx &\lesssim C(\alpha)^p\frac{1}{N^{p-2\alpha p}}\int_{N^{-1}}^1\left(1+\frac{1}{r}\right)^p r\,dr\\
&\lesssim\frac{C(\alpha)^p\ln(N)}{N^{2-2\alpha p}}.
\end{aligned}$$

Here we use that in the proof of $(ii)$ we have already estimated this integral by $N^{p-2}\ln(N)$. $\square$

We now use Lemma 2.13 to show that $\psi$ is in certain Lebesgue and Sobolev spaces. This is important because $\psi(t,x)$ will show up in the non-linear part of the equation later on. We need to be able to make sense of $\psi(t,x)^3$. This is not possible if $\psi(t,x)$ merely a distribution in space and not given by a function. In higher dimensions this happens and as a result one has to renormalize the equation.

**Lemma 2.15.** *Let $T > 0$ and $0 < \alpha < \frac{1}{2}$. The following hold:*

*(i) $\psi(t,x)$ is a centered Gaussian random variable for almost all $t \in [0,T]$ and $x \in B$. We have an estimate*

$$\mathbb{E}\left[|\psi(t,x)|^2\right] \leq C(1 + |\ln(|x|)|).$$

*(ii) There exists $C > 0$ so that for all $p \geq 1$ and $t \in [0,T]$ we have $\psi(t,\cdot) \in L^p(B)$ a.s, $\psi \in L^p([0,T], L^p_{\mathrm{r}}(B))$ a.s. and*

$$\mathbb{E}\left[\int_B |\psi(t,x)|^p \, dx\right] \leq C^p p^p \qquad \mathbb{E}\left[\int_0^T \int_B |\psi(t,x)|^p \, dx \, dt\right] \leq TC^p p^p.$$

*(iii) Let $p \geq 1$ so that $\alpha p < 2$. Then $\langle \nabla \rangle^\alpha \psi(t,x)$ is a centered Gaussian random variable for almost all $t \in [0,T]$ and $x \in B$. We have an estimate*

$$\mathbb{E}\left[|\langle \nabla \rangle^\alpha \psi(t,x)|^2\right] \leq C(\alpha)|x|^{-2\alpha}.$$

*(iv) There exists $C(\alpha) > 0$ so that for $p \geq 1$ with $\alpha p < 2$ and for almost all $t \in [0,T]$ we have $\psi(t,\cdot) \in W^{\alpha,p}_{\mathrm{r}}(B)$ a.s., $\psi \in L^p([0,T] \times W^{\alpha,p}_{\mathrm{r}}(B))$ and*

$$\mathbb{E}\left[\int_B |\langle \nabla \rangle^\alpha \psi(t,x)|^p \, dx\right] < C(\alpha)^p \qquad \mathbb{E}\left[\int_0^T \int_B |\langle \nabla \rangle^\alpha \psi(t,x)|^p \, dx \, dt\right] \leq TC(\alpha)^p.$$

*Proof.* We see that for any $M \in \mathbb{N}$ by independence of the $g_n$

$$\mathbb{E}\left[\left|\sum_{n=1}^{M} \langle \lambda_n \rangle^{2\alpha} c_n(t) e_n(x) g_n(t)\right|^2\right] = \sum_{n=1}^{N} c_n^2(t) e_n(x)^2.$$

We know that on any compact interval $[0,T]$ we have $c_n^2(t) \lesssim \langle \lambda_n \rangle^{-2}$ uniformly in $t$. since by Lemma 2.12 this is finite we know that $\langle \nabla \rangle^\alpha \psi(t,x)$ is a centered Gaussian r.v. in $L^2(\Omega)$. We now define a Gaussian process $X_n(t) = \mathbb{1}_{\{t>T\}} \langle \psi(t), e_n \rangle$. Then we can apply Lemma 2.13 for $N = 1$ to $G$ and get the desired estimates. Since $G$ is indistinguishable from $\psi$ on $[0,T]$ they transfer over to $\psi$. □

We can furthermore find almost sure continuity in certain Sobolev spaces via the Kolmogorov continuity theorem.

**Lemma 2.16.** *Let $T > 0$, $\alpha < \frac{1}{2}$, $1 \leq p < \infty$ so that $\alpha p < 2$ and let $0 < \gamma < \frac{1}{2} - \frac{1}{p}$. Then there exists a modification $\tilde{\psi} \in C^{0,\gamma}([0,T], W^{\alpha,p}_{\mathrm{r}})$ of $\psi$.*

*Proof.* We will use the Kolmogorov continuity theorem. From the previous lemma we already know that $\psi(t,\cdot) \in W_{\mathrm{r}}^{\alpha,p}$ a.s., so to apply the theorem it remains to find a $\beta > 0$ so that for all $t_0 < t_1 \le T$,

$$\mathbb{E}\left[\int_B |\langle\nabla\rangle^\alpha(\psi(t_1,x) - \psi(t_0,x))|^p\,dx\right] \le C(T,p)|t_1 - t_0|^{1+\beta}.$$

For this we again use Lemma 2.13, applying it to

$$G(t,x) = \psi(t+t_1,x) - \psi(t+t_0,x) = \sum_{n=1}^{\infty}(c_n(t+t_1)g_n(t+t_1) - c_n(t+t_0)g_n(t+t_0))e_n(x).$$

We want to estimate the variance of the gaussians on the right side. Using (2.14) and get that for $|t_1 - t_0| < 1$ (an assumption we can make without loss of generality) and any $\eta \in (0,2)$,

$$\mathbb{E}\left[(c_n(t+t_1)g_n(t+t_1) - c_n(t+t_0)g_n(t+t_0))^2\right] \le C|t_1 - t_0|\left[\lambda_n\right]^{-2} \le C\frac{|t_1 - t_0|}{\langle\lambda_n\rangle^2}.$$

Now lemma 2.13 with $N = 1$ and $t = 0$ yields the inequality

$$\mathbb{E}\left[\|\psi(t_1,\cdot) - \psi(t_0,\cdot)\|_{W^{\alpha,p}}^p\,dx\right] \le C(\alpha)^p|t_1 - t_0|^{1+\frac{p-2}{2}}.$$

The Kolmogorov continuity theorem now implies that there for any

$$0 < \gamma < \frac{p-2}{2p} = \frac{1}{2} - \frac{1}{p}.$$

there exists a modification $\tilde{\psi} \in C^{0,\gamma}([0,T], W_{\mathrm{r}}^{\alpha,p})$ of $\psi$.

□

## 2.3 Local Well-Posedness for the Complete Problem

We now suppose that we have are given the solutions $\mathbf{w}$ and $\Psi$ from the previous sections and consider them fixed. For given realizations $\mathbf{w}(\omega)$ and $\Psi(\omega)$ we now have to find a solution $\mathbf{v}(\omega)$ to

$$\partial_t^2 v + \partial_t v + (1-\Delta)v + (w + \psi + v)^3 = 0 \tag{2.19}$$

with initial data 0. In terms of pairs of functions this is equivalent to

$$\begin{aligned} \partial_t \mathbf{v} &= L\mathbf{v} - \begin{pmatrix} 0 \\ (w+\psi+v)^3 \end{pmatrix} \\ \mathbf{v}(0) &= 0. \end{aligned} \tag{2.20}$$

We solve this as a deterministic problem with the idea being that we can assume $\mathbf{w}(\omega) + \Psi(\omega)$ to be an arbitrary function that has a regulartiy we **know** the stochastic processes $\mathbf{w}$ and $\Psi$ to possess almost surely.

We are also interested in the truncated system

$$\partial_t^2 v_N + \partial_t v_N + (1-\Delta)v_N + P_N((w+\psi+v_N)^3) = 0 \tag{2.21}$$

with initial data 0. Here **truncation** refers to the projection $P_N$ onto the first $N$ basis vectors:

**Definition 2.17.** *For $N \in \mathbb{N}$, $\beta \in \mathbb{R}$ and $f \in H^\beta$ define*

$$P_N f = \sum_{n=1}^{N} \hat{f}(n) e_n.$$

**Lemma 2.18.** *For all $k \in \mathbb{N}$ and $\beta_1, \beta_2 \in \mathbb{R}$,*

$$P_N : C^k([0,T], H^{\beta_1}) \longrightarrow C^k([0,T], H^{\beta_2}).$$

*Proof.* Let $f \in C^k([0,T], H^{\beta_1})$. We write down the proof only for $k = 1$:

$$\left\| \frac{P_N f(t+h) - P_N f(t)}{h} - P_N \partial_t f(t) \right\|_{H^{\beta_1}}^2$$

$$= \sum_{n=1}^{N} \underbrace{\langle \lambda_n \rangle^{2\frac{\beta_1}{\beta_2}}}_{\lesssim N^{\left|\frac{\beta_1}{\beta_2}\right|}} \langle \lambda_n \rangle^{2\beta_2} \left\langle \frac{f(t+h) - f(t)}{h} - \partial_t f(t), e_n \right\rangle.$$

$$\leq C(N, \beta_1, \beta_2) \left\| \frac{f(t+h) - f(t)}{h} - \partial_t f(t) \right\|_{H^{\beta_2}}^2$$

□

We will solve(2.19) and (2.21) in a mild sense.

**Definition 2.19** (Mild Solution)**.** *Let $\alpha \in \left[1, \frac{4}{3}\right)$ and $T \in \mathbb{R}_+ \cup \{\infty\}$. Let $w, \psi$ be functions so that*

$$w + \psi \in Y_{[0,T]}^{\alpha-1} := L^6([0,T], W_{\mathrm{r}}^{\alpha-1,6}). \tag{2.22}$$

*A **mild solution in $\mathcal{H}_{\mathrm{r}}^\alpha$ on** $[0,T]$ **to** (2.19) is a function*

$$v \in L^\infty([0,T], \mathcal{H}_{\mathrm{r}}^\alpha)$$

*so that for almost all $t \in [0,T]$,*

$$\mathbf{v}(t) = \int_0^t S(t-s) \begin{pmatrix} 0 \\ -(w+\psi+v)^3(s) \end{pmatrix} ds. \tag{2.23}$$

*We analogously define mild solutions to the truncated system (2.21).*

The justification for thse objects being called solutions to the problem is that they are in fact solutions in quite a strong sense, and in particular also in the weaker sense of distributions. This is shown in lemma 2.23.

Defining the operators

$$H(\mathbf{v}) := t \longmapsto \int_0^t S(t-s) \begin{pmatrix} 0 \\ -(w+\psi+v)^3(s) \end{pmatrix} ds$$

and

$$H_N(\mathbf{v}_N) \coloneqq P_N H(\mathbf{v}_N) t \longmapsto \int_0^t S(t-s) \begin{pmatrix} 0 \\ -P_N((w+\psi+v_N)^3)(s) \end{pmatrix} ds,$$

we can phrase (2.19) as the fixed point problem $H(\mathbf{v}) = \mathbf{v}$, and similarly (2.21) as $H_N(\mathbf{v}_N) = \mathbf{v}_N$. We choose the space $L^\infty([0,T], \mathcal{H}_\mathrm{r}^\alpha)$ where $\alpha < \frac{4}{3}$. Note that $H_N : L^\infty([0,T], \mathcal{H}_\mathrm{r}^\alpha) \longrightarrow L^\infty([0,T], P_N \mathcal{D}'_\mathrm{r}(B))$, so $\mathbf{v}_N$ is an evolution in an $N$-dimensional subspace of the radial distributions.

Let us recall two inequalities that will be needed for the local well-posedness: The fractional Leibnitz and Sobolev inequalities.

**Lemma 2.20** (Fractional Leibnitz Inequality / Fractional Leibnitz Rule [11])**.** *For* $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{p_2}$, $\alpha - 1 \in (0,1)$ *and* $f, g \in \mathcal{S}(\mathbb{R}^n)$

$$\|\langle\nabla\rangle^{\alpha-1}(fg)\|_{L^r} \lesssim \|\langle\nabla\rangle^{\alpha-1} f\|_{L^{p_1}} \|g\|_{L^{q_1}} + \|\langle\nabla\rangle^{\alpha-1} g\|_{L^{p_2}} \|f\|_{L^{q_2}}.$$

*In particular for all* $\alpha - 1 \in [0,1)$ *we have*

$$\|\langle\nabla\rangle^{\alpha-1}(f^3)\|_{L^2} \lesssim \|\langle\nabla\rangle^{\alpha-1} f\|_{L^6} \|f\|_{L^6}^2. \tag{2.24}$$

*Proof.* A proof can be found in [11]. The second inequality for $\alpha - 1 > 0$ follows from it by an application with $\frac{1}{2} = \frac{1}{6} + \frac{1}{3} = \frac{1}{3} + \frac{1}{6}$:

$$\|\langle\nabla\rangle^{\alpha-1}(f^3)\|_{L^2} \lesssim \|\langle\nabla\rangle^{\alpha-1} f\|_{L^6} \|f^2\|_{L^3} + \|\langle\nabla\rangle^{\alpha-1}(f^2)\|_{L^3} \|f\|_{L^6}$$

and another one with $\frac{1}{3} = \frac{1}{6} + \frac{1}{6}$:

$$\|\langle\nabla\rangle^{\alpha-1}(f^2)\|_{L^3} \lesssim \|\langle\nabla\rangle^{\alpha-1} f\|_{L^6} \|f\|_{L^6}.$$

The case $\alpha = 0$ is trivial. □

For the Sobolev inequalities we need to make sure that we actually have the embedding for the Bessel potential spaces $H^{\alpha,p}(\mathbb{R}^2)$, not the Sobolev-Slobodeckij spaces $\overline{W}^{\alpha,p}(\mathbb{R}^2)$. Under [22] one can find embeddings for the Triebel-Lizorkin scale which in our special case yield the following lemma:

**Lemma 2.21** (Fractional Sobolev Embeddings)**.** *Let* $1 < p_1 \leq p_2 < \infty$ *and* $\alpha_1, \alpha_2 \geq 0$. *If*

$$\alpha_1 - \frac{2}{p_1} \geq \alpha_2 - \frac{2}{p_2},$$

*then*

$$W_\mathrm{r}^{\alpha_1,p_1} \hookrightarrow W_\mathrm{r}^{\alpha_2,p_2}.$$

This implies for example that $H_\mathrm{r}^1 \hookrightarrow L_\mathrm{r}^p$ for all $1 \leq p < \infty$. We care about the following special case: Given some $\alpha \in \left([1, \frac{4}{3}\right)$ we choose $\alpha_1 = \alpha$, $p_1 = 2$ and $p_2 = 6$. Then for all $\alpha_2$ so that

$$\alpha_2 \leq \alpha - 1 + \frac{1}{3} = \alpha - \frac{2}{3}$$

we have $\|u\|_{W^{\alpha_2,6}} \lesssim \|u\|_{H^\alpha}$. In particular

$$\|u\|_{W^{\alpha-1,6}} \lesssim \|u\|_{H^1}. \tag{2.25}$$

**Theorem 2.22** (Deterministic Local Well-posedness)**.** *Let $\alpha \in \left[1, \frac{4}{3}\right)$ and $w + \psi \in Y^{\alpha-1}_{[0,T']}$ be given as in (2.22). Then there for some $0 < T(\|w+\psi\|_{Y^{\alpha-1}_{[0,T']}}) \leq T'$ there exist unique mild solutions $\mathbf{v}$ and $\mathbf{v}_N$ on $[0,T]$ to (2.19) and (2.21) respectively. For every $r > 0$ the maps*

$$\begin{aligned} B_r^{Y^{\alpha-1}_{[0,T']}} \times [0, T(r)] &\longrightarrow \mathcal{H}^\alpha_{\mathrm{r}} \\ (w+\psi, t) &\longmapsto \mathbf{v}(t) \\ (w+\psi, t) &\longmapsto \mathbf{v}_N(t) \end{aligned}$$

*are jointly continuous.*

*Proof.* We show this only for (2.19) as all the estimates directly transfer to the truncated case. Let $0 < R \leq \frac{1}{2}$ and suppose that

$$\|\mathbf{v}\|_{L^\infty([0,T],\mathcal{H}^\alpha)} \leq R.$$

From lemma 2.2 we know that $\|S(t)u\|_{\mathcal{H}^\alpha} \leq e^{-\frac{t}{2}}\|u\|_{\mathcal{H}^\alpha}$, so

$$\|H(\mathbf{v})(t)\|_{\mathcal{H}^\alpha} \leq \int_0^t \|(w+\psi+v)^3\|_{H^{\alpha-1}}\, ds.$$

We use (2.24)

$$\lesssim \int_0^t \|\langle\nabla\rangle^{\alpha-1}(w+\psi+v)\|_{L^6}\|w+\psi+v\|^2_{L^6}\, ds$$

and Hölder with $\frac{1}{2}+\frac{1}{6}+\frac{1}{3}=1$:

$$\begin{aligned} &\lesssim T^{\frac{1}{2}}\|w+\psi+v\|_{L^6([0,T],W^{\alpha-1,6})}\|w+\psi+v\|^2_{L^6([0,T],L^6)} \\ &\lesssim T^{\frac{1}{2}}\|w+\psi+v\|^3_{L^6([0,T],W^{\alpha-1,6})} \\ &\lesssim T^{\frac{1}{2}}\left(\|w+\psi\|^3_{Y^{\alpha-1}_{[0,T]}} + \|v\|^3_{L^6([0,T],W^{\alpha-1,6})}\right) \end{aligned}$$

We use that by (2.25)

$$\|v\|_{L^6([0,T],L^6)} \lesssim \|v\|_{L^6([0,T],H^\alpha)} \leq T^{\frac{1}{6}}R$$

and get

$$\|H(\mathbf{v})\|_{L^\infty([0,T],\mathcal{H}^\alpha)} \lesssim T^{\frac{1}{2}}\|w+\psi\|^3_{Y^{\alpha-1}_{[0,T]}} + TR^3.$$

From this we can see that if $T(\|w+\psi\|_{Y^{\alpha-1}_{[0,T']}})$ is small enough we get a selfmap

$$H : B_R^{L^\infty([0,T],\mathcal{H}^\alpha_{\mathrm{r}})} \longrightarrow B_R^{L^\infty([0,T],\mathcal{H}^\alpha_{\mathrm{r}})}.$$

We want to show that $H$ is a contraction on $B_R$, so let $\mathbf{v}_1, \mathbf{v}_2 \in B_R$.

$$\begin{aligned}
\|H(\mathbf{v}_1)(t) - H(\mathbf{v}_2)(t)\|_{\mathcal{H}^\alpha} &\leq \int_0^t \|(w+\psi+v_1)^3 - (w+\psi+v_2)^3\|_{H^{\alpha-1}}\, ds \\
&\leq 3\int_0^t \|(w+\psi)^2(v_1-v_2)\|_{H^{\alpha-1}}\, ds \\
&+ 3\int_0^t \|(w+\psi)(v_1-v_2)^2\|_{H^{\alpha-1}}\, ds \\
&+ \int_0^t \|(v_1-v_2)^3\|_{H^{\alpha-1}}\, ds \\
&= (I) + (II) + (III)
\end{aligned}$$

We apply the fractional Leibnitz inequality several times.

$$\begin{aligned}
(I) &\lesssim \int_0^t \|(w+\psi)^2\|_{W^{\alpha-1,3}}\|v_1-v_2\|_{L^6} + \|(w+\psi)^2\|_{L^3}\|v_1-v_2\|_{W^{\alpha-1,6}}\, ds \\
&\lesssim \int_0^t \big(\|w+\psi\|_{W^{\alpha-1,6}}\|w+\psi\|_{L^6} + \|w+\psi\|_{L^6}^2\big)\|v_1-v_2\|_{W^{\alpha-1,6}}\, ds
\end{aligned}$$

Now we Hölder with $\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1$.

$$\begin{aligned}
&\lesssim T^{\frac{1}{2}}\|v_1-v_2\|_{L^6([0,T],W^{\alpha-1,6})}\|w+\psi\|_{L^6([0,T],W^{\alpha-1,6})}^2 \\
&\lesssim T^{\frac{2}{3}}\|\mathbf{v}_1-\mathbf{v}_2\|_{L^\infty([0,T],\mathcal{H}^\alpha)}\|w+\psi\|_{Y^{\alpha-1}_{[0,T]}}^2 \qquad (2.26)
\end{aligned}$$

Here we $(II)$ we proceed similarly

$$\begin{aligned}
(II) &\lesssim \int_0^t \|w+\psi\|_{W^{\alpha-1,6}}\|(v_1-v_2)^2\|_{L^3} + \|w+\psi\|_{L^6}\|(v_1-v_2)^2\|_{W^{\alpha-1,3}}\, ds \\
&\lesssim \int_0^t \big(\|w+\psi\|_{W^{\alpha-1,6}} + \|w+\psi\|_{L^6}\big)\|v_1-v_2\|_{W^{\alpha-1,6}}^2\, ds \\
&\lesssim T^{\frac{1}{2}}\|v_1-v_2\|_{L^6([0,T],W^{\alpha-1,6})}^2\|w+\psi\|_{L^6([0,T],W^{\alpha-1,6})} \\
&\lesssim T^{\frac{5}{6}}\|\mathbf{v}_1-\mathbf{v}_2\|_{L^\infty([0,T],\mathcal{H}^\alpha)}^2\|w+\psi\|_{Y^{\alpha-1}_{[0,T]}} \qquad (2.27)
\end{aligned}$$

Finally

$$(III) \lesssim \int_0^t \|v_1-v_2\|_{W^{\alpha-1,6}}^3\, ds \lesssim T^{\frac{1}{2}}\|v_1-v_2\|_{L^6([0,T],W^{\alpha-1,6})}^3 \lesssim T\|\mathbf{v}_1-\mathbf{v}_2\|_{L^\infty([0,T],\mathcal{H}^\alpha)}^3. \qquad (2.28)$$

Note that $\|\mathbf{v}_1-\mathbf{v}_2\|_{L^\infty([0,T],\mathcal{H}^\alpha_r)} \leq 1$ since $R \leq \frac{1}{2}$. Because of this we can again choose $T$ depending on $\|w+\psi\|_{Y^{\alpha-1}_{[0,T']}}$ small enough so that $H$ is a contraction. Then by the Banach fixed-point theorem there exists a unique $\mathbf{v} \in B_R$ so that $H(\mathbf{v}) = \mathbf{v}$.

It remains to show the continuity. We first show continuity separately, only in $t$ and only $w+\psi$. For some $r>0$, $w_1+\psi_1, w_2+\psi_2 \in B_r^{Y^{\alpha-1}_{[0,T']}}$ let $t \in [0,T(r)]$. We call $\mathbf{v}_1$ and $\mathbf{v}_2$ the corresponding solutions. Observe that

$$\|\mathbf{v}_1(t) - \mathbf{v}_2(t)\|_{\mathcal{H}^\alpha} \lesssim \int_0^t \|(w_1+\psi_1+v_1)^3 - (w_2+\psi_2+v_1)^3\|_{H^{\alpha-1}} \\ +\|(w_2+\psi_2+v_1)^3 - (w_2+\psi_2+v_2)^3\|_{H^{\alpha-1}}\, ds. \tag{2.29}$$

We now repeat the estimates for the contractivity until we arrive at only expressions with norm $\|\cdot\|_{Y^{\alpha-1}_{[0,t]}}$, but we reverse the role of $w+\psi$ and $v$ in the first instance:

$$\begin{aligned}\lesssim \Big(&\|w_1+\psi_1-w_2-\psi_2\|_{Y^{\alpha-1}_{[0,t]}}\|v_1-v_2\|^2_{Y^{\alpha-1}_{[0,t]}} && +\|v_1-v_2\|_{Y^{\alpha-1}_{[0,t]}}\|w_2+\psi_2\|^2_{Y^{\alpha-1}_{[0,t]}}\\ +&\|w_1+\psi_1-w_2-\psi_2\|^2_{Y^{\alpha-1}_{[0,t]}}\|v_1-v_2\|_{Y^{\alpha-1}_{[0,t]}} && +\|v_1-v_2\|^2_{Y^{\alpha-1}_{[0,t]}}\|w_2+\psi_2\|_{Y^{\alpha-1}_{[0,t]}}\\ +&\|w_1+\psi_1-w_2-\psi_2\|^3_{Y^{\alpha-1}_{[0,t]}} && +\|v_1-v_2\|^3_{Y^{\alpha-1}_{[0,t]}}\Big)\end{aligned}$$

We define

$$\begin{aligned} A &= \|v_1\|_{L^\infty([0,T],H^{\alpha-1})} + \|v_2\|_{L^\infty([0,T],H^{\alpha-1})}\\ B &= \|w_2+\psi_2\|_{Y^{\alpha-1}_{[0,T]}} + \|w_1+\psi_1\|_{Y^{\alpha-1}_{[0,T]}}\\ E &= (B^2+AB+A^2)\\ \alpha(t) &= E\|w_1+\psi_1-w_2-\psi_2\|_{Y^{\alpha-1}_{[0,t]}}.\end{aligned}$$

The inequality that we have derived can be simplified and rewritten as

$$\|\mathbf{v}_1(t)-\mathbf{v}_2(t)\|_{\mathcal{H}^\alpha} \le C\alpha(t) + \int_0^t CE\|\mathbf{v}_1(s)-\mathbf{v}_2(s)\|_{\mathcal{H}^\alpha}\, ds.$$

Now Grönwall's inequality implies that

$$\|\mathbf{v}_1-\mathbf{v}_2\|_{L^\infty([0,T],\mathcal{H}^\alpha)} \lesssim C\alpha(t)e^{CET}.$$

Let $w_n+\psi_n \longrightarrow w+\psi$ in $Y^{\alpha-1}_{[0,T']}$. Since these are bounded in $Y^{\alpha-1}_{[0,T']}$ they are contained in a ball of radius $r$ and we have a time of existence $T(r)$ for all of them. We can also choose a uniform $R$ in the existence result, meaning that $\mathbf{v}_n, \mathbf{v} \in B_R^{L^\infty([0,T(r)],\mathcal{H}^\alpha_r)}$. We can therefore set $A = 2R$ and $B = \|w+\psi\|_Y^{\alpha-1}$. Then we get

$$\|\mathbf{v}_n-\mathbf{v}\|_{L^\infty([0,T],\mathcal{H}^\alpha)} \le e^{CET}CE\|w_n+\psi_n-w-\psi\|_{Y^{\alpha-1}_{[0,T']}} \xrightarrow{n\to\infty} 0. \tag{2.30}$$

Therefore $\mathbf{v}(t)$ is continuous in $w+\psi$, uniformly in $t$. In the subsequence lemma 2.23 we show that $\mathbf{v}$ is continuous in $t$ with respect to $\|\cdot\|_{L^\infty([0,T],\mathcal{H}^\alpha)}$ for all $w+\psi \in Y^{\alpha-1}_{[0,T']}$. Then if $(t_n, w_n+\psi_n) \longrightarrow (t, w+\psi)$ we have

$$\|\mathbf{v}_n(t_n)-\mathbf{v}(t)\|_{\mathcal{H}^\alpha} \le \|\mathbf{v}_n(t_n)-\mathbf{v}(t)\|_{\mathcal{H}^\alpha} + \|\mathbf{v}(t_n)-\mathbf{v}(t)\|_{\mathcal{H}^\alpha} \longrightarrow 0.$$

□

**Lemma 2.23.** *Let* $\mathbf{v}$, $\mathbf{v}_N$ *be a mild solutions to (2.19) and (2.21) respectively. Then*

$$\mathbf{v}, \mathbf{v}_N \in C([0,T], \mathcal{H}^\alpha) \cap C^1([0,T], \mathcal{H}^{\alpha-1})$$

*and the derivatives in the latter space are given by*

$$\partial_t \mathbf{v} = L\mathbf{v} + \begin{pmatrix} 0 \\ -(w+\psi+v)^3 \end{pmatrix}.$$

$$\partial_t \mathbf{v}_N = L\mathbf{v}_N + \begin{pmatrix} 0 \\ -P_N((w+\psi+v_N)^3) \end{pmatrix}.$$

*Conversely, if* $\mathbf{v}$ *and* $\mathbf{v}_N$ *have these properties, then they are mild solutions.*

*Proof.* Again the proof for the case of $\mathbf{v}_N$ is virtually identical to the one for $\mathbf{v}$. Let $0 \le t \le t+h \le T$. Then

$$\begin{aligned}\|\mathbf{v}(t+h) - \mathbf{v}(t)\|_{\mathcal{H}^\alpha} \le & \int_0^t \left\| (S(t+h-s) - S(t-s)) \begin{pmatrix} 0 \\ -(w+\psi+v)^3 \end{pmatrix} \right\|_{\mathcal{H}^\alpha} ds \\ & + \int_t^{t+h} \left\| S(t+h-s) \begin{pmatrix} 0 \\ -(w+\psi+v)^3 \end{pmatrix} \right\|_{\mathcal{H}^\alpha} ds.\end{aligned} \tag{2.31}$$

By the arguments in the local well-posedness, in particular (2.24) and (2.25), we know that $\|(w+\psi+v)^3\|_{L^\infty([0,T],H^{\alpha-1})} < \infty$. Together with $S \in C([0,T], L(\mathcal{H}^\alpha_{\mathrm{r}}, \mathcal{H}^\alpha_{\mathrm{r}}))$ from lemma 2.2 this implies that (2.31) vanishes as $h \to 0$. This proves the continuity.

