UNIVERSITÄT KARLSRUHE
INSTITUT FÜR ANALYSIS
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WS 2007/08
31st Oct 2007

1st Exercise sheet

Functional Analysis

Deadline: Thursday, 8th Nov 2007, 15 p.m.

Exercise 1

Let $1 \leqslant p, q, r \leqslant \infty$.

a) Assume $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, with the convention that $\frac{1}{\infty} = 0$, as in the exercise session. Let $x = (x_n)_n \in \ell^p$ and $y = (y_n)_n \in \ell^q$. Show that $(xy)_n := (x_n y_n)_n$ belongs to ℓ^r and the following generalization of the Hölder inequality holds:

$$\|xy\|_r \leqslant \|x\|_p \|y\|_q.$$

(Hint: Use the common Hölder inequality.)

b) Assume that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. Let $x \in \ell^p$, $y \in \ell^q$, $z \in \ell^r$. Show that $(xyz)_n := (x_n y_n z_n)_n$ belongs to ℓ^1 and $\|xyz\|_1 \leqslant \|x\|_p \|y\|_q \|z\|_r$.

Exercise 2 (C)

Let $n \in \mathbb{N}$ and $1 \leqslant p, q \leqslant \infty$. Denote as usual for $x = (x_k)_{k=1,\ldots,n} \in \mathbb{C}^n$ $\|x\|_p := \left(\sum_{j=1}^n |x_j|^p\right)^{1/p}$.

a) Let $\mathbb{1} := (1, \ldots, 1) \in \mathbb{C}^n$. Show that $\|\mathbb{1}\|_p = n^{\frac{1}{p}}$.

b) Assume that $q \leqslant p$. Show that for all $x \in \mathbb{C}^n$

$$\|x\|_q \leqslant n^{\frac{1}{q} - \frac{1}{p}} \|x\|_p.$$

Show that $n^{\frac{1}{q} - \frac{1}{p}}$ cannot be replaced by a smaller constant.
(Hint: Ex.1 a).)

c) Assume now that $q > p$. Show that for all $x \in \mathbb{C}^n$

$$\|x\|_q \leqslant \|x\|_p.$$

(Hint: Consider first the case $\|x\|_p = 1$.)

Exercise 3

Define

$$Tx = \left(\sum_{k=1}^{\infty} a_{jk} x_k \right)_{j \in \mathbb{N}}$$

where $A = (a_{jk}) \in \mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ and $x = (x_k) \in \mathbb{C}^{\mathbb{N}}$ are such that the sums exist.

a) Show that $T \in B(\ell^1)$ (i.e. $(Tx)_j$ converges for all $x \in \ell^1$ and $j \in \mathbb{N}$, $Tx \in \ell^1$ and $T : \ell^1 \to \ell^1$ is continuous) if and only if $C_1 = \sup_{k\in\mathbb{N}} \sum_{j=1}^{\infty} |a_{jk}| < \infty$. In this case $\|T\| = C_1$.

b) Show that $T \in B(\ell^\infty)$ if and only if $C_\infty = \sup_{j\in\mathbb{N}} \sum_{k=1}^{\infty} |a_{jk}| < \infty$. In this case $\|T\| = C_\infty$.

Exercise 4 (C)

The differential operator $\frac{d}{dt} : C^1([0,1]) \to C([0,1])$ maps every function $f \in C^1([0,1])$ to its derivative $\frac{d}{dt}f = f'$. Show that the following assertions hold:

a) Let us consider $C^1([0,1])$ as a subspace of $(C([0,1]), \|\cdot\|_\infty)$. Then $\frac{d}{dt}$ is not continuous.

b) If $C^1([0,1])$ is endowed with the norm $\|f\| := \|f\|_\infty + \|\frac{d}{dt}f\|_\infty$, then $\frac{d}{dt}$ is continuous and $\|\frac{d}{dt}\| = 1$.

Information about the problem sessions:
Every Thursday, there will be an exercise sheet. The sheets will be available next to office 305 in the mathematics building and on the web page
http://www.mathematik.uni-karlsruhe.de/mi1weis/lehre/fa2007w.
For every exercise sheet, you can hand in the two exercises marked with a C (1 student per sheet). The corresponding box is next to office 328 in the mathematics building. The deadline is on Thursday in general, please verify on each exercise sheet. You can get back the corrected solutions the week after next to the stairs near office 302, mathematics building.

If you have questions concerning the problem sessions, you can come to my office: room 131.2 in the mathematics building (next to S13), or send me an email:
christoph.kriegler(at)math.uni-karlsruhe.de.

Übungsschein (Certificate for successful participation at the exercise sessions)
For each exercise sheet you can get 8 points, in general 4 points for each exercise. To get the Übungsschein you need 50% of all the points of the first 4 sheets, 50% on the sheets 5-8 and 50% on the rest of the sheets.