Now we show the differentiability. We estimate

$$\left\| \frac{\mathbf{v}(t+h) - \mathbf{v}(t)}{h} - L\mathbf{v}(t) - \begin{pmatrix} 0 \\ -(w+\psi+v)^3 \end{pmatrix} \right\|_{\mathcal{H}^{\alpha-1}}$$

$$\begin{aligned}\le & \left\| \int_0^t \left( \frac{S(t+h-s) - S(t-s)}{h} - LS(t-s) \right) \begin{pmatrix} 0 \\ -(w+\psi+v)^3 \end{pmatrix} ds \right\|_{\mathcal{H}^{\alpha-1}} \\ + & \left\| \int_t^{t+h} \left( \frac{S(t+h-s) - \mathrm{Id}}{h} \right) \begin{pmatrix} 0 \\ -(w+\psi+v)^3 \end{pmatrix} ds \right\|_{\mathcal{H}^{\alpha-1}}.\end{aligned}$$

Using the Lipschitz continuity in ($i$) from lemma 2.3 we see that the first integrand has a majorant with respect to $h$. Then ($ii$) from lemma 2.3 implies that it vanishes for every $s$, and so dominated convergence shows that the integral vanishes as $h \to 0$. For the second integral we also use the Lipschitz continuity in ($i$) to see that the integrand has a majorant with respect to $h$. Then the fact that the domain vanishes shows that the integral vanishes as $h \to 0$.

The converse statement that such $\mathbf{v}$ and $\mathbf{v}_N$ are mild solutions is merely a calculation analogous to the derivation of Duhamel's formula. □

Note that as one would expect, the truncated solutions $\mathbf{v}_N$ approximate $\mathbf{v}$.

**Lemma 2.24.** *Let* $\mathbf{v}$, $\mathbf{v}_N$ *be a mild solutions on* $[0,T]$ *to (2.19) and (2.21) respectively. Then there exists a constant* $C > 0$ *so that the following hold:*

*(i) We have an estimate*

$$\|\mathbf{v}-\mathbf{v}_N\|_{L^\infty([0,T],\mathcal{H}^\alpha)} \leq C\|(w+\psi+v)^3 - P_N(w+\psi+v)^3\|_{L^1_t H^{\alpha-1}_x} \times \exp\big(TC(1+\|w+\psi\|_{Y^{\alpha-1}_{[0,T]}})^2(1+\|\mathbf{v}\|_{L^\infty_t\mathcal{H}^\alpha_x}+\|\mathbf{v}_N\|_{L^\infty_t\mathcal{H}^\alpha_x})^3\big).$$

*(ii) There exist constants $C_1, C_2 > 0$ so that if*

$$\|(w+\psi+v)^3 - P_N(w+\psi+v)^3\|_{L^1_t H^{\alpha-1}_x} \leq C_1 e^{-2^p C_1 C_2 T},$$

*then* $\|\mathbf{v}-\mathbf{v}_N\|_{L^\infty([0,T],\mathcal{H}^\alpha)} \leq 1$.

*(iii)* $\|\mathbf{v}-\mathbf{v}_N\|_{L^\infty([0,T],\mathcal{H}^\alpha)} \xrightarrow{N\to\infty} 0$.

*Proof.* We start with $(i)$. Let $t \in [0,T]$. We apply the definition of a mild solution.

$$\begin{aligned}\|\mathbf{v}(t)-\mathbf{v}_N(t)\|_{\mathcal{H}^\alpha} &\leq \left\|\int_0^{t_1} S(t_1-s)\begin{pmatrix}0\\ -(w+\psi+v)^3+P_N(w+\psi+v)^3\end{pmatrix} ds\right\|_{\mathcal{H}^\alpha}\\ &+\left\|\int_0^{t_1} S(t_1-s)\begin{pmatrix}0\\ P_N(-(w+\psi+v)^3+(w+\psi+v_N)^3)\end{pmatrix} ds\right\|_{\mathcal{H}^\alpha}\\ &= (I)+(II)\end{aligned}$$

We define $\alpha_N \geq (I)$ by

$$\alpha_N := \|(w+\psi+v)^3 - P_N(w+\psi+v)^3\|_{L^1([0,T],H^{\alpha-1})} \xrightarrow{N\to\infty} 0.$$

For $(II)$ we can repeat the estimates (2.26), (2.27) and (2.28) with $\mathbf{v_1} = \mathbf{v}$ and $\mathbf{v_2} = \mathbf{v}_N$, except that we keep the time integral in order to apply Grönwall's inequality. We find a constant $C > 0$ so that

$$\begin{aligned}&(I)+(II) \qquad (2.32)\\ &\lesssim_C \|(w+\psi+v)^3 - P_N(w+\psi+v)^3\|_{L^1([t_0,t_1],H^{\alpha-1})}\\ &+(1+\|w+\psi\|_{Y^{\alpha-1}}+\|w+\psi\|^2_{Y^{\alpha-1}})\int_{t_0}^{t_1}\|\mathbf{v}-\mathbf{v}_N\|_{\mathcal{H}^\alpha}+\|\mathbf{v}-\mathbf{v}_N\|^2_{\mathcal{H}^\alpha}+\|\mathbf{v}-\mathbf{v}_N\|^3_{\mathcal{H}^\alpha}\,ds\\ &\lesssim_C \|(w+\psi+v)^3 - P_N(w+\psi+v)^3\|_{L^1([t_0,t_1].H^{\alpha-1})}\\ &+(1+\|w+\psi\|_{Y^{\alpha-1}})^2(1+\|\mathbf{v}\|_{L^\infty_t\mathcal{H}^\alpha_x}+\|\mathbf{v_N}\|_{L^\infty_t\mathcal{H}^\alpha_x})^3\int_{t_0}^{t_1}\|\mathbf{v}-\mathbf{v}_N\|_{\mathcal{H}^\alpha}\,ds.\end{aligned}$$

After applying Grönwall's inequality we have shown $(i)$. For $(ii)$ need further estimates. Rewriting (2.32), we have shown that there exists a summarizing constant $C_1 = C(T,\alpha)$ so that

$$f_N(t) \leq C_1\alpha_N + C_1C_2\int_0^t f_N(s)(1+f_N(s))^{p-1}\,ds, \qquad (2.33)$$

where

$$f_N(t) = \|\mathbf{v}(t) - \mathbf{v}_N(t)\|_{\mathcal{H}^\alpha} \tag{2.34}$$
$$C_2 = (1 + \|w+\psi\|_{Y^{\alpha-1}} + \|w+\psi\|^2_{Y^{\alpha-1}}) \tag{2.35}$$
$$p = 6. \tag{2.36}$$

We fix a large $N \in \mathbb{N}$. Let $g$ be a solution to the ODE

$$g'(s) = 2C_1C_2 g(s)(1+g(s))^{p-1}$$
$$g(0) = C_1\alpha_N.$$

We know by the Picard-Lindelöf theorem that a local solution exists and that it can be extended either to infinity or to a finite time blow-up. By using Grönwall's inequality we now show that if $\alpha_N \leq C_1^{-1}e^{-2^pC_1C_2T}$, then a solution exists until time $T$ and $g(t) \leq 1$ for all $t \leq T$.

Since $\alpha_N > 0$ the solution will be positive on a small time interval and have a positive derivative there, meaning that $g$ is strictly increasing. Since $\alpha_N \longrightarrow 0$ we can assume that $C_1\alpha_N < 1$. If $g(t) \leq 1$ for all $t \leq T$ then we are done, so we take $t < T$ to be first point where $g(t) = 1$. Then for all $s \leq t$ we have

$$g'(s) \leq 2^pC_1C_2g(s),$$

so Grönwall's inequality implies

$$1 = g(t) \leq C_1\alpha_N e^{2^pC_1C_2t} \Longrightarrow C_1e^{2^pC_1C_2t} \geq \alpha_N^{-1} \geq C_1e^{2^pC_1C_2T} \Longrightarrow t \geq T.$$

We now show that $f_N(t) \leq g(t) \leq 1$ for all $t \leq T$.

We define

$$t = \inf\{0 \leq s \leq T : f_N(s) \geq g(s)\}.$$

If $t = \infty$, i.e. the infimum does not exist, then we are done. We therefore assume that $t \leq T$. Applying the definition of $t$ to (2.33) yields

$$g(t) - g(0) = f_N(t) - C_1\alpha_N \leq C_1C_2\int_0^t f_N(s)(1+f_N(s))^{p-1}\,ds \leq C_1C_2\int_0^t g(s)(1+g(s))^{p-1}\,ds.$$

But since

$$g(t) - g(0) = \int_0^t g'(s)\,ds = 2C_1C_2\int_0^t g(s)(1+g(s))^{p-1}\,ds,$$

we have a contradiction. Therefore $t = \infty$ and we have proven that $f_N \leq g \leq 1$.

Statement $(iii)$ is now just an application of $(i)$ with the uniform estimate from $(ii)$. □

Using this we can show that mild solutions in $\mathcal{H}^1_{\mathrm{r}}$ automatically gain $\mathcal{H}^\alpha_{\mathrm{r}}$ regularity for $1 < \alpha < \frac{4}{3}$:

**Lemma 2.25** (Preservation of Regularity)**.** *Let $\mathbf{v} \in L^\infty([0,T], \mathcal{H}^1_{\mathrm{r}})$ be a mild solution to (2.19) for $\alpha = 1$. Let $\alpha' \in \left[1, \frac{4}{3}\right)$. If $w + \psi \in Y^{\alpha'-1}_{[0,T]}$, then $\mathbf{v}, \mathbf{v}_N \in L^\infty([0,T], \mathcal{H}^{\alpha'}_{\mathrm{r}})$ for all $n \in \mathbb{N}$.*

*Proof.* In order to be rigorous, we first derive an estimate for $\mathbf{v}_N$. By the definition of a mild solution yielding the analogous of (2.3) for $\mathbf{v}_N$, the fractional Leibnitz inequality (2.24) and the embeddings $Y^1, H^1 \subset L^6$, we have for $t \in [0,T]$ that

$$\begin{aligned}\|\mathbf{v}_N(t)\|_{\mathcal{H}^{\alpha'}} &\lesssim \int_0^t \|P_N((w+\psi+v_N)^3)\|_{H^{\alpha'-1}}\,dt\\ &\lesssim \int_0^t \|w+\psi+v_N\|_{L^6}^2 \|w+\psi+v_N\|_{W^{\alpha'-1,6}}\,dt\\ &\lesssim \int_0^t (\|w+\psi\|_{W^{\alpha'-1,6}}^2 + \|v_N\|_{H^1}^2)(\|w+\psi\|_{W^{\alpha'-1,6}} + \|\mathbf{v}_N\|_{\mathcal{H}^{\alpha'}})\,dt\\ &\lesssim C + C\int_0^t \|\mathbf{v}_N\|_{\mathcal{H}^{\alpha'}}\,dt,\end{aligned}$$

where $C = C(T, \|w+\psi\|_{Y^{\alpha-1}}, \|v_N\|_{L_t^\infty H_x^1})$ is some large constant. THen By Grönwall's inequality implies

$$\|\mathbf{v}_N(t)\|_{\mathcal{H}^{\alpha'}} \leq Ce^{Ct}.$$

This constant $C$ is uniformly bounded in $N$ since $\|v_N\|_{L_t^\infty H_x^1}$ is uniformly bounded in $N$ (lemma 2.24), and hence $\mathbf{v}_N \in L^\infty([0,T], \mathcal{H}_{\mathrm{r}}^{\alpha'})$ also with a uniform bound. By the Banach-Alaoglu theorem there must exist $\tilde{\mathbf{v}} \in L^\infty([0,T], \mathcal{H}_{\mathrm{r}}^{\alpha'})$ which is the weak limit of a subsequence $\mathbf{v}_{N_k}$. Since lemma 2.24 also implies that $\|\mathbf{v} - \mathbf{v}_N\|_{L^\infty([0,T],\mathcal{H}^1)} \longrightarrow 0$, it must be the case that $\mathbf{v} = \tilde{\mathbf{v}} \in L^\infty([0,T], \mathcal{H}_{\mathrm{r}}^{\alpha'})$. □

**Corollary 2.26** (Stochastic Local Well-posedness)**.** *Let $\alpha \in \left[1, \frac{4}{3}\right)$, $T' > 0$ and $\Psi$ be the stochastic convolution. Let $\mathbf{w}_0 \in L^2(\Omega, L^2(B))$ so that almost surely $\|w\|_{Y^{\alpha-1}_{[0,T']}} < \infty$, where $\mathbf{w} = \pi_1 S(t)\mathbf{w}_0$.*

*Then there exists a random variable $T > 0$ and random variables $\mathbf{v}, \mathbf{v}_N \in L^\infty([0,\infty), \mathcal{H}_{\mathrm{r}}^\alpha)$ which are almost surely mild solutions to (2.19) and (2.21) on $[0,T]$ respectively. Furthemore*

$$\mathbf{u} \coloneqq \mathbf{v} + \mathbf{w} + \Psi \quad \text{and} \quad \mathbf{u}_N \coloneqq \mathbf{v}_N + \mathbf{w} + \Psi$$

*solve (1.5), i.e.*

$$\begin{aligned}\partial_t^2 u + \partial_t u + (1-\Delta)u + u^3 &= \sqrt{2}\xi\\ u(0) &= w_0,\end{aligned}$$

*and*

$$\begin{aligned}\partial_t^2 u_N + \partial_t u_N + (1-\Delta)u_N + u_N^3 &= \sqrt{2}\xi\\ u_N(0) &= w_0\end{aligned}$$

*in the sense of distributions up to time $T$.*

*Proof.* We require that a.s. $w \in Y^{\alpha-1}_{[0,T']}$ and we have shown that a.s. $\psi \in Y^{\alpha-1}_{[0,T']}$ in lemma 2.15. (Note that this uses $\alpha - 1 < \frac{1}{3} < \frac{1}{2}$). Therefore we can just apply Theorem 2.22 pathwise to find the time of existence $T$ as well as $\mathbf{v}$ and $\mathbf{v}_N$ on the interval $[0,T']$. We extend $\mathbf{v}$ and $\mathbf{v}_N$ by zero onto $(T,\infty)$.

Using Theorem 2.6, lemma 2.9 and lemma 2.23, we then have that almost surely in the sense of distributions on the time interval $[0, T]$,

$$\partial_t(\mathbf{v} + \mathbf{w} + \Psi) = L(\mathbf{v} + \mathbf{w} + \Psi) + \begin{pmatrix} 0 \\ -(w + \psi + v)^3 + \sqrt{2}\xi \end{pmatrix}.$$

The proof of the truncated case is again a direct transfer of the one above. □

# 3 Global Well-posedness

## 3.1 Controlling the Growth of $\|\psi\|_{L^p}$

The energy estimates we will derive to get global well-posedness involve $L^p$-norms of $\psi$ for large $p$, so we have to control its growth. What we know so far from lemmas 2.12, 2.15 and 2.14 is that

$$\mathbb{E}\left[\int_B |\psi(t,x)|^p \, dx\right] \leq p^{\frac{p}{2}} \int_B \mathbb{E}\left[|\psi(t,x)|^2\right]^{\frac{p}{2}} \, dx \leq C^p p^{\frac{p}{2}} \int_B (1+|\ln(|x|)|)^{\frac{p}{2}} \, dx \leq C^p p^{\frac{p}{2}}.$$

Using this estimate, we have

$$\mathbb{E}\left[\int_B |\psi(t,x)|^p \, dx\right] \leq C_1^p p^p,$$

and with Jensen's inequality we can find $C_2 > 0$ so that

$$\mathbb{E}\left[\left(\int_B |\psi(t,x)|^p \, dx\right)^2\right] \leq C_2^p p^p p^p$$

and

$$\operatorname{Var}\left(\int_B |\psi(t,x)|^p \, dx\right) \leq C_1^{2p}(2p)^{2p} = C_2^p p^p p^p.$$

What we will now do is prove the existence of such a kind of growth estimate **almost surely** in several different ways.

**Lemma 3.1.** *There exists a $K > 0$ so that for all $t \geq s \geq 0$ the following holds almost surely:*

*(i)* $\exists p_0 > 0 : \forall p \geq p_0, \|\psi(t)\|_{L^p} \leq Kp.$

*(ii)* $\exists p_0 > 0 : \forall p \geq p_0, \|\psi\|_{L^p([s,t]\times B)} \leq (t-s)^{\frac{1}{p}} Kp.$

*(iii)* $\forall p \in \mathbb{N}, \|\psi(t)\|_{L^p} \leq Q(t)^{\frac{1}{p}} Kp$ *where* $Q(t) = \sum_{p=1}^{\infty} \frac{\|\psi(t)\|_{L^p}^p}{K^p p^p} < \infty$ *a.s.*

*(iv)* $\forall p \in \mathbb{N}, \|\psi\|_{L^p([s,t]\times B)} \leq (t-s)^{\frac{1}{p}} Q(s,t)^{\frac{1}{p}} Kp$ *where* $Q(s,t) = \sum_{p=1}^{\infty} \frac{\|\psi\|_{L^p([s,t]\times B)}^p}{K^p p^p} < \infty$ *a.s.*

*Proof.* We write down the proof only for $(i)$ and $(iii)$ as including the time integral for $(ii)$ and $(iv)$ causes no complications. It suffices to show this for $p \in \mathbb{N}$.

We prove $(i)$ via an argument with the Borel-Cantelli lemma and the Chebyshev inequality. Observe that for a large $K > 0$

$$\begin{aligned}
\sum_{p=1}^{\infty} \mathbb{P}\left(\int_B |\psi(t,x)|^p \, dx > 2K^p p^p\right) &\leq \sum_{p=1}^{\infty} \mathbb{P}\left(\left|\int_B |\psi(t,x)|^p - \mathbb{E}\left[\int_B |\psi(t,x)|^p \, dx\right]\right| > K^p p^p\right) \\
&\quad + \sum_{p=1}^{\infty} \mathbb{P}\left(\mathbb{E}\left[\int_B |\psi(t,x)|^p \, dx\right] > K^p p^p\right) \\
&= (I) + (II).
\end{aligned}$$

The sum $(II)$ is zero if $K > C_1$. For $(I)$ we use Chebyshev's inequality:

$$(I) \leq \sum_{p=1}^{\infty} \frac{\mathrm{Var}\left(\int_B |\psi(t,x)|^p \, dx\right)}{K^{2p} p^{2p}} \leq \sum_{p=1}^{\infty} \left(\frac{C_2}{K^2}\right)^p \frac{p^p p^p}{p^p p^p} < \infty,$$

where the result at the end is certainly summable given $K^2 > C_2$. Then the Borel-Cantelli lemma now implies

$$\mathbb{P}\left(\int_B |\psi(t,x)|^p \, dx > 2K^p p^p \text{ for infinitely many } p \in \mathbb{N}\right) = 0.$$

This finishes the proof of $(i)$. While this is a good result, we would like to have the estimate **for all** $p$ as opposed to just large $p$. In return we give up that the constant in the estimate is deterministic. This leads us to statement $(iii)$. Quantitatively, what we have really shown is that

$$\mathrm{Var}(Q) < \infty \text{ where } Q = \sum_{p=1}^{\infty} \frac{\|\psi\|_{L^p}^p}{K^p p^p}.$$

Therefore $Q < \infty$ a.s. and for all $p \in \mathbb{N}$

$$\|\psi\|_{L^p}^p \leq Q K^p p^p.$$

□

## 3.2 The Space of Initial Data

For reasons that will later become apparent in the proof of the global well-posedness, we want an initial data $\mathbf{w}_0$ so that for a given $\alpha \in \left[1, \frac{4}{3}\right)$ and for all $T > 0$ the following two conditions hold:

$$\|\pi_1 S(t)\mathbf{w}_0\|_{L^6([0,T],W^{\alpha-1,6})} < \infty \text{ and } \exists\, K > 0 : \sum_{p=1}^{\infty} \frac{\|\pi_1 S(t)\mathbf{w}_0\|_{L^p([0,T],L^p)}^p}{K^p p^p} < \infty.$$

We have in fact already shown that both of these hold for the stochastic convolution $\psi$ (lemmas 2.15 and 3.1). We have defined a shorthand notation for the first quantity in (2.22) that we adapt:

$$Y^{\alpha-1} := L^6([0,\infty), W_{\mathrm{r}}^{\alpha-1,6}).$$

We would like the second quantity to correspond to some function space. Because of separability issues we have to require something stronger here: Instead of working with a $p$ growth bound we use a slightly weaker $p \ln\ln p$ growth bound. Consider the set

$$B_1^{\mathcal{Z}} := \left\{ f \in L_{\mathrm{r}}^1([0,\infty) \times B) : \sum_{p=p_0}^{\infty} \frac{\|f\|_{L_{t,x}^p}^p}{(p \ln\ln p)^p} \leq 1 \right\},$$

and for all $f \in L^1([0,\infty) \times B)$ define the quantity

$$\|f\|_{\mathcal{Z}} := \left(\sup\{\lambda \geq 0 : \lambda f \in B_1^{\mathcal{Z}}\}\right)^{-1}.$$

The choice of the starting value $p_0$ is rather arbitrary as in any case the norms are equivalent. We choose $p_0 = 7$ as a large $p_0$ will help us in some estimates. To emphasize that this choice is arbitrary we mostly write $p_0$ instead of 7. We define the space $\mathcal{Z}$ to be the sub-Banach space of $L^1([0,\infty) \times B)$ induced by the norm $\|\cdot\|_{\mathcal{Z}}$.

**Lemma 3.2.** *Let*

$$\mathcal{Z} := \{f \in L^1_{\mathrm{r}}([0,\infty) \times B) : \|f\|_{\mathcal{Z}} < \infty\}.$$

*The following hold:*

*(i) For all* $R \geq 0$,

$$\|f\|_{\mathcal{Z}} \leq R \iff \sum_{p=p_0}^{\infty} \frac{\|f\|^p_{L^p_{t,x}}}{R^p (p \ln \ln p)^p} \leq 1.$$

*(ii) There is an equivalent norm:*

$$\frac{1}{2}\|f\|_{\mathcal{Z}} \leq \sup_{p \geq p_0} \frac{\|f\|_{L^p_{t,x}}}{p \ln \ln p} \leq \|f\|_{\mathcal{Z}}.$$

*(iii)* $\mathcal{Z}$ *is well-defined as a Banach space.*

*(iv) We have the embeddings*

$$L^{\exp}([0,\infty), H^1_{\mathrm{r}}) \hookrightarrow L^{\exp}([0,\infty), BMO_Q\big|_{B,\mathrm{r}} \cap L^1_{\mathrm{r}}) \hookrightarrow \mathcal{Z}.$$

*In fact a slightly stronger result holds:*

$$\sup_{p \geq p_0} \frac{\|f\|_{L^p_{t,x}}}{p} \lesssim \|f\|_{L^{\exp}([0,\infty),H^1)}.$$

*Here* $BMO_Q\big|_{B,\mathrm{r}}$ *refers to the subspace of* $BMO_Q$ *for the unit cube* $Q = [0,1]^2$ *of those functions which are radially symmetric and vanish outside of the unit ball* $B$.

*Proof.* (i) If $\|f\|_{\mathcal{Z}} \leq R$, then for any $0 < r^{-1} < R^{-1}$ there exists $\lambda > r^{-1}$ with $\lambda f \in \mathcal{Z}$. Therefore

$$\sum_{p=p_0}^{\infty} \frac{\|f\|^p_{L^p_{t,x}}}{r^p (p \ln \ln p)^p} \leq 1.$$

By letting $r \searrow R$ we get the statement for $R$.

Now suppose that

$$\sum_{p=p_0}^{\infty} \frac{\|f\|^p_{L^p_{t,x}}}{R^p (p \ln \ln p)^p} \leq 1.$$

Then $R^{-1} f \in \mathcal{Z}$, and so $\|f\|_{\mathcal{Z}} \leq R$.

(ii) If $\|f\|_{\mathcal{Z}} \leq R$, then for any $q \in \mathbb{N}$ with $q \geq p_0$,

$$\frac{\|f\|^q_{L^q_{t,x}}}{R^q(q\ln\ln q)^q} \leq \sum_{p=p_0}^{\infty} \frac{\|f\|^p_{L^p_{t,x}}}{R^p(p\ln\ln p)^p} \leq 1,$$

which implies $\sup_{p\geq p_0} \|f\|_{L^p_{t,x}}(p\ln\ln p)^{-1} \leq R$.

Now suppose that $\sup_{p\geq p_0} \|f\|_{L^p_{t,x}}(p\ln\ln p)^{-1} \leq R$. Then for any $r > R$ we have

$$\sum_{p=p_0}^{\infty} \frac{\|f\|^p_{L^p_{t,x}}}{r^p(p\ln\ln p)^p} \leq \sum_{p=p_0}^{\infty} \frac{R^p}{r^p} \leq \frac{1}{1-\frac{R}{r}} - 1.$$

We choose $r = 2R$ so that the right hand side is 1. Then $(i)$ yields $\|f\|_{\mathcal{Z}} \leq 2R$.

(iii) We have to check that $\|f\|_{\mathcal{Z}}$ is a norm. The completeness will then follow trivially from the completeness of the equivalent norm in $(ii)$. Only the triangle inequality is non-trivial. Suppose that $\|f\|_{\mathcal{Z}} + \|g\|_{\mathcal{Z}} \leq R$. Then there exists some $\lambda \in (0,1)$ so that $\|f\|_{\mathcal{Z}} \leq \lambda R$ and $\|g\|_{\mathcal{Z}} \leq (1-\lambda)R$. By convexity of $\|\cdot\|_{L^p}$,

$$\sum_{p=p_0}^{\infty} \frac{\|f+g\|^p_{L^p_{t,x}}}{R^p(p\ln\ln p)^p} \leq \lambda \sum_{p=p_0}^{\infty} \frac{\|f\|^p_{L^p_{t,x}}}{\lambda^p R^p(p\ln\ln p)^p} + (1-\lambda) \sum_{p=p_0}^{\infty} \frac{\|g\|^p_{L^p_{t,x}}}{(1-\lambda)^p R^p(p\ln\ln p)^p}.$$

But $(i)$ implies that this is less or equal to $\lambda 1 + (1-\lambda)1 = 1$, and then again $(i)$ implies that $\|f+g\|_{\mathcal{Z}} \leq R$.

(iv) Let $f \in L^{\exp}([0,\infty), H^1)$. In 2 dimensions it is known that $H^1 \hookrightarrow BMO$ by an application of the Poincaré inequality. At the same time $H^1 \hookrightarrow L^1$ is trivial. Therefore the first embedding holds.

The second embedding is an application of the John-Nirenberg inequality ([13]). It implies that if $f : Q \longrightarrow \mathbb{R}$ is a measurable function in $BMO_Q$, then for all $\lambda > 0$ we have

$$\mu(\{x : |f(x) - \bar{f}| > \lambda\}) \leq Ce^{-c\frac{\lambda}{\|f\|_{BMO}}},$$

where $\mu$ is the Lebesgue measure and $\bar{f}$ the mean of $f$. Let $p \geq p_0$ and $t \geq 0$. Then

$$\int_Q |f(t) - \bar{f}(t)|^p\,dx = \int_0^\infty \mu(\{x \in Q : |f(t,x) - \bar{f}(t)| > \lambda^{\frac{1}{p}}\})\,d\lambda$$

$$= \int_0^\infty p\lambda^{p-1}\mu(\{x : |f(t,x) - \bar{f}(t)| > \lambda\})\,d\lambda \leq C\int_0^\infty p\lambda^{p-1}e^{-c\frac{\lambda}{\|f(t)\|_{BMO}}}\,d\lambda$$

$$= \frac{C}{c^p}\,p\,\Gamma(p)\,\|f(t)\|^p_{BMO} \leq \frac{C}{c^p}\,p^p\,\|f(t)\|^p_{BMO}.$$

Rearranging and integrating this in time, we get

$$\frac{\|f\|_{L^p_{t,x}}}{p} \leq \underbrace{\|\bar{f}\|_{L^p_{t,x}}}_{=\|f\|_{L^p_t L^1_x}} + \frac{C^{\frac{1}{p}}}{c}\|f\|_{L^p([0,\infty),BMO)}$$

$$\overset{(1.8)}{\lesssim} \|f\|_{L^{\exp}([0,\infty),L^1)} + \|f\|_{L^{\exp}([0,\infty),BMO)} \lesssim \|f\|_{L^{\exp}([0,\infty),L^1\cap BMO)}.$$

□

There is a problem with this space: it is not separable. The space $\mathcal{Z}$ that we have defined here is also knwon as the Orlicz-space generated by the $N$-function $M(x) = \sum_{p=2}^{\infty} \frac{|x|^p}{p \ln \ln p}$, and this space is only separable if $M$ fulfills the doubling condition $M(2x) \leq CM(x)$. This is not the case for our $M$. The definition of Orlicz spaces and this result can be found in [15, p.108].

We can get around this by working with the closure of $L^\infty$ in the Orlicz-space, as this subspace is always separable. We define

$$Z = \overline{L^\infty_{\mathrm{r}}([0,T] \times B)}^{\mathcal{Z}}.$$

The separability of this subspace is part of the theory of Orlicz-spaces but we will prove it manually in our case.

Given a measurable set $I \subseteq [0, \infty)$, we also define the spaces $\mathcal{Z}_I$ and $Z_I$ in the same fashion but replacing the domain $[0, \infty)$ in every time integral by $I$. The results of Lemma 3.2 hold analogously. We also define $Y_I^{\alpha-1}$ in the same fashion. It is important that for all $T > 0$ the stochastic convolution is almost surely an element of $Y_{[0,T]}^{\alpha-1}$ and $Z_{[0,T]}$. For the former space we already know this. For the latter space the reason is that

$$Z_{[0,T]} = \left\{ f \in \mathcal{Z}_{[0,T]} : \limsup_{p \in \mathbb{N}} \frac{\|f\|_{L^p([0,T]\times B)}}{p \ln \ln p} = 0 \right\},$$

and we know from Lemma 3.1 that the stochastic convolution almost surely has a growth bound $\|\psi\|_{L^p_{[0,T]\times B}} \leq Kp$.

The following lemma estbalishes some properties of $Z$. We do not use the compactness results because they are not quite strong enoguh to be useful, but they are still worth noting.

**Lemma 3.3.** *The following hold:*

*(i) There exists a countable subset of* $\mathcal{D}_{\mathrm{r}}([0,\infty) \times B)$ *which is dense in* $Z$.

*(ii) We have*

$$Z = \left\{ f \in \mathcal{Z} : \limsup_{p \in \mathbb{N}} \frac{\|f\|_{L^p([0,T]\times B)}}{p \ln \ln p} = 0 \right\}.$$

*(iii) If a bounded set* $E \subseteq L^{\exp}([0,\infty), H^1_{\mathrm{r}})$ *is equicontinuous in* $C([0,\infty), L^p_{\mathrm{r}})$ *for every* $p \in \mathbb{N}$, *then it is relatively compact in* $Z$.

*(iv) Given* $f \in Z_{[0,T]}$ *and a measurable function* $p : [0,T] \longrightarrow [1,\infty)$, *we can estimate*

$$\int_0^T \frac{\|f\|^{p(t)}_{L^{p(t)}_x}}{p(t) \ln \ln p(t)} \, dt \leq 2\pi(T+1)\|f\|_{Z_{[0,T]}}$$

*Proof.* (i) It suffices to show that $L^\infty_{t,x}$ is separable with respect to the equivalent norm $\sup_{p \geq p_0} \frac{\|f\|_{L^p_{t,x}}}{p \ln \ln p}$. It is well known that for every $p \in \mathbb{N}$ there exists a countable subset of test functions $\tilde{E}_p \subset \mathcal{D}_{\mathrm{r}}([0,\infty) \times B)$ which is dense in $L^p_{t,x}$. By adding for every function $g \in \tilde{E}_p$ and $r, q \in \mathbb{Q}_+$ the functions $\rho_r * (f \mathbb{1}_{|f| \leq q})$, where $\rho_r$ is a standard mollifier, to $\tilde{E}_p$, we get another countable dense subset $E_p$ that has the property that any $f \in L^\infty$ can be approximated in $L^p$ by functions $g_n \in E_p$ with $\|g_n\|_{L^\infty} \leq \|f\|_{L^\infty}$. We set $E = \bigcup_{p \in \mathbb{N}} E_p$, which is still countable. We claim that $E$ is dense in $L^\infty_{t,x}$ with respect to the aforementioned norm of $\mathcal{Z}$.

Let $f \in L^\infty_{t,x}$ and $\epsilon > 0$. We choose $q_0 \geq \frac{2\pi \|f\|_{L^\infty_{t,x}}}{\epsilon}, p_0$ and some $g \in E$ so that $\|f - g\|_{L^{q_0}_{t,x}} < \frac{\epsilon}{2\pi}$ and $\|g\|_{L^\infty_{t,x}} \leq \|f\|_{L^\infty_{t,x}}$. Then for $p_0 \leq p \leq q_0$ we have

$$\frac{\|f - g\|_{L^p}}{p \ln \ln p} \leq \pi \|f - g\|_{L^{q_0}_{t,x}} < \frac{\epsilon}{2},$$

and for $p > q_0$ we have

$$\frac{\|f - g\|_{L^p}}{p \ln \ln p} \leq \frac{2\pi \|f\|_{L^\infty_{t,x}}}{q_0} < \frac{\epsilon}{2}.$$

Therefore $Z$ is separable.

(ii) We show $\subseteq$. Let $f \in Z$, $\epsilon > 0$ and choose some $g \in L^\infty$ so that $\|f - g\|_{\mathcal{Z}} < \epsilon$. Then

$$\limsup_{p \geq p_0} \frac{\|f\|_{L^p_{t,x}}}{p \ln \ln p} < \epsilon + \limsup_{p \geq p_0} \frac{\|g\|_{L^p_{t,x}}}{p \ln \ln p} \leq \epsilon + \pi \limsup_{p \geq p_0} \frac{\|g\|_{L^\infty_{t,x}}}{p \ln \ln p} = \epsilon.$$

Since $\epsilon > 0$ is arbitrary the left hand side is zero.

For the direction $\supseteq$, suppose that $f \in \mathcal{Z}$ and that $\limsup_{p \geq p_0} \frac{\|f\|_{L^p_{t,x}}}{p \ln \ln p} = 0$. Then there exists $C > 0$ so that $\|f\|_{L^p} \leq C$ for all $p \geq 1$, and so $f \in L^\infty_{t,x}$.

(iii) This is an addition to (iv) from Lemma 3.2. We claim that $E$ is compact in $L^p([0, \infty), L^p_{\mathrm{r}})$. This follows from the Arzela-Ascoli theorem for metric spaces. In our situation it states that the set $E$ is realtively compact in $C([0, \infty), L^p_x)$ if it is equicontinuous and pointwise relatively compact. We have assumed equicontinuity and get the pointwise relative compactness from the boundedness of $E$ in $L^{\exp}([0, T], H^1_{\mathrm{r}})$, together with the Rellich-Kandrachov theorem, which states that $H^1_{\mathrm{r}} \hookrightarrow L^p_{\mathrm{r}}$ is compact for all $p$. Since $C([0, T], L^p_{\mathrm{r}})$ embeds continuously into $L^p([0, T], L^p_{\mathrm{r}})$ we get the claim. Now we consider the space

$$\mathcal{E} \coloneqq \prod_{p=p_0}^{\infty} \overline{E}^{L^p_{t,x}},$$

which we equip with the product topology. Here $\overline{E}^{L^p_{t,x}}$ refers to the closure of $E$ in $L^p_{t,x}$, equipped with the topolgoy of $L^p_{t,x}$. By Tychonoff's theorem the space $\mathcal{E}$ is itself compact. Then the closed subset

$$\tilde{E} \coloneqq \{F \in \mathcal{E} : \pi_p F = \pi_1 F \ \forall p \geq p_0\}$$

is compact with respect to the product topology. Note that convergence of $F_n$ to $F$ in $\tilde{E}$ is equivalent to convergence of $\pi_1 F_n$ to $\pi_1 F$ in $L^p$ for all $p \geq p_0$. We claim that this topology on $\tilde{E}$ is in fact equivalent to the (seemingly stronger) one given by the metric

$\sup_{p\geq p_0} \frac{\|\pi_1 F-\pi_1 G\|_{L^p_{t,x}}}{p\ln\ln p}$ for $F, G \in \tilde{E}$. One implication is trivial and we now show the other direction:

Since $E$ is bounded in $L^{\exp}([0,\infty), H^1)$ we know by statement $(iv)$ from Lemma 3.2 that there exists a constant $C$ so that $\sup_{p\geq p_0} \frac{\|\pi_1 F\|_{L^p}}{p} \leq C$ for all $F \in \tilde{E}$. Now suppose that $F_n \longrightarrow F$ in $\tilde{E}$. Let $q_0 > p_0$ be large so that $\frac{C}{\ln\ln q_0} < \epsilon$. We have

$$\sup_{p\geq p_0} \frac{\|\pi_1 F_n - \pi_1 F\|_{L^p_{t,x}}}{p\ln\ln p} \leq \sup_{p_0\leq p\leq q_0} \frac{\|\pi_1 F_n - \pi_1 F\|_{L^p_{t,x}}}{p\ln\ln p} + \frac{C}{\ln\ln q_0}.$$

Here the first term converges to 0 since $\pi_1 F_n \longrightarrow \pi_1 F$ in $L^p$ for all $p \leq q_0$, and the second term is less than $\epsilon$, concluding our argument.

We now define

$$E' = \{f \in L^1_{t,x} : (f, f, ...) \in \tilde{E}\}.$$

What we have shown above implies that $E'$ is compact in $\mathcal{Z}$ and in fact is a subset of $Z$, so $E \subset E'$ is relatively compact in $Z$.

(iv) Recall that by Hölder's inequality for any $q \geq p$,

$$\|f\|_{L^p} \leq \|\mathbb{1}_B\|^{\frac{1}{p}}_{L^{\frac{q}{q-p}}} \|f\|_{L^q} \leq \pi\|f\|_{L^q}.$$

Using this, we estimate

$$\begin{aligned}
\int_0^T \frac{\|f(t)\|_{L^{p(t)}_x}}{p(t)\ln\ln p(t)}\, dt &\leq \pi \int_0^T \frac{\lceil p(t)\rceil \ln\ln\lceil p(t)\rceil}{p(t)\ln\ln p(t)} \frac{\|f(t)\|_{L^{\lceil p(t)\rceil}_x}}{\lceil p(t)\rceil \ln\ln\lceil p(t)\rceil}\, dt \\
&\leq 2\pi\|f(t)\|_{Z_{[0,T]}} \int_0^T 1 + \frac{\|f(t)\|^{\lceil p(t)\rceil}_{L^{\lceil p(t)\rceil}_x}}{\|f\|^{\lceil p(t)\rceil}_{Z_{[0,T]}} (\lceil p(t)\rceil \ln\ln\lceil p(t)\rceil)^{\lceil p(t)\rceil}}\, dt \\
&\leq 2\pi\|f\|_{Z_{[0,T]}} \left( T + \sum_{p\geq p_0} \frac{\|f(t)\|^p_{L^p_{t,x}}}{\|f\|^p_{Z_{[0,T]}} (p\ln\ln p)^p} \right) \\
&\overset{\text{lem. 3.2}}{\leq} 2\pi\|f\|_{Z_{[0,T]}}(T+1).
\end{aligned}$$

□

**Definition 3.4.** *For $\alpha \in \mathbb{R}$ we define the space*

$$X^{\alpha-1} := Y^{\alpha-1} \cap Z$$

*with norm*

$$\|f\|_{X^{\alpha-1}} = \|f\|_{Y^{\alpha-1}} + \|f\|_Z.$$

*We define the* ***space of initial data*** *as*

$$\mathcal{X}^{\alpha-1} := \{\mathbf{w}_0 \in \mathcal{H}^{-\infty}_{\mathrm{r}} : \pi_1 S(t)\mathbf{w}_0 \in X^{\alpha-1}\}$$

*with norm*

$$\|\mathbf{w}_0\|_{\mathcal{X}^{\alpha-1}} = \|\pi_1 S(t)\mathbf{w}_0\|_{X^{\alpha-1}}.$$

**Lemma 3.5.** *The following hold:*

*(i)* $X^{\alpha-1}$ *is a separable Banach space. It has a countable and dense subset of radially symmetric test functions.*

*(ii)* $\mathcal{X}^{\alpha-1}$ *is a separable Banach space. It has a countable and dense subset of radially symmetric test functions.*

*(iii) We have continuous inclusions* $\mathcal{H}^1_{\mathrm{r}} \hookrightarrow \mathcal{X}^{\alpha-1} \hookrightarrow \mathcal{H}^{\alpha-1}_{\mathrm{r}}$.

*(iv) All the relevant Borel* $\sigma$*-algebras are identical in the sense that for* $\beta < \beta'$ *the following inclusions are bimeasurable:*

$$\mathcal{H}^{\beta'}_{\mathrm{r}} \hookrightarrow \mathcal{H}^{\beta}_{\mathrm{r}}.$$

$$\mathcal{H}^1_{\mathrm{r}} \hookrightarrow \mathcal{X}^{\alpha-1} \hookrightarrow \mathcal{H}^{\alpha-1}_{\mathrm{r}}.$$

*Proof.* (i) $X^{\alpha-1}$ is the intersection of two Banach spaces in both of which we can find the same countable dense subset of radially symmetric test functions, so it is a separable Banach space.

(ii) Completeness: Let $\mathbf{w}_n$ be a Cauchy sequence in $\mathcal{X}^{\alpha-1}$. This implies that there exists some $F \in X^{\alpha-1}$ so that $\pi_1 S(t)\mathbf{w}_n \xrightarrow{X^{\alpha-1}} F$. Then for all $T>0$ we also have $\pi_1 S(t)\mathbf{w}_n \xrightarrow{L^2_{[0,T]\times B}} F$, and so we know by Lemma 2.7 that $\mathbf{w}_n$ converges to some $\mathbf{w}_0$ in $\mathcal{H}^0_{\mathrm{r}}$. A direct consequence is that $S(t)\mathbf{w}_n \longrightarrow S(t)\mathbf{w}_0 \in C([0,T],\mathcal{H}^0_{\mathrm{r}})$. But then both $F(t,x)$ and $\pi_1 S(t)\mathbf{w}_0(x)$ are the limit of $S(t)\mathbf{w}(x)$ for almost all $(t,x) \in [0,T]\times B$, hence $F = \pi_1 S(t)\mathbf{w}_0$ up to time $T$, which is arbitratily large. Therefore $\mathbf{w}_n \longrightarrow \mathbf{w}_0$ in $\mathcal{X}^{\alpha-1}_T$.

Separability: Define

$$A = \{F \in X^{\alpha-1} : F = \pi_1 S(t)\mathbf{f} \text{ for some } \mathbf{f} \in \mathcal{X}^{\alpha-1}\} \subset X^{\alpha-1}.$$

This is a subspace of a separable metric space (lem. 3.3) and hence separable. Let $F_j$ be a countable dense subset. Since $\mathbf{f} \mapsto \pi_1 S(t)\mathbf{f}$ is surjective from $\mathcal{X}^{\alpha-1}$ to $A$, we can choose $\mathbf{f}_j$ so that $F_j = \pi_1 S(t)\mathbf{f}_j$. Then for any $\mathbf{w}_0 \in \mathcal{X}^{\alpha-1}$ and $\epsilon > 0$ we know that there exists a $j \in \mathbb{N}$ so that $\|\mathbf{w}_0 - \mathbf{f}_j\|_{\mathcal{X}^{\alpha-1}_T} = \|F_j - \pi_1 S(t)\mathbf{w}_0\|_{X^{\alpha-1}_{[0,T]}} < \epsilon$.

(iii) If $\mathbf{f}_n \longrightarrow \mathbf{f} \in \mathcal{H}^1_{\mathrm{r}}$, then $\pi_1 S(t)\mathbf{f}_n \longrightarrow \pi_1 S(t)\mathbf{f}$ in $L^{\exp}([0,T],\mathcal{H}^1)$ by Lemma 2.2 and we can use Lemma 3.2.

If $\mathbf{f}_n \longrightarrow \mathbf{f} \in \mathcal{X}^{\alpha-1}$, then because of the $Y^{\alpha-1}$ norm part we get $\pi_1 S(t)\mathbf{f}_n \longrightarrow \pi_1 S(t)\mathbf{f}$ in $L^2([0,T],\mathcal{H}^{\alpha-1}_{\mathrm{r}})$ for all $T>0$. Now Lemma 2.7 implies that $\mathbf{f}_n \longrightarrow \mathbf{f}$ in $\mathcal{H}^{\alpha-1}_{\mathrm{r}}$.

(iv) By the continuity of these maps it suffices to show that a measurable set in the smaller space is measurable in the larger space. Let $A \in \mathcal{B}(\mathcal{H}^{\beta'}_{\mathrm{r}})$. It is a well-known result that on separable Hilbert spaces the Borel $\sigma$-algebras induced by the weak and strong topology agree, and since the weak topology in $\mathcal{H}^{\beta'}_{\mathrm{r}}$ is weaker than the weak topology in the larger space $\mathcal{H}^{\beta}_{\mathrm{r}}$, we get

$$A \in \mathcal{B}(\mathcal{H}^{\beta'}_{\mathrm{r,weak}}) \subseteq \mathcal{B}(\mathcal{H}^{\beta}_{\mathrm{r,weak}}) = \mathcal{B}(\mathcal{H}^{\beta}_{\mathrm{r}}).$$

Similarly if $A \in \mathcal{B}(\mathcal{H}^1_{\mathrm{r}})$, then

$$A \in \mathcal{B}(\mathcal{H}^{\beta'}_{\mathrm{r,weak}}) \subseteq \mathcal{B}(\mathcal{H}^{0}_{\mathrm{r,weak}}) = \mathcal{B}(\mathcal{H}^{0}_{\mathrm{r}}).$$

But since $A \subseteq \mathcal{X}^{\alpha-1}$ and $\mathcal{X}^{\alpha-1}$ has a stronger topology than $\mathcal{H}^0_{\mathrm{r}}$ we have in particular $A \in \mathcal{B}(\mathcal{X}^{\alpha-1})$.

□

The following lemma allows us to find large compact sets in $\mathcal{X}^{\alpha-1}$. This is useful because it can help prove tightness of sets of measures on $\mathcal{X}^{\alpha-1}$, which can be used to show weak convergence results. We will not use this result later though as we can show the desired weak convergences in a more direct fashion.

**Lemma 3.6.** *We have some compactness results:*

*(i) If a bounded set $E \subseteq L^{\exp}([0,\infty), H^1_{\mathrm{r}})$ is equicontinuous in $C([0,\infty), W^{\alpha-1,6}_{\mathrm{r}})$ and $C([0,\infty), L^p_{\mathrm{r}})$ for every $p \in \mathbb{N}$, then it is relatively compact in $X^{\alpha-1}$.*

*(ii) Let $N \in \mathbb{N}$ and $\beta > 1$. The embeddings*

$$\mathcal{H}^\beta_{\mathrm{r}} \xhookrightarrow{cpt.} \mathcal{X}^{\alpha-1}$$

*and*

$$P_N\mathcal{H}^{-\infty}_{\mathrm{r}} \xhookrightarrow{cpt.} \mathcal{X}^{\alpha-1}$$

*are compact. Here $P_N\mathcal{H}^{-\infty}_{\mathrm{r}}$ is equipped with the norm of $\mathbb{R}^N$.*

*Proof.* The proof of $(i)$ just uses the same argument as that of (iii) in Lemma 3.3. We prove (ii).

We have an embedding first of all since for all finite $n, k \leq N$ we have $(e_n, e_k) \in \mathcal{H}^1 \hookrightarrow \mathcal{X}^{\alpha-1}$.

Let $E \subset \mathcal{H}^\beta_{\mathrm{r}}$ or $E \subset P_N\mathcal{H}^{-\infty}_{\mathrm{r}}$ be bounded. We will differentiate these cases later on.

We want to show that $E$ is relatively compact in $\mathcal{X}^{\alpha-1}$. Note that the topology on $\mathcal{X}^{\alpha-1}$ is induced by the map

$$\mathcal{X}^{\alpha-1} \ni \mathbf{w}_0 \longmapsto \pi_1 S(t)\mathbf{w}_0 \in X^{\alpha-1}.$$

It therefore suffices to show that

$$A := \{\pi_1 S(t)\mathbf{w}_0 : \mathbf{w}_0 \in E\}$$

is relatively compact in $X^{\alpha-1}$. In the case of $E \subseteq P_N\mathcal{H}^{-\infty}_{\mathrm{r}}$ being bounded in the norm of $\mathbb{R}^N$, we use that $A = P_N\pi_1 S(t)E$ is the image of a finite rank and hence compact linear operator applied to a bounded set, and hence relatively compact. Alternatively we could argue that there exists constants $C, \tilde{C} > 0$ so that for all $\beta > 0$ and $\mathbf{w} \in E$,

$$\|\mathbf{w}\|^2_{\mathcal{H}^{\beta+1}} = \sum_{n=1}^{\infty} \langle\lambda_n\rangle^{2\beta}\langle w, e_n\rangle^2 + \langle\lambda_n\rangle^{2\beta-2}\langle w_t, e_n\rangle^2 \leq \tilde{C}\sum_{n=1}^{N}\langle\lambda_n\rangle^{2\beta} + \langle\lambda_n\rangle^{2\beta-2} \leq C.$$

Then by applying Lemma 2.3 we find that

$$\|\pi_1 S(t)\mathbf{w}\|_{L^{\exp}([0,\infty),H^\beta)} \leq C,$$

so $E$ is bounded in $C([0,T],H^1)$. The lemma also implies that

$$\|\pi_1 S(t)\mathbf{w}\|_{C^1([0,\infty),H^\beta)} \leq C\|\mathbf{w}\|_{\mathcal{H}^{\beta+1}},$$

so $E$ is equicontinuous in $C([0,T],H^1)$. In the case of $E \subseteq \mathcal{H}^\beta_{\mathrm{r}}$ being bounded we choose $0 < \gamma < \beta - 1$. Then similarly with Lemma 2.3 we get the boundedness and

$$\|\pi_1 S(t)\mathbf{w}\|_{C^{0,\gamma}([0,\infty),H^{\beta-\gamma})} \leq \sup_{\mathbf{f}\in E} \|f\|_{\mathcal{H}^\beta} \|S\|_{C^{0,\gamma}([0,\infty),L(\mathcal{H}^\beta,\mathcal{H}^{\beta-\gamma}))},$$

so we also have the equicontinuity. Therefore in either case we have that $E$ is equicontinuous and bounded in $C([0,T],H^1)$, and hence it fulfills the conditons of $(i)$. □

**Lemma 3.7.** *The map*

$$\begin{aligned} S : [0,\infty) \times \mathcal{X}^{\alpha-1} &\longrightarrow \mathcal{X}^{\alpha-1} \\ (t,\mathbf{w}_0) &\longmapsto S(t)\mathbf{w}_0 \end{aligned}$$

*is jointly continuous in* $(t,\mathbf{w}_0)$. *It is Lipschitz continuous with constant* $1$ *in* $\mathbf{w}_0$.

*Proof.* Let $t_0 \geq 0$ and $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{X}^{\alpha-1}$. We have

$$\begin{aligned} \|S(t_0)(\mathbf{w}_1 - \mathbf{w}_2)\|_{\mathcal{X}^{\alpha-1}} &= \|t \mapsto \pi_1 S(t_0+t)(\mathbf{w}_1 - \mathbf{w}_2)\|_{X^{\alpha-1}} \\ &\leq \|\pi_1 S(t)(\mathbf{w}_1 - \mathbf{w}_2)\|_{X^{\alpha-1}} = \|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathcal{X}^{\alpha-1}}. \end{aligned}$$

This proves the Lipschitz continuity. As a result $S(t)\mathbf{w}_0$ is continuous in $\mathbf{w}_0$, uniformly in $t$. It therefore suffices to show continuity in $t$ at every $\mathbf{w}_0$ in to get joint continuity.

Now let $\mathbf{w}_0 \in \mathcal{X}^{\alpha-1}$ and $t_n \longrightarrow t \geq 0$. We first assume that $\mathbf{w}_0 \in \mathcal{H}^1_{\mathrm{r}}$. Set $\epsilon_n = t_n - t$ and assume that $n$ is large enough so that $\epsilon_n \geq 0$. Then

$$\begin{aligned} \|(S(t) - S(t_n))\mathbf{w}_0\|_{\mathcal{X}^{\alpha-1}} &= \|\pi_1(\mathrm{Id} - S(\epsilon_n))S(t+s)\mathbf{w}_0\|_{X^{\alpha-1}} \\ &\leq \|\pi_1(\mathrm{Id} - S(\epsilon_n))S(s)\mathbf{w}_0\|_{Y^{\alpha-1}} + \|\pi_1(\mathrm{Id} - S(\epsilon_n))S(s)\mathbf{w}_0\|_Z. \end{aligned}$$

We know that $L^{\exp}([0,\infty),H^\alpha_{\mathrm{r}}) \hookrightarrow Y^{\alpha-1}$ (equation 2.25) and $L^{\exp}([0,\infty),H^1_{\mathrm{r}}) \hookrightarrow Z$ (Lemma 3.2, $(iv)$). Therefore we can estimate the above by

$$\lesssim \sup_{s\in[0,\infty)} \underbrace{e^{\frac{s}{2}}\|S(s)\|_{L(\mathcal{H}^\alpha,\mathcal{H}^\alpha)}}_{\text{bounded in } s \text{ by lem. 2.2}} \|(\mathrm{Id} - S(\epsilon_n))\mathbf{w}_0\|_{\mathcal{H}^\alpha} \xrightarrow{n\to\infty} 0.$$

Now suppose that $\mathbf{w}_0 \in \mathcal{X}^{\alpha-1}$. Let $\epsilon > 0$ and, using the density of radially symmetric test functions, choose some $\mathbf{w}_0' \in \mathcal{H}^1_{\mathrm{r}}$ so that $\|\mathbf{w}_0 - \mathbf{w}_0'\|_{\mathcal{X}^{\alpha-1}} < \frac{\epsilon}{2}$. Then

$$\|(S(t) - S(t_n))\mathbf{w}_0\|_{\mathcal{X}^{\alpha-1}} \leq \|(S(t) - S(t_n))(\mathbf{w}_0 - \mathbf{w}_0')\|_{\mathcal{X}^{\alpha-1}} + \|(S(t) - S(t_n))\mathbf{w}_0'\|_{\mathcal{X}^{\alpha-1}}.$$

We already know that the second term vanishes as $n \nearrow \infty$. For the first term we estimate

$$\|(S(t) - S(t_n))(\mathbf{w}_0 - \mathbf{w}_0')\|_{\mathcal{X}^{\alpha-1}} \leq 2\|\pi_1 S(s)(\mathbf{w}_0 - \mathbf{w}_0')\|_{X^{\alpha-1}} < \epsilon.$$

Therefore $S(t_n)\mathbf{w}_0 \longrightarrow S(t)\mathbf{w}_0$. □

**Theorem 3.8.** *Let $\alpha \in \left[1, \frac{3}{4}\right)$ and note that we have chosen $p_0 = 7$ in the definition of $\mathcal{Z}$. There exists a constant $C > 0$ so that the following hold:*

*(i) If $\mathbf{w}$ is an $L^2_{\mathrm{r}}([0,\infty) \times B)$-valued random variable with estimates*

$$\mathbb{E}\left[\|\mathbf{w}\|^p_{L^p}\right] \le \eta A^p p^p \tag{3.1}$$

*and*

$$\mathbb{E}\left[\|\mathbf{w}\|^6_{Y^{\alpha-1}}\right] \le \eta A^6 \tag{3.2}$$

*for all $p \ge p_0$, some $A > 0$ and some $\eta \in [0,1]$, then $\mathbf{w} \in X^{\alpha-1}$ almost surely and*

$$\mathbb{E}\left[\|\mathbf{w}\|^6_{X^{\alpha-1}}\right] \le C\eta^{\frac{6}{7}} A^6.$$

*(ii) Let $\{X_n(t)\}_{n\in\mathbb{N}} \cup \{X_{t,n}(t)\}_{n\in\mathbb{N}}$ be a family of independent and centered normally distributed random variables. Let $\sigma_n^2, \sigma_{t,n}^2$ be their variances and assume that there exists a constant $\beta > 0$ so*

$$\sigma_n^2 \le \frac{\beta^2}{\langle\lambda_n\rangle^2} \qquad \text{and} \qquad \sigma_{t,n}^2 \le \beta^2$$

*for all $n \in \mathbb{N}$. Now for $N < M \in \mathbb{N}\cup\{\infty\}$ define*

$$\mathbf{w}_N^M := \sum_{n=N}^{M} \left(X_n e_n, X_{t,n} e_n\right)$$

*Then for all $N < M \in \mathbb{N}\cup\{\infty\}$ we have $\mathbf{w}_N^M \in \mathcal{X}^{\alpha-1}$ almost surely and*

$$\mathbb{E}\left[\|\mathbf{w}_N^M\|^6_{\mathcal{X}^{\alpha-1}}\right] \le C^6\beta^6 \left(\frac{1+\ln(N)}{N^{8-6\alpha}}\right)^{\frac{6}{7}} \xrightarrow{N\to\infty} 0.$$

*Proof.* We start with $(i)$. The first given estimate implies that for any $R \ge 2A$

$$\mathbb{E}\left[\sum_{p=p_0}^{\infty} \frac{\|\mathbf{w}\|^p_{L^p_{([0,\infty)\times B)}}}{R^p(p\ln\ln p)^p}\right] \le \mathbb{E}\left[\sum_{p=p_0}^{\infty} \frac{\|\mathbf{w}\|^p_{L^p_{([0,\infty)\times B)}}}{R^p p^p}\right] \le \eta \sum_{p=p_0}^{\infty} \left(\frac{A}{R}\right)^p = \frac{A^{p_0}}{R^{p_0}} \frac{R}{R-A} \le 2\eta \frac{A^{p_0}}{R^{p_0}}. \tag{3.3}$$

Then, using Lemma 3.2 $(i)$, we have

$$\mathbb{P}\left(\|\mathbf{w}\|_Z > R\right) = \mathbb{P}\left(\sum_{p=p_0}^{\infty} \frac{\|\mathbf{w}\|^p_{L^p_{([0,\infty)\times B)}}}{R^p(p\ln\ln p)^p} > 1\right) \le 2\eta \frac{A^{p_0}}{R^{p_0}}.$$

Recall that $p_0 = 7$. We apply the layer-cake formula:

$$\begin{aligned}
\mathbb{E}\left[\|\mathbf{w}\|^6_Z\right] &= \int_0^\infty 6R^5\, \mathbb{P}\left(\|\mathbf{w}\|_Z > R\right) dR \\
&\le \int_0^{2A\eta^{\frac{1}{p_0}}} 6R^5\, dR + 2\eta \int_{2A\eta^{\frac{1}{p_0}}}^\infty 6R^5 \frac{A^{p_0}}{R^{p_0}}\, dR \\
&\le \eta\left(2^6 A^6 \eta^{\frac{6}{p_0}} + 12\eta A^{p_0} \int_{2A\eta^{\frac{1}{p_0}}}^\infty R^{-2}\, dR\right) \\
&\lesssim A^6\eta^{\frac{6}{p_0}} + A^{p_0-1}\eta^{1-\frac{1}{p_0}} \overset{(p_0=7)}{=} A^6\eta^{\frac{6}{7}}.
\end{aligned}$$

Using this and (3.2), we find that there exist constants $C_1, C_2 > 0$, independent of all the other variables, so that

$$\mathbb{E}\left[\|\mathbf{w}\|_{\mathcal{X}^{\alpha-1}}^6\right] \leq C_1\left(\mathbb{E}\left[\|\mathbf{w}\|_{Y^{\alpha-1}}^6\right] + \mathbb{E}\left[\|\mathbf{w}\|_Z^6\right]\right) \leq C_2\eta^{\frac{6}{7}}A^6.$$

We are almost done, but a subtletly remains for our proof to be complete: we know that $\mathbb{E}\left[\|\mathbf{w}\|_Z\right] < \infty$ which implies $\mathbf{w} \in \mathcal{Z}$ almost surely, but for $X \in Z$ by Lemma 3.3, $(ii)$ we need that $\limsup_{p\geq p_0} \frac{\|X\|_{L^p}}{p\ln\ln p} = 0$. We can show this by proving the almost sure existence of a linear growth bound in $p$. This is why we defined the space $\mathcal{Z}$ with a slightly weaker growth bound $p\ln\ln p$. We can improve the left hand side of (3.3) to get

$$\mathbb{E}\left[\sum_{p=p_0}^{\infty} \frac{\|\mathbf{w}\|_{L^p_{([0,\infty)\times B)}}^p}{R^p p^p}\right] < \infty.$$

Then the sum almost surely converges and hence the summands must almost surely have a uniform bound for all $p$, yielding the desired growth bound by $p$. As a consequence it must almost surely be the case that $\limsup_{p\geq p_0} \frac{\|\mathbf{w}\|_{L^p}}{p\ln\ln p} = 0$ and hence $\mathbf{w} \in Z$.

Now we show $(ii)$. This is a combination of $(i)$ and Lemma 2.13. Note first of all that Corollary 2.5 and Lemma 2.2 imply that $\pi_1 S(t)\mathbf{w}_N^M$ is indeed an $L^2_{\mathrm{r}}([0,T]\times B)$-valued random variable. Therefore $(i)$ applies and we only have to show estimates of the type (3.1) and (3.2). We define $G_N^M(t) := \pi_1 S(t)\mathbf{w}_N^M$ and we want this to fulfill the conditions of Lemma 2.13. This means that we have to bound the variance of

$$\langle G_N^M(t), e_n\rangle = \sum_{n=N}^{M} T_n(t)_{1,1}X_n + T_n(t)_{1,2}X_{t,n}.$$

We estimate for $N \leq N \leq M$ that

$$\mathbb{E}\left[\left|\langle G_N^M(t), e_n\rangle\right|^2\right] = \left|T_n(t)_{1,1}^2\sigma_n^2 + T_n(t)_{1,2}^2\sigma_{t,n}^2\right| \lesssim e^{-t}\beta^2\langle\lambda_n\rangle^{-2} + e^{-t}\langle\lambda_n\rangle^{-2}\beta^2.$$

Note that $\|e^{-t}\beta\|_{L^p_([0,\infty)} \leq \beta$. Therefore we can apply Lemma 2.13 (note that $\alpha - 1 < \frac{1}{3} < \frac{1}{2}$ and $6(\alpha-1) < 2$) and get the estimates

$$\mathbb{E}\left[\|G_N^M\|_{L^p_{t,x}}^p\right] \leq \frac{C^p\beta^p(1+\ln(N))}{N^2}$$

$$\mathbb{E}\left[\|G_N^M\|_{L^6_t W_x^{\alpha-1,6}}^6\right] \leq \frac{C^6\beta^6(1+\ln(N))}{N^{2-6(\alpha-1)}}.$$

Now $(i)$ concludes. □

**Lemma 3.9.** *Let $\alpha \in \left[1, \frac{3}{4}\right)$. For all $\gamma \in \left(0, \frac{1}{3}\right)$ there exists a modification $\tilde{\Psi}$ of the stochastic convolution such that $\tilde{\Psi} : [0,\infty) \longrightarrow \mathcal{X}^{\alpha-1}$ is almost surely continuous $\gamma$-Hölder continuous on $[0,T]$ for all $T > 0$.*

*Proof.* Let $T > 0$. We will use the Kolmogorov continuity theorem and find a modification for Hölder-continuity on $[0, T]$. Then a single modification that is almost surely Hölder continuous on all intervals $[0, T]$ can easily be constructed. Let $0 < t_0 < t_1 < T$. We get the estimate needed for the Kolmogorov continuity theorem through Theorem 3.8 applied to $\Psi(t_1) - \Psi(t_0)$. For the variances we have

$$\mathbb{E}\left[\langle \psi(t_1) - \psi(t_0), e_n \rangle^2\right] \lesssim \frac{|t-s|}{\langle \lambda_n \rangle^2}$$

and

$$\mathbb{E}\left[\langle \psi_t(t_1) - \psi_t(t_0), e_n \rangle^2\right] \lesssim |t-s|.$$

by Lemma 2.10. Thince the we have the variance bound $c_n^2(t) \lesssim \langle \lambda_n \rangle^{-2}$ and $g_n(t)^2 \lesssim 1$ we can apply the theorem for $N = 1$ and $M = \infty$, which yields

$$\mathbb{E}\left[\|\Psi(t_1) - \Psi(t_0)\|_{\mathcal{X}^{\alpha-1}}^6\right] \leq C_1 |t_1 - t_0|^3.$$

Kolmogorov's continuity theorem now implies that for all $0 < \gamma < \frac{3-1}{6} = \frac{1}{3}$ there exists a modification $\tilde{\Psi}_T$ of $\Psi$ such that $\tilde{\Psi} \in C^{0,\gamma}([0, T], \mathcal{X}^{\alpha-1})$ almost surely. We can now construct from these $\tilde{\Psi}_T$ a modification $\tilde{\Psi}$ of $\Psi$ which is a.s. continuous in $\mathcal{X}^{\alpha-1}$ on the whole domain $[0, \infty)$. □

From now on we will implicitly assume that $\Psi$ refers to this modification $\tilde{\Psi}$.

### 3.3 Energy Estimates

Let $\mathbf{v} \in \mathcal{H}_{\mathrm{r}}^1$. we define an energy $E : \mathcal{H}_{\mathrm{r}}^1 \longrightarrow [0, \infty)$ by

$$E(\mathbf{v}) = \frac{1}{2} \int_B |v|^2 + |v_t|^2 + |\nabla v|^2 + \frac{1}{2}|v|^4 \, dx.$$

Recall that $H^1$ embeds into all $L^p$ spaces so this is finite. This energy has regularity $E \in C^1(\mathcal{H}_{\mathrm{r}}^1, \mathbb{R})$ with Fréchet derivative

$$DE(\mathbf{v})(\mathbf{f}) = \int_B vf + v_t f_t + \nabla v \nabla f + v^3 f \, dx.$$

Our solutions only have regularity $C^1([0, T], \mathcal{H}_{\mathrm{r}}^{\alpha-1})$ with $\alpha - 1 < 1$, so we can not simply differentiate the energy straight away. Instead we consider the energy of the solutions $\mathbf{v}_N(t)$ to the truncated equation. Then we take a limit to show that $E(\mathbf{v}(t))$ is absolutely continuous in time and even continuously differentiable under additional assumptions.

**Lemma 3.10.** *Let $\mathbf{v}$, $\mathbf{v}_N$ be mild solutions to (2.19) and (2.21) on $[0, T]$. Then $t \mapsto E(\mathbf{v}_N(t)) \in C^1([0, T], \mathbb{R})$ with derivative*

$$\frac{d}{dt} E(\mathbf{v}_N(t)) = \int_B -|v_{N,t}|^2 + v_{N,t}(v^3 - P_N(w + \psi + v_N)^3) \, dx, \tag{3.4}$$

*and $E(\mathbf{v}(t))$ is absolutely continuous in time with a.e. derivative*

$$\frac{d}{dt} E(\mathbf{v}(t)) = \int_B -|v_t|^2 + v_t(v^3 - (w + \psi + v)^3) \, dx. \tag{3.5}$$

*If in addition that $w + \psi \in L^\infty([0, T], L_{\mathrm{r}}^8)$, then also $E(\mathbf{v}(t)) \in C^1([0, T], \mathbb{R})$.*

*Proof.* We first show that the energy is absolutely continuous, then we show that the a.e. derivative is itself continuous, therefore making the energy $C^1$.

We know that $\mathbf{v}_N \in C^1([0,T],\mathcal{H}^{\alpha-1})$, so with Lemma 2.18 we have $\mathbf{v}_N = P_N\mathbf{v}_N \in C^1([0,T],\mathcal{H}^1)$ and can compute

$$\begin{aligned}\frac{d}{dt}E(\mathbf{v}_N(t)) &= DE(\mathbf{v}_N(t))(\partial_t\mathbf{v}_N(t))\\ &= \int_B v_N v_{N,t} + v_{N,t}\partial_t v_{N,t} + \nabla v_N \nabla v_{N,t} + v_N^3 v_{N,t}\,dx\\ &= \int_B v_{N,t}\big(\underbrace{\Delta v_N - \Delta v_N}_{=0} + \underbrace{v_N - v_N}_{=0} - v_{N,t} + v_N^3 - P_N(w+\psi+v_N)^3\big)\,dx.\end{aligned}$$

The above implies that for any $0 \le t_0 < t_1 \le T$,

$$E(\mathbf{v}_N(t_1)) - E(\mathbf{v}_N(t_0)) = \int_{t_0}^{t_1}\int_B -|v_{N,t}|^2 + v_{N,t}v_N^3 - v_{N,t}P_N(w+\psi+v_N)^3\,dx\,ds.$$

Now we let $N \longrightarrow \infty$ on both sides and use dominated convergence. A majorant is given by the absolute value of the integrand. We check its integrability for the last two terms. For the first one:

$$\int_{t_0}^{t_1}\int_B \left|v_{N,t}v_N^3\right|\,dx\,dt \le \|v_{N,t}\|_{L_t^2L_x^2}\|v_N\|^3_{L_t^6L_x^6} \lesssim \|\mathbf{v}_N\|^4_{L_t^\infty\mathcal{H}_x^\alpha} < \infty.$$

For the second one:

$$\begin{aligned}\int_{t_0}^{t_1}\int_B \left|v_{N,t}P_N(w+\psi+v_N)^3\right|\,dx\,dt &\le \|v_{N,t}\|_{L_t^2L_x^2}\|P_N(w+\psi+v_N)^3\|_{L_t^2L_x^2}\\ \lesssim \|\mathbf{v}_N\|_{L_t^\infty\mathcal{H}_x^\alpha}\|w+\psi+v_N\|^3_{L_t^6L_x^6} &\le \|\mathbf{v}_N\|_{L_t^\infty\mathcal{H}_x^\alpha}\|w+\psi+v_N\|_{Y^{\alpha-1}} < \infty.\end{aligned}$$

Since we know by Lemma 2.24 that the left hand side converges to $E(\mathbf{v}(t_1)) - E(\mathbf{v}(t_1))$ almost everywhere, and the integrand in the right hand side also converges almost everyhwere, taking the limit $N \longrightarrow \infty$ yields that $E(\mathbf{v}(t))$ is absolutely continuous in time with a.e. derivative given by (3.5).

Assume now that $w+\psi \in L^\infty([0,T],L_r^8)$ and note that after dropping the projection (since $v_{N,t} = P_N v_{N,t}$), we can write

$$\begin{aligned}\frac{d}{dt}E(\mathbf{v}_N(t)) &= \int_B -v_{N,t}^2 - 3v_{N,t}v_N(w+\psi)^2 - 3v_{N,t}v_N^2(w+\psi) - v_{N,t}(w+\psi)^3\,ds\\ &= (I) + (II) + (III) + (IV).\end{aligned}$$

Consider now the following estimates. For $(I)$:

$$\left|\int_B v_{N,t}^2\,dx - \int_B v_t^2\,dx\right| \le \|v_{N,t} - v\|_{L^\infty([0,T],L^2)} \le \|\mathbf{v}_N - \mathbf{v}\|_{L_t^\infty\mathcal{H}_x^\alpha}.$$

For $(II)$ we apply Hölder with $\frac{1}{2}+\frac{1}{4}+\frac{1}{4}=1$:

$$\left|\int_B v_{N,t}v_N(w+\psi)^2\,dx - \int_B v_t v(w+\psi)^2\,dx\right|$$

$$\leq \|v_{N,t} - v_t\|_{L^2} \|v_N\|_{L^4} \|w + \psi\|_{L^8}^2 + \|v_t\|_{L^2} \|v_N - v\|_{L^4} \|w + \psi\|_{L^8}^2$$
$$\lesssim \|\mathbf{v}_N - \mathbf{v}\|_{L_t^\infty \mathcal{H}_x^\alpha} (\|\mathbf{v}_N\|_{L_t^\infty \mathcal{H}_x^\alpha} + \|\mathbf{v}\|_{L_t^\infty \mathcal{H}_x^\alpha}) \|w + \psi\|_{L_t^\infty L_x^8}^2$$

For $(III)$ we also apply Hölder with $\frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1$:

$$\left| \int_B v_{N,t} v_N^2 (w + \psi) \, dx - \int_B v_t v^2 (w + \psi) \, dx \right|$$
$$\leq \|v_{N,t} - v_t\|_{L^2} \|v_N^2\|_{L^4} \|w + \psi\|_{L^4} + \|v_t\|_{L^2} \|v_N^2 - v^2\|_{L^4} \|w + \psi\|_{L^4} \tag{3.6}$$

Note that

$$\|v_N^2 - v^2\|_{L^4} \left( \int_B (v_N - v)^4 (v_N + v)^4 \, dx \right)^{\frac{1}{4}} \leq \|v_N - v\|_{L^8} \|v_N + v\|_{L^8},$$

so

$$(3.6) \lesssim \|\mathbf{v}_N - \mathbf{v}\|_{L_t^\infty \mathcal{H}_x^\alpha} (\|\mathbf{v}_N\|_{L_t^\infty \mathcal{H}_x^\alpha}^2 + \|\mathbf{v}\|_{L_t^\infty \mathcal{H}_x^\alpha}^2) \|w + \psi\|_{L_t^\infty L_x^4}$$

For $(IV)$ we apply Hölder with $\frac{1}{2} + \frac{1}{2} = 1$:

$$\left| \int_B v_{N,t} (w + \psi)^3 \, dx - \int_B v_t (w + \psi)^3 \, dx \right| \leq \|v_{N,t} - v_t\|_{L^2} \|w + \psi\|_{L^6}^2$$
$$\lesssim \|\mathbf{v}_N - \mathbf{v}\|_{L_t^\infty \mathcal{H}_x^\alpha} \|w + \psi\|_{L_t^\infty L_x^6}^2.$$

Together these imply that

$$\frac{d}{dt} E(\mathbf{v}_N(t)) \longrightarrow \frac{d}{dt} E(\mathbf{v}(t))$$

uniformly in time. Therefore since left hand side is continuous in time for any $N$, so is the right hand side. □

We now derive a global energy estimate.

**Theorem 3.11** (Energy Estimate)**.** *Let* $\mathbf{v}$ *be a mild solution to (2.19) on* $[0, T]$*. Assume in addition to* $w + \psi \in Y_{[0,T]}^{\alpha-1}$ *that* $w + \psi \in Z_{[0,T]}$*. Define* $E(t) = E(\mathbf{v}(t))$ *or* $E(t) = E(\mathbf{v}_N(t))$*.*

*Then there exists a constant* $C > 0$ *such that for all* $0 \leq t_0 < t_1 \leq T$ *with* $\ln \ln \ln E(s) \geq 1$ *for* $s \in [t_0, t_1]$ *we have the estimate*

$$E(t_1) \leq \exp\left( \exp\left( G^{-1}\left( G(\ln \ln(E(t_0))) + C(1 + t_1 - t_0) \|w + \psi\|_{Z([t_0, t_1])} \right) \right) \right).$$

*Here* $G(x) = \frac{x}{\ln(x)}$*, which is invertible on* $[1, \infty)$*.*

*Proof.* The proof for the truncated case of $\mathbf{v}_N$ is again identical to the one for $\mathbf{v}$.

We have calculated before that

$$\frac{d}{dt} E(\mathbf{v}(t)) = \int_B -|v_t|^2 + v_t \big(v^3 - (w + \psi + v)^3\big) \, dx$$
$$\leq \int_B -3 v_t v (w + \psi)^2 - 3 v_t v^2 (w + \psi) - v_t (w + \psi)^3 \, dx$$
$$= (I) + (II) + (III).$$

For $(I)$ we apply Hölder with $\frac{1}{2}+\frac{1}{4}+\frac{1}{4}=1$:

$$|(I)| \lesssim \|v_t\|_{L^2}\|v\|_{L^4}\|w+\psi\|_{L^8}^2 \lesssim E^{\frac{3}{4}}\|w+\psi\|_{L^8}^2$$

For $(III)$ we apply Hölder with $\frac{1}{2}+\frac{1}{2}=1$:

$$|(III)| \lesssim \|v_t\|_{L^2}\|w+\psi\|_{L^6}^3 \lesssim E^{\frac{1}{2}}\|w+\psi\|_{L^6}^3.$$

For $(II)$ we apply Hölder with $\frac{1}{2}+\frac{1}{q}+\frac{1}{p}$ where $p>1$ and $\frac{1}{q}+\frac{1}{p}=1$.

$$(II) \lesssim \|v_t\|_{L^2}\|v\|_{L^{2q}}^2\|w+\psi\|_{L^p}.$$

Now we apply a specical case of the Gagliardo-Nirenberg-Sobolev inequality, also known as a generalization of Ladyzhenskaya's inequality:

$$\|u\|_{L^{2q}} \lesssim \|u\|_{L^4}^{\frac{2}{q}}\|u\|_{H^1}^{1-\frac{2}{q}}.$$

The Gagliardo-Nirenberg-Sobolev inequality is obtained, as the name suggests, by combining the Sobolev inequality and the Gagliardo-Nirenberg interpolation inequality. A detailed but transparent study of this inequality can be found at [4]. We get that

$$(II) \lesssim C\|w+\psi\|_{L^p}\|v_t\|_{L^2}\|v\|_{L^4}^{\frac{4}{q}}\|v\|_{H^1}^{2-\frac{4}{q}} \lesssim \|w+\psi\|_{L^p}E^{\frac{1}{2}+\frac{1}{q}+1-\frac{2}{q}} = \frac{\|w+\psi\|_{L^p}}{p}pE^{1+\frac{1}{p}}.$$

Now we choose $p(s)=\ln(E(s))\geq 1$ and define $\beta(s,r)=\frac{\|w(s)+\psi(s)\|_{L^r}}{r\ln\ln r}$. Applying Young's inequality for products to the estimates for $(I)$ and $(III)$, we can summarize what we have shown as

$$\begin{aligned}E'(s) &\lesssim \beta(s,6)+\beta(s,8)+E(s)+\beta(s,\ln(E(s)))\ln(E(s))\ln\ln\ln(E(s))e^{\ln(E(s))\left(1+\frac{1}{\ln(E(s))}\right)}\\ &= \beta(s,6)+\beta(s,8)+E(s)+\beta(s,\ln(E(s)))eE(s)\ln(E(s))\ln\ln\ln(E(s)).\end{aligned}$$

Then using $E(s)\geq 1$ we have

$$\begin{aligned}\frac{E'(s)}{E(s)} &\lesssim 1+\frac{\beta(s,6)+\beta(s,8)}{E(s)}+\beta(s,\ln(E(s)))\ln(E(s))\ln\ln\ln(E(s))\\ &\lesssim 1+\beta(s,6)+\beta(s,8)+\beta(s,\ln(E(s)))\ln(E(s))\ln\ln\ln(E(s)).\end{aligned}$$

Since ln is Lipschitz continuous on the range of values of $E$ on $[t_0,t_1]$ and $E$ is absolutely continuous, $\ln\circ E$ is absolutely continuous on $[t_0,t_1]$ with derivative given by the left hand side above. Then we get

$$\frac{\ln(E(s))'}{\ln(E(s))} \lesssim 1+\beta(s,6)+\beta(s,8)+\beta(s,\ln(E(s)))\ln\ln\ln(E(s)).$$

Again $f:=\ln\circ\ln\circ E$ is absolutely continuous on $[t_0,t_1]$ so we get the differential inequality

$$f'(s)\leq C\cdot\big(1+\beta(s,6)+\beta(s,8)+\beta(s,\ln(E(s))\ln(f(s))\big).$$

Since $\ln(f) \geq 1$ on $[t_0, t_1]$ the function $\frac{f}{\ln(f)}$ is absolutely continuous and we can calculate

$$\left(\frac{f}{\ln(f)}\right)' = \left(\frac{1}{\ln(f)} - \frac{1}{\ln(f)^2}\right) f'.$$

If $s$ is a time where the left hand side is nonnegative, then so is $f'(s)$ and we get for those times that

$$\left(\frac{f}{\ln(f)}\right)'(s) = \frac{f'(s)}{\ln(f(s))} \leq C \cdot \left(\frac{1 + \beta(s,6) + \beta(s,8)}{\ln(f(s))} + \beta(s, \ln(E(s))\right).$$

Using again $\ln(f(s)) \geq 1$ and setting $G(x) = \frac{x}{\ln(x)}$, the above implies

$$G(f(t_1)) - G(f(t_0)) \leq C \int_{t_0}^{t_1} 1 + \beta(s,6) + \beta(s,8) + \beta(s, \ln(E(s))\, ds.$$

Note that $G'(x) = \ln(x)^{-1} - \ln(x)^{-2}$ is strictly positive on $[1, \infty)$ and so on the range of values of $f$ on $[t_0, t_1]$. Therefore $G$ is invertible on the relevant interval. Using $(iv)$ from Lemma 3.3, we get

$$E(t_1) \leq \exp\left(\exp\left(G^{-1}\left(G(\ln\ln(E(t_0))) + C(1 + t_1 - t_0)\|w + \psi\|_{Z([t_0,t_1])}\right)\right)\right).$$

Note that

$$x\ln(x) \leq G^{-1}(x) \leq 2x\ln(x)$$

for all $\ln\ln\ln x \geq 1$. For the first inequality we check that

$$G^{-1}(x\ln(x)) = \frac{x\ln(x)}{\ln(x\ln(x))} \leq x.$$

For the second one we have

$$G^{-1}(2x\ln(x)) = \frac{2x\ln(x)}{\ln(2x\ln(x))} \geq x$$

if $x^2 \geq 2x\ln(x)$. This is the case since $x$ is sufficiently large. □

**Lemma 3.12.** *Let $T^* > 0$ and $\alpha \in \left[1, \frac{4}{3}\right)$. Let $\mathbf{v}$ be a mild solution to (2.19) or (2.21)) on $[0, T]$ for all $T \in [0, T^*)$. Then the following are equivalent:*

*(i)* $\sup_{t \in [0,T^*)} E(t) < \infty$.

*(ii)* *$\mathbf{v}$ can be extended to a mild solution on $[0, T']$ for some $T' > T^*$. This extension is given by*

$$\mathbf{v}(T^* + t) = \tilde{\mathbf{v}}(t) + S(t)\mathbf{v}(T^*),$$

*where $\tilde{\mathbf{v}}$ is a mild solution on $[0, T' - T^*]$ for an initial data and stochastic convolution part given by*

$$\tilde{w}(t) + \tilde{\psi}(t) = w(T^* + t) + S(t)\mathbf{v}(T^*) + \psi(T^* + t).$$

*Proof.* We first show that $(i) \implies (ii)$. Note that $(i)$ implies $\mathbf{v} \in L^\infty([0,T^*],\mathcal{H}^1_{\mathrm{r}})$, and so by Lemma 2.25 we have $\mathbf{v} \in L^\infty([0,T^*],\mathcal{H}^\alpha_{\mathrm{r}})$. Lemma 2.23 implies that it is continuous on $[0,T^*]$.

By Theorem 2.22 there exists for some $T > 0$ a mild solution $\tilde{\mathbf{v}}$ on $[0,T]$ with initial data and stochastic convolution part given by

$$\tilde{w}(t) + \tilde{\psi}(t) = w(T^* + t) + S(t)\mathbf{v}(T^*) + \psi(T^* + t).$$

Since Lemma 2.23 yields regularity, we can make the following explicit computation to see that we have a solution in a strong sense, and hence particularly a mild solution:

$$\partial_t(\tilde{\mathbf{v}}(t)+S(t)\mathbf{v}(T^*)) = L(\tilde{\mathbf{v}}(t)+S(t)\mathbf{v}(T^*)) - \begin{pmatrix} 0 \\ (w(T^*+t)+\psi(T^*+t)+\tilde{\mathbf{v}}(t)+S(t)\mathbf{v}(T^*))^3 \end{pmatrix},$$

$$\tilde{\mathbf{v}}(0) + S(0)\mathbf{v}(T^*) = \mathbf{v}(T^*).$$

We have shown that we can continue $\mathbf{v}$ to a mild solution on $[0, T^* + T]$.

The direction $(ii) \implies (i)$ is trivial as $(ii)$ implies that $\mathbf{v} \in L^\infty([0,T'],\mathcal{H}^\alpha_{\mathrm{r}})$, and this bounds the energy. □

**Lemma 3.13.** *Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be two mild solutions to (2.19) or (2.21) on $[0,T']$ for a given $w+\psi$ as in the definition of a mild solution. Then $\mathbf{v}_1 = \mathbf{v}_2$.*

*Proof.* The local well-posedness result only gives us uniqueness up to some fixed time $T > 0$ determined by the estimates for the fixed point argument. Define

$$T^* = \sup\{0 \le T_1 \le T' : \mathbf{v}_1(t) = \mathbf{v}_2(t)\ \forall t \in [0,T_1]\}.$$

By the continuity of the solutions (Lemma 2.23) we directly get $\mathbf{v}_1(T^*) = \mathbf{v}_2(T^*)$. If $T^* = T'$ then we are done, so suppose that $T^* < T'$. Consider the maps

$$\tilde{\mathbf{v}}_1(t) = \mathbf{v}_1(T^*+t) - S(t)\mathbf{v}_1(T^*)$$
$$\tilde{\mathbf{v}}_2(t) = \mathbf{v}_2(T^*+t) - S(t)\mathbf{v}_2(T^*).$$

Since Lemma 2.23 yields regularity, we can make the following computation:

$$\partial_t\tilde{\mathbf{v}}_1(t) = L\mathbf{v}_1(t) - \begin{pmatrix} 0 \\ (w(T^*+t)+S(t)\mathbf{v}_1(T^*)+\psi(T^*+t)+\tilde{\mathbf{v}}_1(t))^3 \end{pmatrix},$$

$$\tilde{\mathbf{v}}_1(0) = \mathbf{v}_1(T^*+0) - S(0)\mathbf{v}_1(T^*) = 0.$$

Therefore $\tilde{\mathbf{v}}_1$ and $\tilde{\mathbf{v}}_2$ are themselves mild solutions to (2.19) with $w(t)+\psi(t)$ replaced by $w(T^*+t)+S(t)\mathbf{v}_1(T^*)+\psi(T^*+t)$. The local uniqueness result implies that $\tilde{\mathbf{v}}_1 = \tilde{\mathbf{v}}_2$ on $[0,T]$. But by the uniqueness of the linear problem $S(t)\mathbf{v}_1(T^*) = S(t)\mathbf{v}_2(T^*)$, hence $\mathbf{v}_1 = \mathbf{v}_2$ on $[T^*,(T^*+T)\wedge T']$. By the definition of $T^*$ this implies that $T^* = T'$, a contradiction. □

**Theorem 3.14** (Deterministic Global Well-posedness)**.** *If $w+\psi \in X^{\alpha-1}_{[0,T]}$ for all $T > 0$, then there exist unique global solutions $\mathbf{v}$ and $\mathbf{v}_N$ to (2.19) and (2.21). For any $T > 0$ the maps*

$$X^{\alpha-1}_{[0,T]} \times [0,T] \longrightarrow \mathcal{H}^\alpha_{\mathrm{r}}$$
$$w+\psi \longmapsto \mathbf{v}$$
$$w+\psi \longmapsto \mathbf{v}_N$$

*are jointly continuous in $(w+\psi, t)$.*

*Proof.* The proof for (2.21) is completely analogous to the one presented below. Since $w+\psi \in X^{\alpha-1}_{[0,T]}$ we have $\|w+\psi\|_{Y^{\alpha-1}_{[0,T]}} + \|w+\psi\|_{Z_{[0,T]}} < \infty$ for any $T > 0$, so the assumptions of the theorems and lemmas that we use below are fulfilled.

Define

$$T^* = \sup\{T > 0 : \text{ (2.19) has a mild solution } \mathbf{v}^T \text{ on } [0,T] \}.$$

If $T_0, T_1 < T^*$, then by the uniqueness in 3.13 we have $\mathbf{v}^{T_1}\big|_{[0,T_1\wedge T_2]} = \mathbf{v}^{T_2}\big|_{[0,T_1\wedge T_2]}$. We can therefore just write $\mathbf{v}$ to refer to a unique solution that exists on the interval $[0,T^*)$. If $T^* = \infty$ then we are done, so we suppose that $T^* < \infty$. The energy estimate in Theorem 3.11 implies that $\sup_{t\in[0,T^*)} E(t) < \infty$, and so by Lemma 3.12 we can extend $\mathbf{v}$ to a mild solution on a larger interval, contradicting the maximality of $T^*$.

The continuity works the same way as the one in the local well-posedness Theorem 2.22, except that we can use the energy inequality to perform the argument for a uniform $T > 0$ independent of the size of $\|w+\psi\|_{Y^{\alpha-1}_{[0,T]}}$. □

**Corollary 3.15** (Stochastic Global Well-posedness)**.** *Let $\alpha \in \left[1, \frac{4}{3}\right)$ and $\Psi$ be the stochastic convolution. Let $\mathbf{w}_0 \in \mathcal{X}^{\alpha-1}$.*

*Then there exist random variables $\mathbf{v}$ and $\mathbf{v}_N$ which a.s. are mild solutions (2.19) and (2.21) respectively on any interval $[0,T]$. Furthermore*

$$\mathbf{u} \coloneqq \mathbf{v} + \mathbf{w} + \Psi \quad \textit{and} \quad \mathbf{u}_N \coloneqq \mathbf{v}_N + \mathbf{w} + \Psi$$

*solve (1.5), i.e.*

$$\begin{gathered}\partial_t^2 u + \partial_t u + (1-\Delta)u + u^3 = \sqrt{2}\xi \\ (u, u_t)(0) = \mathbf{w}_0\end{gathered}$$

*and*

$$\begin{gathered}\partial_t^2 u_N + \partial_t u_N + (1-\Delta)u_N + u_N^3 = \sqrt{2}\xi \\ (u_N, u_{N,t})(0) = \mathbf{w}_0\end{gathered}$$

*almost surely globally in time in the sense of distributions.*

*Proof.* We have shown in lemmas 2.15 and 3.1 that $\psi \in X^{\alpha-1}$. The assumption $\mathbf{w}_0 \in \mathcal{X}^{\alpha-1}$ then implies $w + \psi \in X^{\alpha-1}$, so we can use Theorem 3.14 to find $\mathbf{v}$ and $\mathbf{v}_N$.

We apply the stochastic local well-posedness (Corollary 2.26) to see that $\mathbf{u}$ and $\mathbf{u}_N$ are distributional solutions on any $[0,T]$, hence globally in time. □

# 4 The Invariant Measure

## 4.1 The Stochastic Flows $\Phi$ and $\Phi_N$

Given the global well-posedness result we can now define stochastic flows on $\mathcal{X}^{\alpha-1}$ by

$$\begin{aligned} \Phi : \mathbb{R}_+ \times \Omega \times \mathcal{X}^{\alpha-1} &\longrightarrow \mathcal{X}^{\alpha-1} \\ (t, \xi, \mathbf{w}_0) &\longmapsto \mathbf{u}(t) = S(t)\mathbf{w}_0 + \Psi(\xi, t) + \mathbf{v}(t) \end{aligned}$$

and analogously

$$\begin{aligned} \Phi_N : \mathbb{R}_+ \times \Omega \times \mathcal{X}^{\alpha-1} &\longrightarrow \mathcal{X}^{\alpha-1} \\ (t, \xi, \mathbf{w}_0) &\longmapsto \mathbf{u}_N(t) = S(t)\mathbf{w}_0 + \Psi(\xi, t) + \mathbf{v}_N(t), \end{aligned}$$

where $\mathbf{v}$ and $\mathbf{v}_N$ are the unique global solutions from Theorem 3.14 for $w + \psi = \pi_1 S(t)\mathbf{w}_0 + \pi_1 \Psi(t, \xi)$.

Here we use a variable $\xi \in \Omega$, which is already the notation for the radially symmetric space-time white noise $\xi$. The reason for this is that we can write $\Psi(\xi)$ to signify that $\Psi$ depends on the white noise $\xi$. We overload the notation and sometimes write $\Psi(\xi')$ for a different white noise $\xi'$ to mean that this white noise is instead used in the definition of the stochastic convolution.

**Lemma 4.1.** *$\Phi$ and $\Phi_N$ are continuous stochastic flows in the sense that the following holds almost surely (i.e. for almost all $\xi \in \Omega$):*

*(i)* $\Phi(0, \xi, \mathbf{w}_0) = \Phi_N(0, \xi, \mathbf{w}_0) = \mathbf{w}_0$ *for all* $\mathbf{w}_0 \in \mathcal{X}^{\alpha-1}$*,*

*(ii)* $\Phi(t, \xi, \mathbf{w}_0)$ *and* $\Phi_N(t, \xi, \mathbf{w}_0)$ *are continuous with respect to* $(t, \mathbf{w}_0)$*.*

*(iii) Let $\xi_1$, $\xi_2$ and $\xi$ independent instances of radially symmetric space-time white noise. Then for all $s, t \geq 0$ and $\mathbf{w}_0 \in \mathcal{X}^{\alpha-1}$ we have*

$$\mathrm{Law}(\Phi(s, \xi_1, \Phi(t, \xi_2, \mathbf{w}_0)) = \mathrm{Law}(\Phi(t + s, \xi, \mathbf{w}_0)$$

*and*

$$\mathrm{Law}(\Phi_N(s, \xi_1, \Phi_N(t, \xi_2, \mathbf{w}_0)) = \mathrm{Law}(\Phi_N(t + s, \xi, \mathbf{w}_0).$$

*Proof.* Statement $(i)$ is trivial. We show $(ii)$. We have seen in Lemma 3.7 that $S(t)\mathbf{w}_0$ is continuous in $(t, \mathbf{w}_0)$. We have seen in Lemma 3.9 that also $\Psi$ is a.s. continuous in $t$. Lastly, we have shown in Lemma 2.22 that the the mild solution $\mathbf{v}$ and $\mathbf{v}_N$ for given $w + \psi$ are continuous from $X^{\alpha-1}_{[0,T]} \times [0, T]$ to $\mathcal{H}^\alpha_{\mathrm{r}}$ for any $T > 0$. Since the latter space embeds into $\mathcal{X}^{\alpha-1}$ and since the map

$$\begin{aligned} \mathcal{X}^{\alpha-1} &\longrightarrow X^{\alpha-1}_{[0,T]} \\ \mathbf{w}_0 &\longmapsto w := \pi_1 S(t)\mathbf{w}_0 \end{aligned}$$

is by the definition of $\mathcal{X}^{\alpha-1}$ continuous, we are done.

Now we show $(iii)$ for the case of $\Phi$. We have

$$\Phi(t, \xi_1, \mathbf{w}_0) = S(t)\mathbf{w}_0 + \Psi(\xi_1, t) + \mathbf{v}_1(t)$$

where $\mathbf{v}_1$ is a mild solution corresponding to $[w+\psi](t) = \pi_1 S(t)\mathbf{w}_0 + \psi(\xi_1, t)$, and

$$\Phi(s, \xi_2, \Phi(t, \xi_1, \mathbf{w}_0)) = S(s)\Phi(t, \xi_1, \mathbf{w}_0) + \psi(\xi_2, s) + \mathbf{v}_2(s)$$

where $\mathbf{v}_2$ is a mild solution corresponding to

$$\begin{aligned}[w+\psi](s) &= \pi_1 S(s)\Phi(t, \xi_1, \mathbf{w}_0) + \pi_1\psi(\xi_2, s) \\ &= \pi_1 S(t+s)\mathbf{w}_0 + \pi_1 S(s)\Psi(\xi_1, t) + \pi_1 S(s)\mathbf{v}_1(t) + \psi(\xi_2, s).\end{aligned}$$

For $f \in \mathcal{D}_{\mathrm{r}}(\mathbb{R} \times B)$ We define

$$\langle \xi_3, f \rangle \coloneqq \langle \xi_1, \mathbb{1}_{(-\infty,t]} f \rangle + \langle \xi_2, \mathbb{1}_{[0,\infty)} f(\cdot + t) \rangle.$$

This means that essentially $\xi_3$ is $\xi_1$ up to time $t$ and afterwards $\xi_3$ is $\xi_2$. Since $\xi_1$ and $\xi_2$ are independent it is easily verified that $\xi_3$ is again a radially symmetric space-time white noise. Note that by definition 2.8 for $\mathbf{f} \in \mathcal{D}^2_{\mathrm{r}}(\mathbb{R}_+ \times B)$ we have

$$\begin{aligned}\langle \Psi(\xi_3, t+s), \mathbf{f} \rangle &= \left\langle \xi_3, \mathbb{1}_{[0,t+s]}(r)\sqrt{2}\pi_2 S^*(t+s-r)\mathbf{f}(r) \right\rangle \\ &= \left\langle \xi_1, \mathbb{1}_{[0,t]}(r)\sqrt{2}\pi_2 S^*(t+s-r)\mathbf{f}(r) \right\rangle \\ &\quad + \left\langle \xi_2, \mathbb{1}_{[0,s]}(r)\sqrt{2}\pi_2 S^*(s-r)\mathbf{f}(r) \right\rangle \\ &= \langle S(s)\Psi(\xi_1, t) + \Psi(\xi_2, s), \mathbf{f} \rangle .\end{aligned}$$

We now proceed similarly as in Lemma 3.12, defining

$$\mathbf{v}(s) = \begin{cases} \mathbf{v}_1(s), & s \le t \\ S(s-t)\mathbf{v}_1(t) + \mathbf{v}_2(s-t), & s \ge t \end{cases}.$$

Since Lemma 2.23 yields regularity can make the following calculation: For $s \le t$ we have

$$\partial_t \mathbf{v}(s) = L\mathbf{v}_1(s) + \begin{pmatrix} 0 \\ -\big(\pi_1 S(s)\mathbf{w}_0 + \psi(\xi_1, s) + v_1(s)\big)^3 \end{pmatrix}.$$

On the other hand for $s \ge 0$ we have

$$\partial_t \mathbf{v}(t+s) = L(\mathbf{v}_2(s) + S(s)\mathbf{v}_1(t)) \begin{pmatrix} 0 \\ -\big(\pi_1 S(t+s)\mathbf{w}_0 + \psi(\xi_3, s+t) + \pi_1 S(s)\mathbf{v}_1(t) + v_2(s)\big)^3 \end{pmatrix}.$$

Since $\Psi(\xi_1, s) = \Psi(\xi_3, s)$ almost everywhere for $s \le t$ we see that $\mathbf{v}$ is a mild solution corresponding to $[w+\psi](s) = \pi_1 S(s)\mathbf{w}_0 + \psi(\xi_3, s)$. But at the same time

$$\Phi(t+s, \xi, \mathbf{w}_0) = S(t+s)\mathbf{w}_0 + \Psi(\xi, t+s) + \tilde{\mathbf{v}}_0(t+s)$$

where $\tilde{\mathbf{v}}_0$ is a mild solution corresponding to $[w+\psi](s) = \pi_1 S(s)\mathbf{w}_0 + \psi(\xi, s)$. Let $T > t+s > 0$. Since $\xi_3$ and $\xi$ are both radially symmetric space-time white noises, we have

$$\mathrm{Law}(\psi(\xi)) = \mathrm{Law}(\psi(\xi_3)) \qquad \text{as measures on } Y^{\alpha-1}_{[0,T]}$$

Then due to the continuous dependence of the mild solution with respect to the "initial data + stochastic part" $w+\psi$ established in Theorem 3.14 this implies that

$$\mathrm{Law}(\mathbf{v}(t+s)) = \mathrm{Law}(\tilde{\mathbf{v}}(t+s)) \qquad \text{as measures on } \mathcal{X}^{\alpha-1}.$$

Therefore

$$\mathrm{Law}(\Phi(s, \xi_2, \Phi(t, \xi_1, \mathbf{w}_0))) = \mathrm{Law}(\Phi(t+s, \xi, \mathbf{w}_0)).$$

□

## 4.2 Limits of Invariant Measures

Our goal is to find a measure that is invariant under the flow of $\Phi$. What this means is that if the random initial data is distributed according to this distribution, then the distribution stays constant as our initial data evolves under the flow.

**Definition 4.2.** *A measure $\rho$ on $\mathcal{X}^{\alpha-1}$ is called* ***invariant under*** $\Phi$ *if for every Lipschitz continuous and bounded $F : \mathcal{X}^{\alpha-1} \longrightarrow \mathbb{R}$ and $t \geq 0$ we have*

$$\int_{\mathcal{X}^{\alpha-1}} \mathbb{E}_\xi \left[F(\Phi(t, \xi, \mathbf{w}_0))\right] d\rho(\mathbf{w}_0) = \int_{\mathcal{X}^{\alpha-1}} F(\mathbf{w}_0)\, d\rho(\mathbf{w}_0).$$

We face two challenges: Constructing such a measure, and showing that it is actually invariant. We will first spend some time exploring different measures on $\mathcal{X}^{\alpha-1}$ that we can define as the laws of certain random variables. Then we will construct the an invariant measure as a limit of invariant measures for finite dimensional equations, where we can use finite dimensional methods to prove the invariance.

This limit will be achieved in the weak topology on the space of probability measure on $\mathcal{X}^{\alpha-1}$. We will metrize this topology with the Wasserstein metric because it aids us in computations. Let $p \geq 1$. Given two probability measure $\mu, \nu$ on a metric space $(E, d)$ with finite $p$-th moment, their $p$-**Wasserstein distance** is defined as

$$W_p^{(E,d)}(\mu, \nu) \coloneqq \inf_{\substack{X \sim \mu,\, X_N \sim \nu \\ \text{prob. space } (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})}} \mathbb{E}_{\tilde{P}} \left[d(X, X_N)^p\right]^{\frac{1}{p}}. \tag{4.1}$$

By the Kantorovich-Rubinstein Theorem [16] there is an alternative representation in the case $p = 1$:

$$W_1^{(E,d)}(\mu, \nu) = \sup_{[F]_{\text{Lip}} \leq 1} \int F(\, d\mu - \, d\nu), \tag{4.2}$$

which we will use later. The Wasserstein metric is related to the theory of optimal transport because it represents the minimal "cost" associated with moving an amount of mass distributed according to $\mu$ into the distribution $\nu$. The reader may find out more about this metric in [25, Def. 6.4]. Theorem 6.18 in this book establishes that if $(E, d)$ is complete, then the $p$-Wasserstein metric is a complete metric on the space of probability measures on $E$ with finite $p$-th moment.

Convergence of $\mu_N$ to $\mu$ in the $p$-Wasserstein metric is equivalent to weak convergence plus convergence of the $p$-th moment ([25, Thm. 3.9]), meaning there there exists some $x_0 \in E$ so that

$$\int_E d(x, x_0)^p \, d\mu_N(x) \longrightarrow \int_E d(x, x_0)^p \, d\mu(x).$$

This means that if we are in a bounded setting $d \leq 1$, then the Wasserstein metric metrizes the weak topology.

The following lemma is a crucial element of our strategy. It gives us a way to get an invariant measure for a flow $\Phi$ from a sequence of invariant measures for flows $\Phi_N$.

**Lemma 4.3.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(E, d)$ a Polish space. Suppose that we are given measurable functions*

$$\Phi, \Phi_N : [0, \infty) \times \Omega \times E \longrightarrow E$$

*with $\Phi(0, \xi, x) = \Phi_N(0, \xi, x) = x$ for all $N \in \mathbb{N}$. Then a probability measure $\mu$ on $E$ is invariant under $\Phi$ if the following hypotheses are given:*

*(H1) There exists a function $L(t, \xi, x, y)$ which is locally bounded in $(x, y) \in E^2$ and fulfills*

$$d(\Phi_N(t, \xi, x), \Phi_N(t, \xi, y)) \le L(t, \xi, x, y) d(x, y) \tag{4.3}$$

*for all $N \in \mathbb{N}$ and $x, y \in E$.*

*(H2) For all $t > 0$ we have*

$$\int_E \mathbb{E}_\xi \left[ d(\Phi(t, \xi, x), \Phi_N(t, \xi, x)) \right] \, d\mu(x) \xrightarrow{N \to \infty} 0. \tag{4.4}$$

*(H3) There exists a sequence of probability measures $(\mu_N)_{N \in \mathbb{N}}$ on $E$, where each $\mu_N$ is invariant under the flow $\Phi_N$, so that $\mu_N$ converges weakly to $\mu$.*

*Proof.* By the Skorokhod representation Theorem [2, Thm. 3.8.6] there exists a probability space $(\Omega_2, \mathcal{F}_2, \mathcal{P}_2)$ on which we have $E$-valued random variables $X_N, X$, so that $\mathrm{Law}(X_N) = \mu_N$, $\mathrm{Law}(X) = \mu$ and almost surely $X_N \longrightarrow X$. Dominated convergence then implies that

$$\int_{\Omega_2} \tilde{d}(X_N(\omega), X(\omega)) \, d\,\mathbb{P}_2(\omega) \xrightarrow{N \to \infty} 0.$$

Now let $F : (E, d) \longrightarrow \mathbb{R}$ be Lipschitz continuous with constant $L' > 0$ and absolutely bounded by $b > 0$. We have

$$\begin{aligned}
\int_E \mathbb{E}_\xi \left[ F(\Phi(t, \xi, x)) \right] \, d\mu(x) = &\int_{\Omega_2} \mathbb{E}_\xi \left[ F(\Phi_N(t, \xi, X_N(\omega))) \right] \, d\,\mathbb{P}_2(\omega) \\
&+ \int_{\Omega_2} \mathbb{E}_\xi \left[ F(\Phi_N(t, \xi, X(\omega))) - F(\Phi_N(t, \xi, X_N(\omega))) \right] \, d\,\mathbb{P}_2(\omega) \\
&+ \int_{\Omega_2} \mathbb{E}_\xi \left[ F(\Phi(t, \xi, X(\omega))) - F(\Phi_N(t, \xi, X(\omega))) \right] \, d\,\mathbb{P}_2(x).
\end{aligned}$$

The first term is equal to $\int_E F(x) \, d\mu_N(x)$ due to the invariance of $\mu_N$ and $\mathrm{Law}(X_N) = \mu_N$. Since $\mu_N$ converges weakly to $\mu$ and $F$ is also continuous and bounded with respect to $\tilde{d}$, the first term converges to $\int_E F(x) \, d\mu(x)$. It therefore suffices to show that the other terms vanish as $N \to \infty$.

The third term vanishes directly due to Lipschitz continuity of $F$ and assumption (H2). For the second term we use (H1) to estimate

$$|F(\Phi(t, \xi, X(\omega))) - F(\Phi_N(t, \xi, X_N(\omega)))| \le b \wedge L' L(t, \xi, X(\omega), X_N(\omega)) d(X(\omega), X_N(\omega)).$$

For a given $t$, $\xi$ and $\omega$ this almost surely converges to 0 as $N$ goes to infinity because $L(t, \xi, X(\omega), X_N(\omega))$ is bounded on the bounded set $\{(X(\omega), X_N(\omega)) : N \in \mathbb{N}\} \subset E^2$ and $d(X_N, X) \longrightarrow 0$ almost surely. Therefore the integral also vanishes by dominated convergence. □

### 4.3 Measures on $\mathcal{X}^{\alpha-1}$

Let us now define and analyze a number of measures on the space of initial data that we will use in the subsequent section. In Section 2.1 we have previously considered random initial data of the form

$$\mathbf{w}_0 = \sum_{n=1}^{\infty} \begin{pmatrix} a_n X_n e_n \\ b_n X_{t,n} e_n \end{pmatrix}.$$

Here $a_n, b_n \in \mathbb{R}$ and $\{X_n\}_{n\in\mathbb{N}}$, $\{X_{t,n}\}_{n\in\mathbb{N}}$ are families of independent standard normal random variables. We are interested in the case where $a_n = \langle\lambda_n\rangle^{-1}$ and $b_n = 1$. Corollary 2.5 tells us that then

$$\mathbb{E}\left[\|\mathbf{w}_0\|_{\mathcal{H}^\beta}\right] < \infty \iff \mathbf{w}_0 \in \mathcal{H}^\beta \text{ a.s.} \quad \iff \beta < \frac{1}{2}.$$

We can therefore consider

$$\mu := \mathrm{Law}(\mathbf{w}_0)$$

as a measure on $\mathcal{H}^0_{\mathrm{r}}$. It is in fact a Gaussian measure in the sense of Gaussian measures on infinite dimensional Hilbert spaces, but we do not need to use this and hence do not define it.

The choice of $\mathcal{H}^0_{\mathrm{r}}$ is rather arbitrary here as $\mu$ and the following measures are in fact concentrated on and have finite expectation in the smaller space $\mathcal{X}^{\alpha-1}$. Once we have shown this fact we can immediately switch to considering them as measure on the space of initial data $\mathcal{X}^{\alpha-1}$. Due to *(iv)* from Lemma 3.5, the Borel $\sigma$-algebras are, in a sense, identical.

Since our plan is to work with finite-dimensional approximations, we define the measures

$$\mu_{N,M} = \mathrm{Law}\left(\sum_{n=N+1}^{M} \begin{pmatrix} \langle\lambda_n\rangle^{-1} X_n e_n \\ X_{t,n} e_n \end{pmatrix}\right)$$

for $M > N$. We also define this for $M = \infty$ so that we can, for example, write $\mu = \mu_{0,\infty}$. Unless explicitly stated however, we always assume $M < \infty$. These measures are Gaussian measures on $\mathbb{R}^{M-N}$ and hence

$$\mu_{N,M} = \mathcal{F}^{-1}_{\#}\left(\frac{1}{\Gamma_{N,M}} \exp\left(-\frac{1}{2}\sum_{n=N+1}^{M} \left(\langle\lambda_n\rangle^2 y_n^2 + y_{t,n}^2\right)\right) dy\right),$$

where

$$\Gamma_{N,M} = \int_{\mathbb{R}^{M-N}} \exp\left(-\frac{1}{2}\sum_{n=N+1}^{M} \left(\langle\lambda_n\rangle^2 y_n^2 + |y_{t,n}|^2\right)\right) dy$$

and

$$\begin{aligned} \mathcal{F}^{-1} : \ell^2(\mathbb{N}) &\longrightarrow \mathcal{H}^0_{\mathrm{r}} \\ (x_n)_{n\in\mathbb{N}} &\longmapsto \sum_{n=1}^{\infty} x_n e_n. \end{aligned}$$

Here $\mathcal{F}$ is the analogue of the Fourier transform for our radial and 2-dimensional situation, defined in section 1. Note that we use the indices $y = (y_{N+1}, ..., y_M)$ to make the notation more adaptable to changes of $M$ and $N$. For example, we want to make it clear to the reader that

$$\mathcal{F}^{-1}(y) = \sum_{n=N+1}^{M} y_n e_n$$

is the correct interpretation. We see that $\mu_{N,M}$ is a measure concentrated on the smaller subspace $(P_M - P_N)\mathcal{H}^0_{\mathrm{r}}$.

We can heuristically write

$$\mu = \mathcal{F}^{-1}_{\#}\left(\frac{1}{\Gamma}\exp\left(-\frac{1}{2}\sum_{n=1}^{\infty}\left(\langle\lambda_n\rangle^2 y_n^2 + y_{t,n}^2\right)\right) dy\right),$$

where

$$\Gamma = \int_{\mathbb{R}^\infty}\exp\left(-\frac{1}{2}\sum_{n=1}^{\infty}\left(\langle\lambda_n\rangle^2 y_n^2 + |y_{t,n}|^2\right)\right) dy.$$

Of course this is not rigorous as there exists no Lebesgue measure on $\mathbb{R}^\infty$.

Our first goal is to show that $\mu$ and $\mu_{N,M}$ are concentrated on and have finite expectation in $\mathcal{X}^{\alpha-1}$. Recall that the latter does not trivially imply the former due to the subtlety in the difference between the spaces $Z$ and $\mathcal{Z}$. The case of $\mu_{N,M}$ is rather trivial as

$$\int_{\mathcal{H}^0}\|\mathbf{x}\|_{\mathcal{X}^{\alpha-1}}\, d\mu_{N,M}(\mathbf{x}) = \int_{\mathbb{R}^{M-N}}\left\|\sum_{n=N+1}^{M} y_n e_n\right\|_{\mathcal{X}^{\alpha-1}} \frac{1}{\Gamma_{N,M}} e^{-\frac{1}{2}\sum_{n=N+1}^{M}\left(\langle\lambda_n\rangle^2 y_n^2 + y_{t,n}^2\right)}\, dy < \infty.$$

The finiteness just follows from the fact that $e_n \in \mathcal{H}^1_{\mathrm{r}} \subset \mathcal{X}^{\alpha-1}$ for all $n \in \mathbb{N}$. Since $\mu_{N,M}$ is the law of a random variable which a.s. has values in $\mathcal{X}^{\alpha-1}$, it is concentrated on this set.

The case of $\mu$ is not as easy. We will show the following lemma which establishes that in fact $\mu$ is the limit of $\mu_{N,M}$ as $M$ goes to infinity with respect to the 6-Wasserstein metric $W_6^{\mathcal{X}^{\alpha-1}}$. This then implies that in particular $\mu(\mathcal{X}^{\alpha-1}) = 1$ and $\mathbb{E}\left[\|\mathbf{w}_0\|^6_{\mathcal{X}^{\alpha-1}}\right] < \infty$.

**Lemma 4.4.** *For all $N \in \mathbb{N}$ we have $(1 - P_N)\mathbf{w}_0 \in \mathcal{X}^{\alpha-1}$ almost surely and*

$$\mathbb{E}\left[\|(1 - P_N)\mathbf{w}_0\|^6_{\mathcal{X}^{\alpha-1}}\right] < \infty.$$

*Furthermore*

$$W_6^{\mathcal{X}^{\alpha-1}}(\mu_{N,M}, \mu_{N,\infty}) \xrightarrow{M\to\infty} 0.$$

*Proof.* Let $N < M_1 < M_2 \in \mathbb{N}$. For $M \in \mathbb{N}$, consider the random variables

$$\mathbf{w}^M := \sum_{n=1}^{M}\big(X_n e_n, X_{t,n} e_n\big).$$

Since the variances of $X_n$ and $X_{t,n}$ have appropriate bounds we can apply Theorem 3.8 and for some constant $C > 0$, independent of $N$, $M_1$ and $M_2$, get a certain estimate. Together with definition (4.1) we then have

$$W_6^{\mathcal{X}^{\alpha-1}}(\mu_{N,M_1}, \mu_{N,M_2}) \leq \mathbb{E}\left[\left\|(1 - P_N)(\mathbf{w}^{M_1} - \mathbf{w}^{M_2})\right\|^6_{\mathcal{X}^{\alpha-1}}\right]$$

$$= \mathbb{E}\left[\left\|\sum_{n=M_1+1}^{M_2}\left(X_n e_n, X_{t,n} e_n\right)\right\|_{\mathcal{X}^{\alpha-1}}^6\right] \leq C\left(\frac{(1+\ln(M_1))}{{M_1}^{8-6\alpha}}\right)^{\frac{6}{7}} \xrightarrow{M_1\to\infty} 0.$$

Therefore $\mu_{N,M}$ is a Cauchy sequence in $M$ with respect to $W_6^{\mathcal{X}^{\alpha-1}}$, and hence it converges to some measure $\tilde{\mu}_{N,\infty}$ with finite sixth moment. The estimate above also shows that $\mathbf{w}_{N,M} \longrightarrow (1-P_N)\mathbf{w}_0$ in $L^6(\Omega, \mathcal{X}^{\alpha-1})$, where $\Omega$ is our arbitrarily chosen probability space. Therefore there exists a subsequence $(1-P_N)\mathbf{w}^{M_k}$ which almost surely converges to $(1-P_N)\mathbf{w}_0$ in $\mathcal{X}^{\alpha-1}$. As convergence in the $p$-Wasserstein metric implies weak convergence and almost sure convergence implies weak convergence, by the uniqueness of weak limits we know that $\tilde{\mu}_{N,\infty} = \mathrm{Law}((1-P_N)\mathbf{w}_0) = \mu_{N,\infty}$. □

There are some more measures we need to consider. Given that we have defined a number of measures $\mu_{N,M}$ and $\mu$ on $\mathcal{X}^{\alpha-1}$, we can now define new measures on this space which are absolutely continuous with respect to $\mu$ or $\mu_{N,M}$.

We define the measure

$$\rho := \frac{1}{\tilde{\Gamma}} e^{-\frac{1}{4}\|u\|_{L^4}^4}\, d\mu(\mathbf{u})$$

with

$$\tilde{\Gamma} := \int_{\mathcal{X}^{\alpha-1}} e^{-\frac{1}{4}\|u\|_{L^4}^4}\, d\mu(\mathbf{u}).$$

For this measure to be well-defined we need that the density is $\mu$-a.s. finite. This is the case since $\mathbb{E}\left[\|w_0\|_{L^4}^4\right] < \infty$ by lemma 2.13 applied to $w_0$, which has $\mathrm{Law}(\mathbf{w}_0) = \mu$. Similarly for $N < M$ with $M \in \mathbb{N}\cup\{\infty\}$ we define

$$\rho_{N,M} := \frac{1}{\tilde{\Gamma}_{N,M}} e^{-\frac{1}{4}\|P_N u\|_{L^4}^4}\, d\mu_{0,M}(\mathbf{u}),$$

with

$$\tilde{\Gamma}_{N,M} := \int_{\mathcal{X}^{\alpha-1}} e^{-\frac{1}{4}\|P_N u\|_{L^4}^4}\, d\mu_{0,M}(\mathbf{u}).$$

Note that just like for $\mu_{N,M}$ if $M < \infty$ we can write it as an absolutely continuous measure with respect to the $M$-dimensional Lebesgue measure:

$$\rho_{N,M} = \mathcal{F}_{\#}^{-1}\left(\frac{1}{\tilde{\Gamma}_{N,M}} \exp\left(-\frac{1}{4}\left\|\sum_{n=1}^{N} y_n e_n\right\|_{L^4}^4 - \frac{1}{2}\sum_{n=1}^{M}\left(\langle\lambda_n\rangle^2 y_n^2 + y_{t,n}^2\right)\right) dy\right).$$

We also define

$$\rho_N := \rho_{N,\infty} \qquad \text{and } \tilde{\Gamma}_N := \tilde{\Gamma}_{N,\infty}.$$

Note that for $F \in C_b(\mathcal{X}^{\alpha-1})$ we have

$$\begin{aligned}\int_{\mathcal{X}^{\alpha-1}} F(\mathbf{u})\, d\rho_N(\mathbf{u}) &= \mathbb{E}\left[F(\mathbf{w}_0) e^{-\frac{1}{4}\|P_N w_0\|_{L^4}^4}\right] \\ &= \mathbb{E}\left[F(P_N\mathbf{w}_0 + (1-P_N)\mathbf{w}_0) e^{-\frac{1}{4}\|P_N w_0\|_{L^4}^4}\right] \\ &= \int_{P_N\mathcal{X}^{\alpha-1}} \int_{(1-P_N)\mathcal{X}^{\alpha-1}} F(\mathbf{z}+\mathbf{y})\, d\rho_{N,N}(\mathbf{z})\, d\mu_{N,\infty}(\mathbf{y}),\end{aligned}$$

so

$$\rho_N = \rho_{N,N} \times \mu_{N,\infty}.$$

We now show that also $\rho_N$ converges to $\rho$ in a Wasserstein distance as $N$ goes to infinity.

Define $\tilde{d}(\mathbf{w}_1, \mathbf{w}_2) := \|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathcal{X}^{\alpha-1}} \wedge 1$ for $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{X}^{\alpha-1}$. This metric is equivalent to the normal one on $\mathcal{X}^{\alpha-1}$.

**Lemma 4.5.** *We have*

$$W_1^{(\mathcal{X}^{\alpha-1}, \tilde{d})}(\rho_N, \rho) \xrightarrow{N\to\infty} 0.$$

*Proof.* Here we use the dual formulation of the 1-Wasserstein distance given by (4.2). Let $F : (\mathcal{X}^{\alpha-1}, \tilde{d}) \longrightarrow \mathbb{R}$ have Lipschitz constant less or equal to 1. Since $\tilde{d} \leq 1$ this implies $|F| \leq 1$. Then

$$\begin{aligned}\int_{\mathcal{X}^{\alpha-1}} F(\mathbf{u})(\, d\rho_N - \, d\rho)(\mathbf{u}) &= \int_{\mathcal{X}^{\alpha-1}} F(\mathbf{u}) \left( \frac{e^{-\frac{1}{4}\|P_N u\|_{L^4}^4}}{\tilde{\Gamma}_{N,\infty}} - \frac{e^{-\frac{1}{4}\|u\|_{L^4}^4}}{\tilde{\Gamma}} \right) d\mu(\mathbf{u}) \\ &\leq \mathbb{E}\left[ \left| \frac{e^{-\frac{1}{4}\|P_N w_0\|_{L^4}^4}}{\tilde{\Gamma}_{N,\infty}} - \frac{e^{-\frac{1}{4}\|w_0\|_{L^4}^4}}{\tilde{\Gamma}} \right| \right].\end{aligned}$$

We want to show that this vanishes as $N \longrightarrow \infty$. Since $\tilde{\Gamma}_{N,\infty} \longrightarrow \tilde{\Gamma} > 0$, the following computation, using $e^{-y}|1 - e^{-x}| \leq x$ for $x, y \geq 0$ and Lemma 2.13, suffices:

$$\begin{aligned}\mathbb{E}\left[ e^{-\frac{1}{4}\|P_N w_0\|_{L^4}^4} \left( 1 - e^{-\frac{1}{4}\left(\|w_0\|_{L^4}^4 - \|P_N w_0\|_{L^4}^4\right)} \right) \right] &\lesssim \mathbb{E}\left[ \|w_0\|_{L^4}^4 - \|P_N w_0\|_{L^4}^4 \right] \\ \overset{A.5}{\lesssim} \mathbb{E}\left[ \|(1 - P_N) w_0\|_{L^4}^4 (\|w_0\|_{L^4}^4 + \|P_N w_0\|_{L^4}^4) \right] & \\ \leq \mathbb{E}\left[ \|(1 - P_N) w_0\|_{L^4}^8 \right] \left( \mathbb{E}\left[ \|w_0\|_{L^4}^8 \right] + \mathbb{E}\left[ \|P_N w_0\|_{L^4}^8 \right] \right) & \\ \lesssim \mathbb{E}\left[ \|(1 - P_N) w_0\|_{L^8}^8 \right] \left( \mathbb{E}\left[ \|w_0\|_{L^8}^8 \right] + \mathbb{E}\left[ \|P_N w_0\|_{L^8}^8 \right] \right) & \\ \overset{lem.2.13}{\lesssim} \frac{1 + \ln(N)}{N^2} \xrightarrow{N\to\infty} 0. &\end{aligned}$$

□

## 4.4 Invariance of $\rho$ under $\Phi$

**Lemma 4.6.** *The following hold:*

*(i) The flows $\Phi$ and $\Phi_N$ fulfill the hypotheses (H1), (H2) and (H3) for $\rho$.*

*(ii) Fix $N \in \mathbb{N}$ and consider for $M > N$ the sequence of functions $(\tilde{\Phi}_{N,M})_{M\in\mathbb{N}} := (P_M - P_N)\Phi_N$ as well as their limit $\tilde{\Phi}_N := (1 - P_N)\Phi_N$ as flows on $(1 - P_N)\mathcal{X}^{\alpha-1}$. These fulfill the hypotheses and (H1), (H2) and (H3) for $\mu_{N,\infty}$.*

*Proof.* (i) We have shown (H3) in Lemma 4.5. We can get (H1) as a slight improvement over the continuity results in Theorem 2.22. First of all, observe that

$$\|\Phi_N(t, \xi, \mathbf{w}_1) - \Phi_N(t, \xi, \mathbf{w}_2)\|_{\mathcal{X}^{\alpha-1}} \leq \|S(t)(\mathbf{w}_1 - \mathbf{w}_2)\|_{\mathcal{X}^{\alpha-1}} + \|\mathbf{v}_{N,1}(t) - \mathbf{v}_{N,2}(t)\|_{\mathcal{H}^{\alpha}}.$$

By the Lipschitz continuity in Lemma 3.7 the first term easily obeys the desired estimate. For the second term we have to look into the proof of 2.22. Note that first of all (2.29) holds for $\mathbf{v}_{N,1}$ and $\mathbf{v}_{N,2}$ as we wrote it there for $\mathbf{v}_1$ and $\mathbf{v}_2$, with no constant depending on $N$ being necessary. We therefore have the analogue of (2.30), which is almost the desired inequality for a suitable choice of $L$. We know that $\|\psi\|_{Y^{\alpha-1}_{[0,t]}}$ is almost surely continuous in $t$, and it trivially holds that $\|w_i\|_{Y^{\alpha-1}_{[0,t]}} \leq \|\mathbf{w}_i\|_{\mathcal{X}^{\alpha-1}}$ for $i = 1, 2$. We also have to estimate $\|\mathbf{v}_{N,i}\|_{\mathcal{H}^\alpha}$ by $\|\mathbf{w}_i\|_{\mathcal{X}^{\alpha-1}}$, which is possible due to the global energy estimate and preservation of regularity (Theorem 3.11 and Lemma 2.25). Then we get (2.30) with some factor that depends continuously on $\mathbf{w}_1$ and $\mathbf{w}_2$. In particular this factor locally bounded with respect to these variables.

Assumption (H2) is a direct consequence of Lemma 2.24 and Lipschitz continuity of $F$, which establish convergence to 0 for all $t$ and almost all $(\xi, x)$. Then due to boundedness of $F$ dominated convergence concludes the argument.

(ii) We have shown (H3) in Lemma 4.4. Assumption (H1) for $(\tilde{\Phi}_{N,M})_M$ follows from the same result for $(\Phi_N)_N$ which we have shown in the stronger norm of $\mathcal{H}^\alpha_{\mathrm{r}}$. We simply estimate

$$\|(P_M - P_N)(\Phi_N(t,\xi,\mathbf{w}_1) - \Phi_N(t,\xi,\mathbf{w}_2))\|_{\mathcal{X}^{\alpha-1}} \leq \|\Phi_N(t,\xi,\mathbf{w}_1) - \Phi_N(t,\xi,\mathbf{w}_2)\|_{\mathcal{H}^\alpha}.$$

Assumption (H2) is also straightforward: Given $t$, $\xi$ and $\mathbf{w} \in \mathcal{X}^{\alpha-1}$, we compute

$$\begin{aligned}\|(1-P_N)\Phi_N(t,\xi,\mathbf{w}) - (P_M - P_N)\Phi_N(t,\xi,\mathbf{w})\|_{\mathcal{X}^{\alpha-1}} &= \|(1-P_M)\Phi_N(t,\xi,\mathbf{w})\|_{\mathcal{X}^{\alpha-1}},\\ &\leq \|(1-P_M)S(t)\mathbf{w}\|_{\mathcal{X}^{\alpha-1}} + \|(1-P_M)\Psi(t,\xi)\|_{\mathcal{X}^{\alpha-1}} + \|(1-P_M)\mathbf{v}_N\|_{\mathcal{X}^{\alpha-1}}.\end{aligned}$$

We have to estimate this under the integral against $\mu_{N,\infty}$ and the expectation $\mathbb{E}_\xi$. Since $P_N\mathbf{v}_N = \mathbf{v}_N$ the third term vanishes as soon as $M > N$. For the first term note that $\mu_{N,\infty} = \mathrm{Law}((1-P_N)\mathbf{w}_0)$ and that $(1-P_N)S(t)\mathbf{w}_0$ fulfills the variance estimates needed to apply Theorem 3.8. We get

$$\begin{aligned}\mathbb{E}_{\mathbf{w}_0}\left[\|(1-P_M)(1-P_N)S(t)\mathbf{w}\|_{\mathcal{X}^{\alpha-1}}\right] &= \mathbb{E}_{\mathbf{w}_0}\left[\|(1-P_M)S(t)\mathbf{w}\|_{\mathcal{X}^{\alpha-1}}\right]\\ &\leq C\left(\frac{(1+\ln(M))}{M^{8-6\alpha}}\right)^{\frac{6}{7}} \xrightarrow{M\to\infty} 0.\end{aligned}$$

For the second term we also apply Lemma 3.8 and get

$$\mathbb{E}_\xi\left[\|(1-P_M)\Psi(t,\xi)\|_{\mathcal{X}^{\alpha-1}}\right] \leq C\left(\frac{(1+\ln(M))}{M^{8-6\alpha}}\right)^{\frac{6}{7}} \xrightarrow{M\to\infty} 0.$$

□

**Theorem 4.7** (Invariant Measures)**.** *For $N \in \mathbb{N}$ and $M > N$ the following hold:*

*(i) $\rho_{N,N}$ is invariant under the flow $P_N\Phi_N$ on the space $P_N\mathcal{X}^{\alpha-1}$.*

*(ii) $\mu_{N,M}$ is invariant under the flow $(P_M - P_N)\Phi_N$ on the space $(P_M - P_N)\mathcal{X}^{\alpha-1}$.*

*(iii) $\mu_{N,\infty}$ is invariant under the flow $(1-P_N)\Phi_N$ on the space $(1-P_N)\mathcal{X}^{\alpha-1}$.*

*(iv)* $\rho_N$ *is invariant under the flow* $\Phi_N$ *on the space* $\mathcal{X}^{\alpha-1}$*.*

*(v)* $\rho$ *is invariant under the flow* $\Phi$ *on the space* $\mathcal{X}^{\alpha-1}$*.*

*Proof.* (i) We fix some $N \in \mathbb{N}$ and do not always mention it in the notation. Define $\mathbf{z} \coloneqq P_N \mathbf{u}_N$. This solves the equation

$$\begin{aligned} z_{tt} + z_t + (1-\Delta)z + P_N(z^3) &= \sqrt{2} P_N \xi \\ (z, z_t)(0) &= P_N \mathbf{w}_0 \end{aligned}$$

or equivalently

$$\partial_t \mathbf{z} = L\mathbf{z} - \begin{pmatrix} 0 \\ P_N(z^3) \end{pmatrix} + \begin{pmatrix} 0 \\ \sqrt{2} P_N \xi \end{pmatrix}.$$

in the sense of distributions. We can write this as a system of distributional equations in time: for some $n \le N$ let $f \in \mathcal{D}(\mathbb{R}_+)$ and test the equation with $f(t)e_n(x)$ in each component. This yields

$$\partial_t \widehat{\mathbf{z}}(n) = \begin{pmatrix} 0 & 1 \\ -\langle \lambda_n \rangle^2 & -1 \end{pmatrix} \widehat{\mathbf{z}}(n) - \begin{pmatrix} 0 \\ \sum_{i,j,k=1}^N \widehat{z}(i)\widehat{z}(j)\widehat{z}(k)\langle e_i e_j e_k, e_n \rangle \end{pmatrix} + \begin{pmatrix} 0 \\ \sqrt{2}\langle \xi, e_n \rangle_x \end{pmatrix}, \tag{4.5}$$

which is to be understood as an equation of time distributions in $\mathcal{D}'(\mathbb{R}_+)$.

Recall that we have seen in Lemma 1.3 that

$$\langle \langle \xi, e_n \rangle_x, f \rangle_t = \int_0^\infty f(t)\, dW_n(t).$$

for some collection of independent Brownian motions $W_n$. We now test (4.5) with $f = \mathbb{1}_{[0,t]}$ and get

$$\begin{aligned} &\begin{pmatrix} \widehat{z}(n,t) \\ \widehat{z}_t(n,t) \end{pmatrix} - \begin{pmatrix} \widehat{z}(n,0) \\ \widehat{z}_t(n,0) \end{pmatrix} \\ &= \begin{pmatrix} \widehat{z}_t(n,s) \\ -\langle \lambda_n \rangle^2 \widehat{z}(n,s) - \widehat{z}_t(n,s) - \sum_{i,j,k=1}^N \widehat{z}(i,s)\widehat{z}(j,s)\widehat{z}(k,s)\langle e_i e_j e_k, e_n \rangle \end{pmatrix} dt \\ &+ \begin{pmatrix} 0 \\ \int_0^t \sqrt{2}\, dW_n(t) \end{pmatrix}. \end{aligned}$$

We write this as a stochastic differential equation

$$d\,\mathbf{Z}(t) = b(\mathbf{Z}(t))\, dt + \sigma\, d\mathbf{W}(t) \tag{4.6}$$

where $\mathbf{Z}(t) = (\widehat{\pi_1 \mathbf{z}}(n,t))_{n=1,\dots,N}, \widehat{\pi_2 \mathbf{z}}(n,t))_{n=1,\dots,N}) \in \mathbb{R}^{2N}$, the function $b : \mathbb{R}^{2N} \longrightarrow \mathbb{R}^{2N}$ is defined by

$$b(z_1, \dots, z_N, z_{t,1}, \dots, z_{t,N}) \coloneqq \begin{pmatrix} (z_{t,n})_{n=1,\dots,N} \\ \big( -\langle \lambda_n \rangle^2 z_n - z_{t,n} - \sum_{i,j,k=1}^N z_i z_j z_k \langle e_i e_j e_k, e_n \rangle \big)_{n=1,\dots,N} \end{pmatrix},$$

$\sigma \in \mathbb{R}^{2N\times 2N}$ is the matrix

$$\sigma = \begin{pmatrix} 0 & & \dots & \dots & & 0 \\ & \ddots & & & & \\ \vdots & & 0 & & & \vdots \\ \vdots & & & \sqrt{2} & & 0 \\ & & & & \ddots & \vdots \\ 0 & & \dots & 0 & \dots & \sqrt{2} \end{pmatrix},$$

and $\mathbf{W} = (V, W)$ where $V$ is another $N$-dimensional Brownian motion independent of $W$.

The Fokker-Planck equations (A.3) state that the law of $\mathbf{Z}(t)$ has a probability density $p(t) : \mathbb{R}^{2N} \longrightarrow [0, \infty)$ which weakly solves

$$\begin{aligned} \partial_t p(t, z, z_t) &= \sum_{n=1}^{N} -\partial_{z_n}(b(z)_n p(t, z, z_t)) - \partial_{z_{t,n}}(b(z)_{N+n} p(t, z, z_t)) \\ &+ \sum_{n=1}^{N}\sum_{m=1}^{N} \big(\partial_{z_n}\partial_{z_m} D_{n,m} + \partial_{z_{t,n}}\partial_{z_m} D_{N+n,m} \\ &\qquad + \partial_{z_n}\partial_{z_{t,m}} D_{n,N+m} + \partial_{z_{t,n}}\partial_{z_{t,m}} D_{N+n,N+m}\big) p(t, z, z_t) \\ &= -Np(t, z, z_t) - b(z)\nabla_{(z,z_t)} p(t, z, z_t) + \Delta_{z_t} p(t, z, z_t), \end{aligned} \tag{4.7}$$

where

$$D_{i,j} = \frac{1}{2}\sum_{k=1}^{2N} \sigma_{i,k}\sigma_{j,k} = \begin{cases} 1, & i, j \geq N+1 \\ 0, & \text{else} \end{cases}.$$

We invite the reader to read up on its derivation in the Appendix if this is a surprise. It is a rather direct consequence of Ito's formula.

To find an invariant measure, we hypothesize that the energy

$$E(\mathbf{z}) = \int_B \frac{1}{2}|z|^2 + \frac{1}{2}|\nabla z|^2 + \frac{1}{2}|z_t|^2 + \frac{1}{4}|z|^4\,dx = \int_B \frac{1}{2}|\langle\nabla\rangle z|^2 + \frac{1}{2}|z_t|^2 + \frac{1}{4}|z|^4\,dx$$

is in fact conserved. Then the measure

$$\mathcal{F}_{\#}\rho_N = e^{-E(\mathcal{F}^{-1}(\mathbf{z}))}\,dz\,dz_t = \tilde{\Gamma}_N^{-1} e^{-\frac{1}{4}\sum_{n=1}^{\infty}\left\langle\left(\sum_{j=1}^{N} z_j e_j\right)^2, e_n\right\rangle^2 - \frac{1}{2}\sum_{n=1}^{N}\langle\lambda_n\rangle^2 z_n^2 - \frac{1}{2}|z_t|^2}\,dz\,dz_t,$$

should, heuristically, not change along trajectories generated by the flow $P_N\Phi_N$, so we have some hope that the measure is invariant under the evolution of $\mathbf{z}$ (this is nevertheless only an intuition). Note that

$$\sum_{n=1}^{\infty}\left\langle\left(\sum_{j=1}^{N} z_j e_j\right)^2, e_n\right\rangle^2 = \left\|\sum_{n=1}^{N} z_n e_n\right\|_{L^4}^4.$$

The measure $\mathcal{F}_{\#}\rho_N$ is a measure on $\mathbb{R}^{2N}$, so we are in a finite-dimensional setting and can check the invariance by a calculation with the Fokker-Planck equation. Let $p_0$ be the density of $\mathcal{F}_{\#}\rho_N = \mathrm{Law}(\widehat{\mathbf{z}}(0))$. Now observe the following computations which we can perform in the sense of distributions:

$$\Delta_{z_t} p_0(z, z_t) = (|z_t|^2 + N) p_0(z, z_t)$$

and

$$\begin{aligned}
& b(z)\nabla_{(z,z_t)} p_0(z, z_t) \\
&= p_0(z, z_t) \left( \begin{matrix} (z_{t,n})_{n=1,\dots,N} \\ \left( -\langle\lambda_n\rangle^2 z_n - z_{n,t} - \left\langle \left(\sum_{j=1}^{\infty} z_j e_j\right)^3 e_k, e_n \right\rangle \right)_{n=1,\dots,N} \end{matrix} \right) \\
&\times \left( \begin{matrix} -\langle\lambda_k\rangle^2 z_k - \left( \sum_{n=1}^{\infty} \left\langle \left( \sum_{j=1}^{N} z_j e_j \right)^2, e_n \right\rangle \left\langle \left( \sum_{j=1}^{N} z_j e_j \right) e_k, e_n \right\rangle \right)_{k=1,\dots,N} \\ -z_t \end{matrix} \right) \\
&= p_0(z, z_t) \left( z_t \cdot (\langle\lambda_n\rangle z_n)_{n=1,\dots,N} - \left\langle \left( \sum_{j=1}^{N} z_j e_j \right)^2, \left( \sum_{j=1}^{N} z_j e_j \right) \left( \sum_{k=1}^{N} z_{t,k} e_k \right) \right\rangle_x \right. \\
&\left. -z_t \cdot (\langle\lambda_n\rangle z_n)_{n=1,\dots,N} + |z_t|^2 + \left\langle \left( \sum_{j=1}^{N} z_j e_j \right)^3, \left( \sum_{k=1}^{N} z_{t,k} e_k \right) \right\rangle_x \right) \\
&= |z_t|^2 p_0(z, z_t).
\end{aligned}$$

We see that $\mathcal{F}_{\#}\rho_N$ is a weak solution to the stationary Fokker-Planck equation, i.e. the right hand side of (4.7) is zero. If we know that weak solutions to the Fokker-Planck equation of a class of measures that **both** $\mathrm{Law}(Z_t)$ and $\mathcal{F}_{\#}\rho_N$ belong to **are unique**, then it follows that $\mathcal{F}_{\#}\rho_N$ is indeed an invariant measure under the flow of $\mathbf{Z}$, that is

$$\mathrm{Law}(\mathbf{Z}(t)) = \mathrm{Law}(\mathbf{Z}(0)) = \mathcal{F}_{\#}\rho_N.$$

It then follows directly that $\rho_N$ is invariant under the flow of $\mathbf{z}$, that is $P_N \Phi_N$.

Now it only remains to show this uniqueness of weak solutions to the Fokker-Planck equation given by measures. We use Lemma A.4. What we have to check is that for all $1 \le i \le 2N$,

$$\int_0^T \int_{\mathbb{R}^{2N}} \frac{1}{1+|x|^2} + \frac{|b^i(x)|}{1+|x|} \, d\,\mathrm{Law}(Z_t)\, dt < \infty.$$

As $|b^i(x)| \le |x|$ for $i \le N$ it suffices to look at the cases $N+1 \le i \le 2N$. Here $|b^i(x)| \lesssim 1 + |x|^3$ and we use that $\int_0^T \mathbb{E}\left[\|\mathbf{Z}_t\|_2^3\right] dt < \infty$, as each $Z_t$ is a Gaussian random variable and their variances are locally bounded in $t$.

(ii) We now deal with the other part of the equation for $\mathbf{u}_N$. Define $\mathbf{y} = (1 - P_N)\mathbf{u}_N$ so that $\mathbf{u}_N = \mathbf{z} + \mathbf{y}$, and define $\mathbf{y}^M = P_M \mathbf{y}$ for $M > N$. This solves

$$\begin{aligned}
y_{tt}^M + y_t^M + (1-\Delta) y^M &= \sqrt{2}(P_M - P_N)\xi \\
(y^M, y_t^M)(0) &= (P_M - P_N)\mathbf{w}_0,
\end{aligned}$$

which again corresponds to a a finite ($(M-N)$ -) dimensional SDE for $\widehat{\mathbf{y}}$. We will now sometimes write $\mathbf{y} \in \mathbb{R}^{2(M-N)}$ and use the notation

$$\mathbf{y} = (y, y_t) = (y_{N+1}, ..., y_M, y_{t,N+1}, ..., y_{t,M}).$$

We can write the SDE as

$$d\,\mathbf{Y}^M(t) = b(\mathbf{Y}^M(t))\,dt + \sigma\,d\mathbf{W}^M(t), \tag{4.8}$$

where

$$b(y_{N+1}, ..., y_M, y_{t,N+1}, ..., y_{t,M}) = \begin{pmatrix} (y_{t,n})_{n=N+1,...,M} \\ (-\langle\lambda_n\rangle^2 y_n - y_{n,t})_{n=N+1,...,M} \end{pmatrix}$$

and $\sigma \in \mathbb{R}^{2(M-N),2(M-N)}$ is defined analogously to the previous case. The Brownian motion is given as $\mathbf{W}^M = (V, W)$ where $W$ is a $(M-N)$ dimensional Brownian motion which is a modification of $\langle \xi, \mathbb{1}_{[0,t]} e_k \rangle$ with $k \in \{N+1, ..., M\}$, and $V$ is any $M-N$ dimensional Brownian motion independent of $W$. Note that the Brownian motions in (4.6) and (4.8) are independent.

Now we can again use the Fokker-Planck equations to verify with an analogous and simpler calculation to the one in $(i)$ that

$$\mathcal{F}_{\#}\mu_{N,M} = \Gamma_{N,M}^{-1} e^{-\frac{1}{2}\sum_{n=N+1}^{M}\langle\lambda_n\rangle^2 y_n^2 - \frac{1}{2}|y_t|^2}\,dy\,dy_t.$$

is an invariant measure on $\mathbb{R}^{2(M-N)}$ under the flow of $\mathbf{Y}^M$, that is $\mathcal{F}(P_M - P_N)\Phi_N$. Then it follows that $\mu_{N,M}$ is invariant under the flow of $\mathbf{y}$, that is $(P_M - P_N)\Phi_N$.

(iii) This is now simply an application of Lemma 4.3. We have shown in Lemma 4.6 that the necessary conditions are fulfilled.

(iv) Since $\rho_N = \rho_{N,N} \times \mu_{N,\infty}$, the invariance of $\rho_N$ follows from $(i)$ and $(iii)$.

(v) This is again an application of Lemma 4.3. We have shown in Lemma 4.6 that the necessary conditions are fulfilled.

□

## 4.5 An Outlook: The Flow as a Feller Semigroup

We can frame the flows $\Phi$ and $\Phi_N$ as Markovian semigroups in the sense of [7, Chp. 2 Prop. 2.1.1, Chp. 3].

**Definition 4.8** (Markovian Transition Function)**.** *Let $(E, d)$ be a Polish space. We say that a family of functions $(\mathcal{P}_t)_{t\in[0,\infty)}$ with*

$$\mathcal{P}_t : E \times \mathcal{B}(E) \longrightarrow \mathbb{R}$$

*is a **Markovian transition function** on $E$ if the following hold for every $t, s \geq 0$, $x \in E$ and $A \in \mathcal{B}(E)$:*

*(i) $\mathcal{P}_t(x, -)$ is a probability measure on $E$.*

*(ii) $\mathcal{P}_t(-, A)$ is a measurable function on $E$.*

*(iii)* $\mathcal{P}_0(x, A) = \mathbb{1}_A(x)$.

*(iv)* *We have Chapman-Kolmogorov equations*

$$\mathcal{P}_{t+s}(x, A) = \int_E \mathcal{P}_s(y, A)\mathcal{P}_t(x, dy).$$

In our instance we take $\mathcal{X}^{\alpha-1}$ as the Polish space and define

$$\mathcal{P}_t(\mathbf{w}_0, A) \coloneqq \left[\Phi(t, -, \mathbf{w}_0)_{\#}\, \mathbb{P}_\xi\right](A) = \mathbb{P}_\xi(\{\xi : \Phi(t, \xi, \mathbf{w}_0) \in A\}).$$

$$\mathcal{P}_{N,t}(\mathbf{w}_0, A) \coloneqq \left[\Phi_N(t, -, \mathbf{w}_0)_{\#}\, \mathbb{P}_\xi\right](A) = \mathbb{P}_\xi(\{\xi : \Phi_N(t, \xi, \mathbf{w}_0) \in A\}).$$

**Lemma 4.9.** $(\mathcal{P}_t)_{t\geq 0}$ *and* $(\mathcal{P}_{N,t})_{t\geq 0}$ *are Markovian transition functions on* $\mathcal{X}^{\alpha-1}$.

*Proof.* We only write down the proof for $(\mathcal{P}_t)_{t\geq 0}$. $(i)$ follows directly from the fact that $\mathcal{P}_t(\mathbf{w}_0, -)$ is the push-forward of a measure. For $(ii)$, write

$$\mathbb{P}_\xi(\{\xi : \Phi(t, \xi, \mathbf{w}_0) \in A\}) = \mathbb{E}_\xi\left[\mathbb{1}_{\Phi(t,\xi,\mathbf{w}_0)\in A}\right].$$

Since $\Phi(t, -, -)$ is measurable the indicator function in the expectation is measurable with respect to $\xi$ and $\mathbf{w}_0$. Then Fubini's theorem implies that the expectation with respect to $\xi$ is still measurable with respect to $\mathbf{w}_0$. Statement $(iii)$ directly follows from $(i)$ of Lemma 4.1, and lastly $(iv)$ follows from $(iii)$ of that lemma in the following fashion:

Note that given three independent copies $\xi$, $\xi_1$ and $\xi_2$ of the white noise, $(iv)$ is equivalent to the statement that for all $\mathbf{w}_0 \in \mathcal{X}^{\alpha-1}$ and $F \in C_b(\mathcal{X}^{\alpha-1})$,

$$\begin{aligned}\mathbb{E}_\xi\left[F(\Phi(t, \xi, \mathbf{w}_0))\right] &= \mathbb{E}_{\xi_2}\left[\int_{\mathcal{X}^{\alpha-1}} F(\Phi(s, \xi_2, \mathbf{v}_0)\, d\big[\Phi(t, -, \mathbf{w}_0)_{\#}\, \mathbb{P}_{\xi_1}\big](\mathbf{v}_0)\right] \\ &= \mathbb{E}_{\xi_2}\left[\mathbb{E}_{\xi_1}\left[F(\Phi(s, \xi_2, \Phi(t, \xi_1, \mathbf{w}_0))\right]\right].\end{aligned}$$

It is now evident that this is equivalent to $(iii)$ from Lemma 4.1. □

**Definition 4.10** (Markovian Feller Semigroup, [7])**.** *A markovian transition function* $(\mathcal{P}_t)_{t\geq 0}$ *on a Polish space* $(E, d)$ *is a* ***Markovian Feller semigroup*** *if it is stochastically continuous, meaning that for all* $F \in C_b(E)$ *and* $x \in E$ *we have*

$$\lim_{t\to 0} \int_E F(y)\, d\mathcal{P}_t(x,\, dy) = F(x),$$

*and furthermore for all* $F \in C_b(E)$

$$\left[x \longmapsto \int_E F(y)\, d\mathcal{P}_t(x,\, dy)\right] \in C_b(E).$$

**Lemma 4.11.** $(\mathcal{P}_t)_{t\geq 0}$ *and* $(\mathcal{P}_{N,t})_{t\geq 0}$ *are Markovian Feller semigroups on* $\mathcal{X}^{\alpha-1}$.

*Proof.* We again only write down the proof for $(\mathcal{P}_t)_{t\geq 0}$. We first show the stochastic continuity. Let $F \in C_b(\mathcal{X}^{\alpha-1})$, $\mathbf{w}_0 \in \mathcal{X}^{\alpha-1}$ and $t \geq 0$ and note that

$$\int_{\mathcal{X}^{\alpha-1}} F(\mathbf{v}_0)\, d\mathcal{P}_t(\mathbf{w}_0,\, d\mathbf{v}_0) = \mathbb{E}_\xi \left[ F(\Phi(t, \xi, \mathbf{w}_0) \right].$$

We know from Lemma 4.1 that $\Phi(t, \xi, \mathbf{w}_0)$ is almost surely continuous in $t$ and so dominated convergence implies that the limit of the above as $t \searrow 0$ is $F(\mathbf{w}_0)$.

The second property we have to show is similarly easy to prove. Clearly if $F$ is bounded then so is $\mathbb{E}_\xi \left[ F(\Phi(t, \xi, \mathbf{w}_0) \right]$. At the same time we know that $\Phi(t, \xi, \mathbf{w}_0)$ is almost surely continuous in $\mathbf{w}_0$, and so dominated convergence implies that the expectation is also continuous in $\mathbf{w}_0$. □

Having establishes this, we could now use the machinery in works such as [7] to tackle problems such as uniqueness of the invariant measure and ergodicity. One of the more difficult parts of the standard approach involves showing the strong Feller property, which states that the flow $\Phi$ is regularizing in the sense that if $F$ is a measurable and bounded function on $\mathcal{X}^{\alpha-1}$, then for all $t > 0$, $\mathbb{E}\left[\Phi(t, \xi, \mathbf{w}_0)\right]$ is a continuous and bounded function in $\mathbf{w}_0$. This straight forward approach does not work though as the Markov semigroup does in fact not have the strong Feller property, although we have not shown this. In [19] a very similar stochastic nonlienar wave equation is treated and at the beginning of section 5 an argument for the failure of the strong Feller property is provided. An alternative technique using a modified Feller property is used to show unique Ergodicity. Transferring this approach over to our equation would be a matter worth investigating.

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# A Appendix

**Lemma A.1.** *Let $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ be real random variables. Suppose that for some $\simeq \in \{\le, =, \ge\}$ we have the following "Gaussian" moment estimates: For any finite set $I \subset \mathbb{N}$ and $n, k \in \mathbb{N}^I$*

$$\mathbb{E}\left[\prod_{i\in I} X_i^{n_i} Y_i^{k_i}\right] \begin{cases} \simeq \prod_{i\in I} \frac{(n_i+k_i)!}{\left(\frac{n_i+k_i}{2}\right)!} & \text{if } \forall i \in I,\ n_i + k_i \text{ even} \\ = 0 & \text{if } \exists i \in I : n_i + k_i \text{ odd}. \end{cases}$$

*(For example this is satisfied with $\simeq$ being $\le$ if $\{(X_n, Y_n)\}_{n\in\mathbb{N}}$ is an independent family of centered normal random vectors with variances $\le 1$, see [20][Thm 3.1]). Let $a_n, b_n \ge 0$ be real sequences and define*

$$A = \sum_{n=1}^{\infty} a_n X_n \qquad \text{and} \qquad B = \sum_{k=1}^{\infty} b_k Y_k$$

*Let $p, q \in \mathbb{N}$. If $p + q$ is odd, then*

$$\mathbb{E}\left[A^p B^q\right] = 0.$$

*If $p + q = 2r$ for some $r \in \mathbb{N}$, then*

$$\mathbb{E}\left[A^p B^q\right] \simeq \frac{p!q!}{r!} \sum_{s=0\vee(p-r)}^{r\wedge\lfloor\frac{p}{2}\rfloor} \binom{r}{s, p-2s, r-p+s} 2^{p-2s} \|a\|_{\ell^2}^{2s} \langle a, b\rangle_{\ell^2}^{p-2s} \|b\|_{\ell^2}^{2(r-p+s)} \tag{A.1}$$

*Proof.* For now we let $N \in \mathbb{N}$ and consider

$$\mathbb{E}\left[\left(\sum_{n=1}^{N} a_n X_n\right)^p \left(\sum_{k=1}^{N} b_n Y_k\right)^q\right]$$

We apply the multinomial theorem:

$$= \mathbb{E}\left[\left(\sum_{n_1+\ldots+n_N=p} \binom{p}{n_1, \ldots, n_N} \prod_{i=1}^{N} (a_i X_i)^{n_i}\right) \left(\sum_{k_1+\ldots+k_N=q} \binom{q}{k_1, \ldots, k_N} \prod_{i=1}^{N} (b_i Y_i)^{k_i}\right)\right]$$

$$= \sum_{n_1+\ldots+n_N=p} \sum_{k_1+\ldots+k_N=q} \binom{p}{n_1, \ldots, n_N} \binom{q}{k_1, \ldots, k_N} \mathbb{E}\left[\prod_{i=1}^{N} a_i^{n_i} b_i^{k_i} X_i^{n_i} Y_i^{k_i}\right]. \tag{A.2}$$

We now apply our moment estimates. This makes all terms vanish where one of the $n_i + k_i$ is odd. This immediately implies that if $p + q$ is odd there are no terms remaining and the sum is zero. From now on we assume that $p + q$ is even and define $r = \frac{p+q}{2}$. So we get

$$(A.2) \simeq \sum_{\substack{n_1+\ldots+n_N=p \\ k_1+\ldots+k_N=q \\ n_i+k_i \text{ even}}} \frac{p!}{n_1! \ldots n_N!} \frac{q!}{k_1! \ldots k_N!} \prod_{i=1}^{N} \frac{(n_i+k_i)!}{\left(\frac{n_i+k_i}{2}\right)!} a_i^{n_i} b_i^{k_i}.$$

$$= \frac{p!q!}{r!} \sum_{\substack{n_1+\ldots+n_N=p \\ k_1+\ldots+k_N=q \\ n_i+k_i \text{ even} \\ m_i:=(n_i+k_i)/2}} \binom{r}{m_1,\ldots,m_N} \prod_{i=1}^{N} \frac{(n_i+k_i)!}{n_i!k_i!} a_i^{n_i} b_i^{k_i}$$

. The rest of the proof is just combinatorics. We can rewrite this in terms of indices $m$ and $n$ instead of $k$ and $n$:

$$= \frac{p!q!}{r!} \sum_{m_1+\ldots+m_N=r} \binom{r}{m_1,\ldots,m_N} \underbrace{\sum_{\substack{(k_i,n_i),1\le i\le N: \\ 2m_i=n_i+k_i \\ n_1+\ldots+n_N=p}} \prod_{i=1}^{N} \binom{2m_i}{n_i} a_i^{n_i} b_i^{k_i}}_{(\star)}.$$

We now compute $(\star)$ for a fixed decomposition $m_1 + \cdots + m_N = r$. Define

$$q_i = \frac{a_i}{a_i+b_i}.$$

With this $(\star)$ can be written as

$$= \prod_{i=1}^{N} \sum_{n_i=1}^{2m_i} \binom{2m_i}{n_i} q_i^{n_i} (1-q_i)^{2m_i-n_i} (a_i+b_i)^{2m_i} \mathbb{1}_{n_1+\ldots+n_N=p}.$$

$$= \prod_{i=1}^{N} (a_i+b_i)^{2m_i} \prod_{i=1}^{N} \sum_{n_i=1}^{2m_i} \binom{2m_i}{n_i} q_i^{n_i} (1-q_i)^{2m_i-n_i} (a_i+b_i)^{2m_i} \mathbb{1}_{n_1+\ldots+n_N=p}.$$

$$= \prod_{i=1}^{N} (a_i+b_i)^{2m_i} \, \mathbb{P}_{\tilde{\Omega}} \left( X_1 + \ldots + X_N = p \right)$$

where $X_i \sim \text{Bin}(2m_i, q_i)$ are independent random variables on some new probability space $\tilde{\Omega}$. Luckily there are closed form expressions for the distribution of such a sum of Binomial variables. To use them, we rewrite each $X_i$ as independent Bernoulli variables. Note that in the sum $X_1 + \ldots + X_N$ only at most $2m_1 + \ldots + 2m_N = 2r = p + q$ variables can be nonzero with the rest having distribution $Bin(0, a_i)$, so the sum really only has at most $2r$ terms for the indices $\{1 \le i \le N : m_i > 0\}$. Then

$$\mathbb{P}_{\tilde{\Omega}} \left( X_1 + \ldots + X_N = p \right) = \mathbb{P}_{\tilde{\Omega}} \left( \sum_{j=1}^{2r} Y_j = p \right)$$

where the $Y_j$ are independent random variables with

$$Y_j \sim \text{Ber}(\tilde{q}_j) \text{ where } \tilde{q}_j = q_i \text{ whenever } \underbrace{2m_1 + \ldots + 2m_{i-1} + 1 \le j \le 2m_1 + \ldots + 2m_i}_{\text{for some } 1\le i\le N \text{ and } 1\le j\le 2r}$$

so that

$$X_i = \sum_{j=2m_1+\ldots+2m_{i-1}+1}^{2m_1+\ldots+2m_i} Y_j.$$

The random variable $\sum_{j=1}^{2r} Y_j$ is now distributed according to a **Poisson binomial distribution**. There exists a closed form expression for its distribution using the discrete Fourier transform [9]:

$$\mathbb{P}_{\tilde{\Omega}}\left(\sum_{j=1}^{2r} Y_j = x\right) = \frac{1}{2r+1}\sum_{l=0}^{2r} e^{-lx\frac{i2\pi}{2r+1}} \prod_{j=1}^{2r}(\tilde{q}_j e^{l\frac{i2\pi}{2r+1}} + (1-\tilde{q}_j)).$$

We define $R = 2r+1$, plug in $x = p$ and rewrite the above

$$= \frac{1}{R}\sum_{l=0}^{2r} e^{-lp\frac{i2\pi}{R}} \prod_{i=1}^{N}(q_i e^{l\frac{i2\pi}{R}} + (1-q_i))^{2m_i}.$$

Let's recap what we have done so far: We have shown that

$$\mathbb{E}\left[\left(\sum_{n=1}^{N} a_n X_n\right)^p \left(\sum_{k=1}^{N} b_n Y_k\right)^q\right]$$

$$\simeq \sum_{m_1+...+m_N=r} \binom{r}{m_1,...,m_N} \frac{p!q!}{r!} \prod_{i=1}^{N}(a_i+b_i)^{2m_i}\, \mathbb{P}_{\tilde{\Omega}}\left(X_1+...+X_N = p\right)$$

$$= \frac{p!q!}{r!}\frac{1}{R}\sum_{l=0}^{2r} e^{-lp\frac{i2\pi}{R}} \sum_{m_1+...+m_N=r} \binom{r}{m_1,...,m_N} \prod_{i=1}^{N}\left((a_i+b_i)(q_i e^{l\frac{i2\pi}{R}} + (1-q_i))\right)^{2m_i}. \quad \text{(A.3)}$$

We can simplify:

$$(a_i+b_i)^2(q_i e^{l\frac{i2\pi}{R}} + (1-q_i))^2 = (a_i^2 e^{l\frac{i2\pi}{R}} + b_i^2)^2.$$

Now we apply the multinomial theorem in reverse:

$$= \frac{p!q!}{r!}\frac{1}{R}\sum_{l=0}^{2r} e^{-lp\frac{i2\pi}{R}} \left(\sum_{i=1}^{N}(a_i e^{l\frac{i2\pi}{R}} + b_i)^2\right)^r$$

$$= \frac{p!q!}{r!}\frac{1}{R}\mathcal{F}_R\{Y_l^r\}_p,$$

where

$$Y_l = \sum_{i=1}^{N}(a_i e^{l\frac{i2\pi}{R}} + b_i)^2$$

and

$$\mathcal{F}_R, \mathcal{F}_R^{-1} : \mathbb{C}^R \longrightarrow \mathbb{C}^R$$

$$(x_l)_{1\leq l\leq R} \longmapsto \left(\mathcal{F}_R\{x_l\}_k\right)_{1\leq k\leq R} = \left(\sum_{l=0}^{R-1} e^{-l\frac{i2\pi}{R}} x_l\right)_{1\leq k\leq R}$$

$$(x_l)_{1\leq l\leq R} \longmapsto \left(\mathcal{F}_R^{-1}\{x_l\}_k\right)_{1\leq k\leq R} = \left(\frac{1}{R}\sum_{l=0}^{R-1} e^{l\frac{i2\pi}{R}} x_l\right)_{1\leq k\leq R}$$

is the discrete Fourier transform and its corresponding inverse. We rewrite

$$\frac{1}{R}\mathcal{F}_R\{Y_l^r\}_p = \mathcal{F}_R^{-1}\{Y_{R-l}^r\}_p.$$

We want to pull the powers $Y_l^r$ in the Fourier transform to the outside. To do this we use the convolution theorem for the discrete Fourier Transform:

$$\mathcal{F}_R^{-1}\{A \cdot B\}_x = \frac{1}{R}\sum_{l=0}^{R-1} \mathcal{F}_R^{-1}\{A\}_l \cdot \mathcal{F}_R^{-1}\{B\}_{(x-l)\%R}.$$

Here % denotes the modulo operation. Applying this repeatedly yields

$$\begin{aligned}\mathcal{F}_R^{-1}\left\{Y_{R-l}^r\right\}_p = \frac{1}{R}^{R-1} \sum_{l_1,\dots,l_{r-1}=0}^{R-1} & \mathcal{F}_R^{-1}\{Y_{R-l}\}_{l_1} \\ & \cdot\mathcal{F}_R^{-1}\{Y_{R-l}\}_{(l_2-l_1)\%R} \\ & \vdots \\ & \cdot\mathcal{F}_R^{-1}\{Y_{R-l}\}_{(l_{p-1}-l_{p-2})\%R} \\ & \cdot\mathcal{F}_R^{-1}\{Y_{R-l}\}_{(p-l_{p-1})\%R} \\ = \sum_{\substack{0\le l_1,\dots,l_{r-1}\le R-1 \\ l_0=0,\, l_r=p}} \prod_{t=1}^{r} & \mathcal{F}_R\{Y_l\}_{(l_t-l_{t-1})\%R}.\end{aligned}$$

To continue we now compute $\mathcal{F}_R\{Y_l\}$:

$$\begin{aligned}\mathcal{F}_R\{Y_l\}_x &= \sum_{l=0}^{R-1} e^{-lx\frac{i2\pi}{R}} \sum_{i=1}^{N}(a_i e^{l\frac{i2\pi}{R}} + b_i)^2 \\ &= \sum_{i=1}^{N}\sum_{l=0}^{R-1} e^{-lx\frac{i2\pi}{R}} \left(a_i^2 e^{2l\frac{i2\pi}{R}} + 2a_i b_i e^{l\frac{i2\pi}{R}} + b_i^2\right) \\ &= \sum_{i=1}^{N} a_i^2\delta_{x,2} + 2a_i b_i \delta_{x,1} + b_i^2\delta_{x,0}.\end{aligned}$$

Here $\delta_{u,v} = \mathbb{1}_{\{u=v\}}$ is the Kronecker delta. With this we have

$$(A.3) = \frac{p!q!}{r!}\frac{1}{R}\mathcal{F}_R\{Y_l^r\}_p$$

$$= \frac{p!q!}{r!} \sum_{\substack{0\le l_1,\dots,l_{r-1}\le R-1 \\ l_0=0,\, l_p=p}} \prod_{t=1}^{r}\sum_{i_t=1}^{N}\left(a_{i_t}^2\delta_{(l_t-l_{t-1})\%R,2} + 2a_{i_t}b_{i_t}\delta_{(l_t-l_{t-1})\%R,1} + b_{i_t}^2\delta_{(l_t-l_{t-1})\%R,0}\right).$$

Suppose we are given $l_{t-1}$. What are the possible values $l_t$ can take so that we get a non-zero term? It has to be the case that $(l_t - l_{t-1})\%R \in \{0, 1, 2\}$. As $l_t \le R-1$ it can never happen that $l_t - l_{t-1} \ge R$. It can however happen that $l_t - l_{t-1} < 0$. We split $\{0, ..., R-1\}$ in three regions:

- If $l_{t-1} \in [0, R-3]$ then $l_t - l_{t-1} \in [3-R, 2]$ and so only if $l_t \in l_{t-1} + \{0,1,2\}$ can we have a nonzero term. One could say that the following "moves" are available to us:

$$l_t \in \{l_{t-1}, l_{t-1}+1, l_{t-1}+2\}.$$

- If $l_{t-1} = R-2$ then $l_t - l_{t-1} \in [2-R, 1]$. Besides choosing $l_t = l_{t-1}$ or $l_t = l_{t-1}+1$ we can choose $l_t = 0$ so that $(l_t - l_{t-1})\%R = 2$. We have the following moves:

$$l_t \in \{l_{t-1}, l_{t-1}+1, 0\}$$

  If $l_{t-1} = R-1$ then $l_t - l_{t-1} \in [1-R, 2]$. Besides choosing $l_t = l_{t-1}$ we can choose $l_t = 0$ so that $(l_t - l_{t-1})\%R = 1$ and $l_t = 1$ so that $(l_t - l_{t-1})\%R = 2$. We have the following moves:

$$l_t \in \{l_{t-1}, 0, 1\}.$$

The behvaiour that this corresponds to is that every tuple $(l_0, l_1, ..., l_{r-1}, l_r)$ describes a path in $\mathbb{Z}/(2r+1)\mathbb{Z} = \mathbb{Z}/(p+q+1)\mathbb{Z}$ that starts in 0, ends in $p$, and has $r = \frac{p+q}{2}$ steps, each consisting of one of the actions +0, +1 or +2. The longest possible path consists of adding +2 each time and so has a total length of $2r$. Therefore there can not exist such a path from 0 to $p$ that goes "around the clock" more than zero times (such a path would need a length of at least $2r+1$). As a result there is no tuple $(l_1, ..., l_{r-1})$ which produces a non-zero term in the sum above where also $(l_t - l_{t-1})\%R \neq l_t - l_{t-1}$, i.e. the $\%R$ is superfluous. We rewrite the previous result as

$$= \frac{p!q!}{r!} \sum_{i_1,...,i_r=1}^{N} \sum_{\substack{l_0,...,l_r \in \mathbb{N} \\ 0=l_0\leq ... \leq l_r = p \\ l_t - l_{t-1} \in \{0,1,2\}}} \prod_{t=1}^{r} \Big( a_{i_t}^2 \delta_{l_t - l_{t-1},2} + 2a_{i_t} b_{i_t} \delta_{l_t - l_{t-1},1} + b_{i_t}^2 \delta_{l_t - l_{t-1},0} \Big)$$

$$= \frac{p!q!}{r!} \sum_{i_1,...,i_r=1}^{N} \sum_{\substack{h_1,...,h_r \in \{0,1,2\} \\ h_1 + ... + h_r = p}} \prod_{t=1}^{r} \Big( a_{i_t}^2 \delta_{h_t,2} + 2a_{i_t} b_{i_t} \delta_{h_t,1} + b_{i_t}^2 \delta_{h_t,0} \Big).$$

To further simplify this combinatorically, we carefully observe that the map

$$\begin{matrix} \{h \in \{0,1,2\}^r : & \longrightarrow & \{(I_1, I_2) \in \mathcal{P}(\{1,...,r\})^2 : \\ h_1 + ... + h_r = p\} & & I_1 \cap I_2 = \varnothing, |I_0| + 2|I_2| = p\} \end{matrix}$$

given by

$$h \longmapsto (\{i : h_i = 1\}, \{i : h_i = 2\})$$

is a bijection. This is because

$$(I_1, I_2) \longmapsto h_i \equiv \begin{cases} 2, & i \in I_2 \\ 1, & i \in I_1 \\ 0, & i \notin I_1 \cup I_2 \end{cases}$$

is an inverse. We get

$$(A.3) = \frac{p!q!}{r!} \sum_{\substack{I_1, I_2 \subset \{1,\dots,r\} \\ I_1 \cap I_2 = \varnothing \\ |I_1|+2|I_2|=p}} \sum_{i_1,\dots,i_r=1}^{N} \prod_{t \in I_2} a_{i_t}^2 \cdot \prod_{t \in I_1} 2a_{i_t} b_{i_t} \cdot \prod_{t \notin I_1 \cup I_2} b_{i_t}^2.$$

Suppose that $I_1,\ I_2$ is one such decomposition of $\{1,...,r\}$ and $J_1,\ J_2$ is another one with $|I_1| = |J_1|$ and $|I_2| = |J_2|$. Let $\sigma$ be a permutation of $\{1,...,r\}$ so that $\sigma(I_1) = J_1$ and $\sigma(I_2) = J_2$. The existence of at least one such permutation is guaranteed since the cardinalities match. Then since for any function $f : \mathbb{N}^r \longrightarrow \mathbb{R}$,

$$\sum_{i_1,\dots,i_r=1}^{N} f(i_1,...,i_r) = \sum_{i_1,\dots,i_r=1}^{N} f(\sigma(i_1),...,\sigma(i_r)),$$

we have

$$\sum_{i_1,\dots,i_r=1}^{N} \prod_{t \in I_2} a_{i_t}^2 \cdot \prod_{t \in I_1} 2a_{i_t} b_{i_t} \cdot \prod_{t \notin I_1 \cup I_2} b_{i_t}^2$$

$$= \sum_{i_1,\dots,i_r=1}^{N} \prod_{t \in I_2} a_{\sigma(i_t)}^2 \cdot \prod_{t \in I_1} 2a_{\sigma(i_t)} b_{\sigma(i_t)} \cdot \prod_{t \notin I_1 \cup I_2} b_{\sigma(i_t)}^2$$

$$\sum_{i_1,\dots,i_r=1}^{N} \prod_{t \in J_2} a_{i_t}^2 \cdot \prod_{t \in J_1} 2a_{i_t} b_{i_t} \cdot \prod_{t \notin J_1 \cup J_2} b_{i_t}^2.$$

As a result in

$$\frac{p!q!}{r!} 2^r \sum_{\substack{I_1, I_2 \subset \{1,\dots,r\} \\ I_1 \cap I_2 = \varnothing \\ |I_1|+2|I_2|=p}} \sum_{i_1,\dots,i_p=1}^{N} \prod_{t \in I_2} a_{i_t}^2 \cdot \prod_{t \in I_1} 2a_{i_t} b_{i_t} \cdot \prod_{t \notin I_1 \cup I_2} b_{i_t}^2.$$

the second sum does not depend on the exact shape of the sets $I_1$ and $I_2$, but instead only on $|I_2|$ and $|I_1| = p - 2|I_2|$. Given $r$, the possible values for $s = |I_2|$ so that a corresponding $I_1$ can exist are $s \in \{0 \vee (p-r), ..., r \wedge \lfloor \frac{p}{2} \rfloor\}$. This is becuase for each of these $s$, there exist

$$\binom{r}{s}\binom{r-s}{p-2s} = \binom{r}{s, p-2s, r-p+s} \qquad \left( = \binom{|\{1,...,r\}|}{|I_2|, |I_1|, |\{1,...,r\} \setminus (I_1 \cup I_2)|} \right)$$

ways to choose the sets $I_1$ and $I_2$. If $p > q$ then $p > r$ and binomial coefficients with negative numbers appear. Using the convention $\binom{x}{y} = 0$ for $y < 0$ and $y > x$ the expression above is correct and also gives the aforementioned bounds for $s$, but we can mitigate confusion by simply assuming WLOG that $p \leq q$ and so $p \leq r$. Therefore our sum becomes

$$= \frac{p!q!}{r!} \sum_{i_1,\dots,i_p=1}^{N} \sum_{s=0 \vee (p-r)}^{r \wedge \lfloor \frac{p}{2} \rfloor} \binom{r}{s, p-2s, r-p+s} \prod_{t=1}^{s} a_{i_t}^2 \cdot \prod_{t=s+1}^{p-s} 2a_{i_t} b_{i_t} \cdot \prod_{t=p-s+1}^{r} b_{i_t}^2.$$

$$= \frac{p!q!}{r!} \sum_{s=0\vee(p-r)}^{r\wedge\lfloor\frac{p}{2}\rfloor} 2^{p-2s} \binom{r}{s, p-2s, r-p+s}$$

$$\sum_{i_1,\dots,i_s=1}^{N} \prod_{t=1}^{s} a_{i_t}^2 \sum_{i_{s+1},\dots,i_{p-s}=1}^{N} \prod_{t=s+1}^{p-s} a_{i_t} b_{i_t} \sum_{i_{p-s+1},\dots,i_r=1}^{N} \prod_{t=p-s+1}^{r} b_{i_t}^2$$

$$= \frac{p!q!}{r!} \sum_{s=0\vee(p-r)}^{r\wedge\lfloor\frac{p}{2}\rfloor} 2^{p-2s} \binom{r}{s, p-2s, r-p+s} \left(\sum_{n=1}^{N} |a_n|^2\right)^s \left(\sum_{n=1}^{N} |a_n b_n|\right)^{p-2s} \left(\sum_{n=1}^{N} |b_n|^2\right)^{r-p+s}$$

To conclude the proof we let $N \longrightarrow \infty$. Then depending on what relation $\simeq$ is, if one side converges the other does as well and if the above converges it converges to

$$\frac{p!q!}{r!} \sum_{s=0\vee(p-r)}^{r\wedge\lfloor\frac{p}{2}\rfloor} 2^{p-2s} \binom{r}{s, p-2s, r-p+s} \|a\|_{\ell^2}^{2s} \langle a, b\rangle_{\ell^2}^{p-2s} \|b\|_{\ell^2}^{2(r-p+s)}$$

$\square$

**Lemma A.2.** *Let $(W_n)_{n\in\mathbb{N}}$ be a sequence of independent one-dimensional Brownian motions. Let $e_n$ be any ONB of $L^2_{\mathrm{r}}$. Consider a stochastic process $\psi$ of the form*

$$\psi(t) = \sum_{n=0}^{\infty} \int_0^t f_n(s)\, dW_n(s) \cdot e_n$$

*for functions $f_n \in C(\mathbb{R}_+)$. Let $\alpha \geq 0$ and $T \geq 0$. Then the following are equivalent:*

*(i)* $\psi \in C([[0,T], H^\alpha_{\mathrm{r}})$ *a.s.*

*(ii)* $\mathbb{P}(\psi \in C([[0,T], H^\alpha_{\mathrm{r}})) > 0$,

*(iii)* $\mathbb{E}\left[\|\psi(t)\|^2_{C([0,T],H^\alpha)}\right] < \infty$,

*(iv)* $\left(\langle\lambda_n\rangle^\alpha \|f_n\|_{L^2([0,T])}\right)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$.

*In particular we have the estimate*

$$\mathbb{E}\left[\|\psi(t)\|^2_{C([0,T],H^\alpha)}\right] \leq 2 \left\|\left(\langle\lambda_n\rangle^\alpha \|f_n\|_{L^2([0,T])}\right)_{n\in\mathbb{N}}\right\|_{\ell^2(\mathbb{N})} = 2\,\mathbb{E}\left[\|\psi(T)\|^2_{H^\alpha}\right]. \tag{A.4}$$

*Proof.* We start by showing $(iii) \iff (iv)$ and the estimate.
Observe that we can trivially move the supremum inside the sum, and the sum out of the integral by Fubini-Tonelli:

$$\mathbb{E}\left[\|\psi(t)\|^2_{C([0,T],H^\alpha)}\right] = \mathbb{E}\left[\sup_{0\leq t\leq T} \sum_{n=0}^{\infty} \left|\int_0^t f_n(s)\, dW_n(s)\right|^2 \langle\lambda_n\rangle^{2\alpha}\right]$$

$$\leq \sum_{n=0}^{\infty} \langle\lambda_n\rangle^{2\alpha}\, \mathbb{E}\left[\left|\sup_{0\leq t\leq T} \int_0^t f_n(s)\, dW_n(s)\right|^2\right].$$

Since $\int_0^t f_n(s)\, dW_n(s)$ is a square integrable martingale we can use Doob's $L^2$ inequality and then Itô-isometry:

$$\leq \sum_{n=0}^{\infty} \langle \lambda_n \rangle^{2\alpha} \cdot 2 \sup_{0 \leq t \leq T} \mathbb{E}\left[ \left| \int_0^t f_n(s)\, dW_n(s) \right|^2 \right]$$
$$= 2 \sum_{n=0}^{\infty} \langle \lambda_n \rangle^{2\alpha} \int_0^T |f_n(s)|^2\, ds.$$

For the reverse inequality we do the same steps except that the supremum is now absent and hence no inequalities are needed.

Clearly $(i) \implies (ii)$. Regarding $(iii) \implies (i)$, observe that for $t_1 < t_2 \in [0, T]$ we can calculate

$$\|\psi(t_2) - \psi(t_1)\|_{H^\alpha}^2 = \sum_{n=0}^{\infty} \langle \lambda_n \rangle^{2\alpha} \int_{t_1}^{t_2} |f_n(s)|^2\, ds,$$

and since $(iv)$ states that

$$\sum_{n=0}^{\infty} \langle \lambda_n \rangle^{2\alpha} \int_0^T |f_n(s)|^2\, ds < \infty$$

we get that $\psi$ is both continuous and bounded with respect to $\|\cdot\|_{H^\alpha}$.

It now only remains to show $(ii) \implies (iii)$. We can apply estimate (A.4) to the case where only finitely many $f_n$ are non-zero. The result is that to show $(iii)$ it suffices to show

$$\mathbb{E}\left[ \sum_{n=1}^{\infty} \langle \lambda_n \rangle^{2\alpha} |\widehat{\phi}(n)|^2 \right] < \infty. \tag{A.5}$$

where $\phi = \psi(T)$. Since the convergence of the sum in the expectation is a tail-event for the sigma algebra $\sigma(\{\widehat{\phi}(n)\}_{n\in\mathbb{N}})$, Kolmogorov's 0-1 law and $(ii)$ imply $(i)$. Then

$$\sum_{n=1}^{N} \langle \lambda_n \rangle^{\alpha} \widehat{\phi}(n)$$

is a sequence of gaussian random variables which is Cauchy in $L^2(\Omega)$ and hence converges to another gaussian random variable. As a consequence we know that the second moment of the limit is finite, i.e. (A.5). □

**Lemma A.3** (Fokker-Planck Equation)**.** *Let $d \in \mathbb{N}$, $b \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ and $\sigma \in C^2(\mathbb{R}^d, R^{d\times d})$. Suppose on some probability space we have a d-valued Brownian Motion $B_t$ and a stochastic process $X_t$ that solves the SDE*

$$dX_t = b(X_t)\, dt + \sigma(X_t)\, dB_t$$

*with initial distribution* $\mathrm{Law}(X_0) = p_0(x)\, dx$. *Then there exists a density $p : [0,\infty) \times \mathbb{R}^d \longrightarrow [0,\infty)$ so that* $\mathrm{Law}(X_t) = p(t,x)\, dx$ *and $p$ is a weak solution to the Fokker Planck Equation*

$$\partial_t p(t,x) = -\partial_{x_j}(b^j(x) p(t,x)) + \frac{1}{2} \partial^{x_j} \partial_{x_i}(\sigma_k^j(x) \sigma_i^k(x) p(s,x)) \tag{A.6}$$
$$p(0,x) = p_0(x)\, dx. \tag{A.7}$$

*Proof.* Clearly $\mathbb{P}(X_t \in A) = 0$ if $\int_A dx = 0$ for all measurable $A \subseteq \mathbb{R}^d$, so $\mathrm{Law}(X_t)$ is absolutely continuous with respect to $dx$ for all $t \geq 0$. Then the Radon-Nikodym theorem gives us the function $p(t,x)$. Since the process $X_t$ is continuous and Gaussian we know that $p$ is continuous in $t$ and $x$. Now let $f : [0,\infty) \times \mathbb{R}^d)$ be compactly supported and continuously differentiable once in time and twice in space. An application of Ito's formula yields

$$f(t,X_t) - f(0,X_0) = \int_0^t \partial_s f(s,X_s)\,ds + \int_0^t \partial_{x_j} f(s,X_s)(b^j(X_s)\,ds + \sigma_i^j(X_s)\,dB^i(s)) + \int_0^t \frac{1}{2}\partial^{x_i}\partial_{x_j} f(s,X_s)\sigma_k^j(X_s)\sigma_i^k(X_s)\,ds.$$

Then

$$\int_{\mathbb{R}^d} f(t,x)p(t,x) - f(0,x)p(0,x)\,dx = \mathbb{E}\left[f(X_t) - f(X_0)\right]$$
$$= \int_0^t \int_{\mathbb{R}^d} \left[\partial_s f(s,x) + \partial_{x_j} f(s,x)b^j(x) + \frac{1}{2}\partial^{x_i}\partial_{x_j} f(s,x)\sigma_k^j(x)\sigma_i^k(x)\right] p(s,x)\,dx\,ds.$$

□

**Lemma A.4** (Uniqueness for Fokker-Planck Equation)**.** *[1, Thm. 9.8.9.] Fix $T > 0$ and $\mu_0$ be a measure on $\mathbb{R}^d$. Let $\sigma \in \mathbb{R}^{d\times d}$ be symmetric and $b \in C(\mathbb{R}^d, \mathbb{R}^d)$ so that on any set of the form $B_r(x_0) \times (0,T)$ the continuity of $b$ with respect to $x$ is uniform in $t$. Then there exists at most one family of measures $\{\mu_t\}_{t\in[0,T]}$ on $\mathbb{R}^d$ so that the following hold:*

*(i) For all $1 \leq i \leq d$,*

$$\int_0^T \int_{\mathbb{R}^d} \frac{1}{1+|x|^2} + \frac{|b^i|}{1+|x|}\,d\mu_t(x)\,dt < \infty.$$

*(ii) The Fokker-Planck equation is solved weakly with initial data $\mu_0$, meaning that for any $f \in C_c^\infty([0,T) \times \mathbb{R}^d)$ we have*

$$\int_0^T \int_{\mathbb{R}^d} \left[\partial_t f(t,x) + \partial_{x_j} f(t,x)b^j(x) + \frac{1}{2}\partial^{x_i}\partial_{x_j} f(t,x)\sigma_k^j(x)\sigma_i^k(x)\right] d\mu_t(x)\,dt$$
$$= -\int_{\mathbb{R}^d} f(0,x)\,d\mu_0(x).$$

*Proof.* This is merely a direct simplification of [1, Thm. 9.8.9.] to the case of a constant diffusion matrix $\sigma$. □

**Lemma A.5.** *Let $F_n$ be the $n$-th Fibonacci number and $a, b \in \mathbb{R}$. For all $p \in \mathbb{N}_{\geq 1}$,*

$$\left|a^p - b^p\right| \leq F_p|a-b|(|a|+|b|)^{p-1}. \tag{A.8}$$

*Proof.* We show this by induction on $p$.

- Case $p = 1$: This is trivial

$$\left|a - b\right| \leq F_1|a-b|(|a|+|b|)^0.$$

- Case $p = 2$: This is a binomial identity.

$$\left|a^2 - b^2\right| \leq |a - b|(|a| + |b|)$$

- Case $p > 2$: We assume (A.8) holds for $p \leq 2$. Then

$$\left|a^p - b^p\right| = \left|(a + b)(a^{p-1} - b^{p-1}) - ab\,(a^{p-2} - b^{p-2})\right|$$

$$\leq (|a| + |b|)F_{p-1}|a - b|(|a| + |b|)^{p-2} + (|a| + |b|)^2 F_{p-2}|a - b|(|a| + |b|)^{p-3}$$

$$\leq (F_{p-1} + F_{p-2})|a - b|(|a| + |b|)^{p-1}.$$

□